Math. Proc. Camb. Phil. Soc. (2008), **145**, 601
doi:10.1017/S0305004108001552 *Printed in the United Kingdom*
First published online 4 July 2008

A Cohen–Macaulay algebra has only finitely many semidualizing modules

BY LARS WINTHER CHRSTENSEN†

Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042*, U.S.A.*
e-mail: lars.w.christensen@ttu.edu

AND SEAN SATHER–WAGSTAFF‡

Department of Mathematics, 300 *Minard Hall, North Dakota State University, Fargo, ND* 58105-5075*, U.S.A.*
e-mail: Sean.Sather-Wagstaff@ndsu.edu

(*Received* 1 *May* 2007; *revised* 10 *July* 2007)

Abstract

We prove the result stated in the title, which answers the equicharacteristic case of a question of Vasconcelos.

In this paper, R is a commutative noetherian local ring. A finitely generated R-module C is *semidualizing* if the homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. Examples include R itself and (if one exists) a dualizing R-module in the sense of Grothendieck. Semidualizing modules have properties similar to those of dualizing modules [**2**] and arise in several contexts.

Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules. Vasconcelos, calling these modules "spherical," asked whether $\mathfrak{S}_0(R)$ is finite when R is Cohen–Macaulay and whether it has even cardinality when it contains more than one element [**8**, p. 97]. In [**7**] affirmative answers to these questions are given, e.g., for certain determinantal rings. Here we prove:

THEOREM 1. *If R is Cohen–Macaulay and equicharacteristic, then $\mathfrak{S}_0(R)$ is finite.*

Remark 2. This result also yields an answer to the parity part of Vasconcelos' question for certain Cohen–Macaulay rings. Let R be as in Theorem 1. If R is Gorenstein, then $\mathfrak{S}_0(R)$ has exactly one element, namely the isomorphism class of R; see [**2**, corollary (8·6)]. On the other hand, if R has a dualizing module and is not Gorenstein, then $\mathfrak{S}_0(R)$ has even cardinality. Indeed, in the derived category of R, every semidualizing R-complex is isomorphic to a shift of a semidualizing R-module; see [**4**, corollary 3·4]. Thus, the set of shift-isomorphism

† This work was done while visiting University of Nebraska–Lincoln, partly supported by grants from the Danish Natural Science Research Council and the Carlsberg Foundation.
‡ Partially supported by NSF grant NSF 0354281.

classes of semidualizing R-complexes is finite, and so [**2**, proposition (3·7)] shows that it has even cardinality.

In preparation for the proof of Theorem 1, we recall some recent results that allow us to reduce it to a problem in representation theory over artinian k-algebras.

Fact 3. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite flat dimension, i.e., such that S has finite flat dimension as an R-module via φ. If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module by [**2**, proposition (5·7)]. If C and C' are semidualizing R-modules such that $S \otimes_R C$ and $S \otimes_R C'$ are isomorphic as S-modules, then C and C' are isomorphic as R-modules; see [**3**, theorem 4·5 and 4·9]. Thus, the functor $S \otimes_R -$ induces an injective map $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(S)$.

Lemma 4. Assume there are local ring homomorphisms of finite flat dimension

$$R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q \xrightarrow{\tau} Q'$$

such that ρ is surjective with kernel generated by a Q-regular sequence. Then there are inequalities of cardinalities $|\mathfrak{S}_0(R)| \leqslant |\mathfrak{S}_0(\widehat{Q})| \leqslant |\mathfrak{S}_0(\widehat{Q'})|$.

Proof. The completion morphism $\epsilon\colon R \to \widehat{R}$ conspires with the completions of the given maps to yield the following

$$\mathfrak{S}_0(R) \xhookrightarrow{\mathfrak{S}_0(\hat{\varphi}\epsilon)} \mathfrak{S}_0(\widehat{R'}) \xleftarrow{\mathfrak{S}_0(\hat{\rho})} \mathfrak{S}_0(\widehat{Q}) \xhookrightarrow{\mathfrak{S}_0(\hat{\tau})} \mathfrak{S}_0(\widehat{Q'}).$$

The injectivity of the induced maps is justified in Fact 3; the surjectivity of $\mathfrak{S}_0(\hat{\rho})$ follows from [**3**, proposition 4·2] because $\widehat{Q}$ is complete and $\hat{\rho}$ is surjective with kernel generated by a $\widehat{Q}$-regular sequence. The desired inequalities now follow.

Proof of Theorem 1. Let $\boldsymbol{x}$ be a system of parameters for R and set $R' = R/(\boldsymbol{x})$ with $\varphi\colon R \to R'$ the natural surjection. Using Lemma 4 with $Q' = Q = R' \cong \widehat{R'}$, we may replace R with R' in order to assume that R is artinian.

There is a flat homomorphism of artinian local rings $R \to R''$, such that R'' has algebraically closed residue field; see [**5**, proposition 0·(10·3·1)]. By Fact 3 we may replace R by R'' to assume that its residue field k is algebraically closed. As R is equicharacteristic and artinian, Cohen's structure theorem implies that R is a k-algebra.

For an R-module M, let $\nu(M)$ denote the minimal number of generators of M. Let C be a semidualizing R-module and let E be the injective hull of k. Hom-evaluation [**1**, proposition 5.3] and the homothety map yield a sequence of isomorphisms

$$C \otimes_R \operatorname{Hom}_R(C, E) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, C), E) \cong \operatorname{Hom}_R(R, E) \cong E.$$

Hence, there is an inequality $\nu(C) \leqslant \nu(E)$. This gives the second inequality below; the first is from a surjection $R^{\nu(C)} \twoheadrightarrow C$.

$$\operatorname{length}_R C \leqslant \nu(C) \cdot \operatorname{length} R \leqslant \nu(E) \cdot \operatorname{length} R$$

By [**6**, proof of first proposition in section 3] there are only finitely many isomorphism classes of R-modules M with $\operatorname{Ext}_R^{\geqslant 1}(M, M) = 0$ and $\operatorname{length}_R M \leqslant \nu(E) \cdot \operatorname{length} R$. From the displayed inequalities it follows that $\mathfrak{S}_0(R)$ is a finite set.

Finally we illustrate how Theorem 1 and Lemma 4 apply to answer Vasconcelos' finiteness question for certain rings of mixed characteristic.

Example 5. Let Q be a complete Cohen–Macaulay local ring with residue field of characteristic $p > 0$. If p is Q-regular, then $\mathfrak{S}_0(R)$ is finite for every local ring R such that $\widehat{R} \cong Q$ or $\widehat{R} \cong Q/(p^n)$ for some $n \geqslant 2$: Theorem 1 implies that $\mathfrak{S}_0(Q/(p))$ is finite, and Lemma 4 applies to the diagram $R \to \widehat{R} \leftarrow Q \to Q/(p)$.

Acknowledgments

We thank Ragnar–Olaf Buchweitz, Hubert Flenner, Graham Leuschke and Diana White for useful conversations related to this work. We also thank the referee for thoughtful suggestions.

REFERENCES

[**1**] H. CARTAN and S. EILENBERG. Homological algebra, Princeton Landmarks in Mathematics (Princeton University Press, 1999), With an appendix by D. A. Buchsbaum. Reprint of the 1956 original. MR 1731415

[**2**] L. W. CHRISTENSEN. Semi-dualizing complexes and their Auslander categories. *Trans. Amer. Math. Soc.* **353** (2001), no. 5, 1839–1883. MR 1813596

[**3**] A. FRANKILD and S. SATHER–WAGSTAFF. Reflexivity and ring homomorphisms of finite flat dimension. *Comm. Algebra* **35** (2007), no. 2, 461–500. MR 2294611

[**4**] A. FRANKILD and S. SATHER–WAGSTAFF. The set of semidualizing complexes is a nontrivial metric space. *J. Algebra* **308** (2007), no. 1, 124–143. MR 2290914

[**5**] A. GROTHENDIECK. Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents I. *Inst. Hautes Études Sci. Publ. Math.* (1961), no. 11, 167. MR 0163910

[**6**] D. HAPPEL. Selforthogonal Modules, Abelian groups and modules (Padova, 1994), *Math. Appl.*, vol. **343** (Kluwer Acad. Publ., 1995), pp. 257–276. MR 1378204

[**7**] S. SATHER–WAGSTAFF. Semidualizing modules and the divisor class group. *Illinois J. Math.* **51** (2007), no. 1, 255–285. MR 2346197

[**8**] W. V. VASCONCELOS. Divisor Theory in Module Categories (North-Holland Publishing Co., 1974). North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR 0498530