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The set of semidualizing complexes is a nontrivial metric space ☆

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Abstract

We show that the set $\mathfrak{S}(R)$ of shift-isomorphism classes of semidualizing complexes over a local ring R admits a nontrivial metric. We investigate the interplay between the metric and several algebraic operations. Motivated by the dagger duality isometry, we prove the following: If K, L are homologically bounded below and degreewise finite R-complexes such that $K \otimes^{\mathbf{L}}_R K \otimes^{\mathbf{L}}_R L$ is semidualizing, then K is shift-isomorphic to R. In investigating the existence of nontrivial open balls in $\mathfrak{S}(R)$, we prove that $\mathfrak{S}(R)$ contains elements that are not comparable in the reflexivity ordering if and only if it contains at least three distinct elements.

Keywords: Semidualizing complexes; Gorenstein dimensions; Metric spaces; Bass numbers; Betti numbers; Curvature; Local homomorphisms; Gorenstein rings; Fixed points

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Introduction

Much research in commutative algebra is devoted to duality. One example of this is the work of Grothendieck and Hartshorne [17] which includes an investigation of the duality properties of finite modules and complexes with respect to a dualizing complex. A second example is the work of Auslander and Bridger [1,2] where a class of modules is identified, those of finite G-dimension, having good duality properties with respect to the ring.

These examples are antipodal in the sense that each one is devoted to the reflexivity properties of finite modules and complexes with respect to a semidualizing complex. See 1.5 for precise definitions. Examples of semidualizing complexes include the ring itself and the dualizing complex, if it exists. Another useful example is the dualizing complex of a local homomorphism of finite G-dimension, as constructed by Avramov and Foxby [5]. The study of the general situation was initiated by Foxby [12], Golod [16], and Christensen [9].

We denote by $\mathfrak{S}(R)$ the set of shift-isomorphism classes of semidualizing complexes over a local ring R; the class of a given semidualizing complex K is denoted $[K]$. The work in the current paper is part of an ongoing research effort on our part to analyze the structure of the set $\mathfrak{S}(R)$ in its entirety. That $\mathfrak{S}(R)$ has more structure than other collections of complexes is demonstrated by the fact that one can inflict upon $\mathfrak{S}(R)$ an ordering given by reflexivity; see 1.5 and 1.7. Further structure is demonstrated in [18] where it is observed that, when R is a Cohen–Macaulay normal domain, the set $\mathfrak{S}(R)$ is naturally a subset of the divisor class group $\mathrm{Cl}(R)$. The analysis of this inclusion yields, for instance, a complete description of $\mathfrak{S}(R)$ for certain classes of rings; see, e.g., Example 5.2.

The main idea in the present work is to use numerical data from the complexes in $\mathfrak{S}(R)$ that are comparable under the ordering to give a measure of their proximity. The distance between two arbitrary elements $[K], [L]$ of $\mathfrak{S}(R)$ is then described via chains of pairwise comparable elements starting with $[K]$ and ending with $[L]$. Details of the construction and its basic properties are given in Section 2. One main result, advertised in the title, is contained in Theorem 2.9.

Theorem A. *The set $\mathfrak{S}(R)$ is a metric space.*

Theorem 3.5, stated next, shows that the metric is not equivalent to the trivial one, unless $\mathfrak{S}(R)$ is itself almost trivial. It also implies that $\mathfrak{S}(R)$ quite frequently contains elements that are noncomparable in the ordering.

Theorem B. *For a local ring R the following conditions are equivalent*:

(i) *There exist elements of $\mathfrak{S}(R)$ that are not comparable.*
(ii) *$\mathfrak{S}(R)$ has cardinality at least* 3.
(iii) *There exists $[K] \in \mathfrak{S}(R)$ and $\delta > 0$ such that the open ball $B([K], \delta)$ satisfies $\{[K]\} \subsetneq B([K], \delta) \subsetneq \mathfrak{S}(R)$.*

This result follows in part from an analysis motivated by Proposition 3.1: if R admits a dualizing complex D, then the map $\mathfrak{S}(R) \to \mathfrak{S}(R)$ given by $[K] \mapsto [\mathbf{R}\mathrm{Hom}_R(K, D)]$ is an isometric involution. This fact led us to investigate the fixed points of this involution. Corollary 3.4 shows that the existence of such a fixed point implies that R is Gorenstein; it is a consequence of Theorem 3.2, stated next.

Theorem C. *Let R be a local ring and K, L homologically bounded below and degreewise finite R-complexes. If $K \otimes^{\mathbf{L}}_R K \otimes^{\mathbf{L}}_R L$ is semidualizing, then K is shift-isomorphic to R in the derived category $\mathcal{D}(R)$.*

Section 4 describes the behavior of the metric with respect to change of rings along a local homomorphism of finite flat dimension. In particular, these operations give a recipe for constructing noncomparable semidualizing complexes; see Corollary 4.7. We conclude with Section 5, which consists of explicit computations.

1. Complexes

This section consists of background and includes most of the definitions and notational conventions used throughout the rest of this paper.

Throughout this work, $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ are local Noetherian commutative rings.

An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

with $\partial^X_i \partial^X_{i+1} = 0$ for each i. We work in the derived category $\mathcal{D}(R)$ whose objects are the R-complexes; references on the subject include [14,17,19,20]. For R-complexes X and Y the left derived tensor product complex is denoted $X \otimes^{\mathbf{L}}_R Y$ and the right derived homomorphism complex is $\mathbf{R}\mathrm{Hom}_R(X, Y)$. For an integer n, the nth *shift* or *suspension* of X is denoted $\Sigma^n X$ where $(\Sigma^n X)_i = X_{i-n}$ and $\partial^{\Sigma^n X}_i = (-1)^n \partial^X_{i-n}$. The symbol "$\simeq$" indicates an isomorphism in $\mathcal{D}(R)$ and "$\sim$" indicates an isomorphism up to shift.

The *infimum*, *supremum*, and *amplitude* of a complex X are, respectively,

$$\inf(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\},$$
$$\sup(X) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\},$$
$$\mathrm{amp}(X) = \sup(X) - \inf(X),$$

with the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. The complex X is *homologically finite*, respectively *homologically degreewise finite*, if its total homology module $\mathrm{H}(X)$, respectively each individual homology module $\mathrm{H}_i(X)$, is a finite R-module.

The *ith Betti number* and *Bass number* of a homologically finite complex of R-modules X are, respectively,

$$\beta^R_i(X) = \mathrm{rank}_k\big(\mathrm{H}_{-i}\big(\mathbf{R}\mathrm{Hom}_R(X, k)\big)\big) \quad \text{and} \quad \mu^i_R(X) = \mathrm{rank}_k\big(\mathrm{H}_{-i}\big(\mathbf{R}\mathrm{Hom}_R(k, X)\big)\big).$$

The *Poincaré series* and *Bass series* of X are the formal Laurent series

$$P^R_X(t) = \sum_{i \in \mathbb{Z}} \beta^R_i(X) t^i \quad \text{and} \quad I^X_R(t) = \sum_{i \in \mathbb{Z}} \mu^i_R(X) t^i.$$

The projective, injective, and flat dimensions of X are denoted $\mathrm{pd}_R(X)$, $\mathrm{id}_R(X)$, and $\mathrm{fd}_R(X)$, respectively; see [11].

The Bass series of a local homomorphism of finite flat dimension is an important invariant that will appear in several contexts in this work.

1.1. A ring homomorphism $\varphi: R \to S$ is local when $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. In this event, the flat dimension of φ is defined as $\mathrm{fd}(\varphi) = \mathrm{fd}_R(S)$, and the *depth* of φ is $\mathrm{depth}(\varphi) = \mathrm{depth}(S) - \mathrm{depth}(R)$. When $\mathrm{fd}(\varphi)$ is finite, the *Bass series* of φ is the formal Laurent series with nonnegative integer coefficients $I_\varphi(t)$ satisfying the formal equality

$$I_S^S(t) = I_R^R(t) I_\varphi(t).$$

The existence of $I_\varphi(t)$ is given by [7, (5.1)] or [5, (7.1)]. The map φ is *Gorenstein* at $\mathfrak{n}$ if $I_\varphi(t) = t^d$ for some integer d (in which case $d = \mathrm{depth}(\varphi)$) equivalently, if $I_\varphi(t)$ is a Laurent polynomial.

Our metric utilizes the curvature of a homologically finite complex, as introduced by Avramov [3]. It provides an exponential measure of the growth of the Betti numbers of the complex. Let $F(t) = \sum_{n \in \mathbb{Z}} a_n t^n$ be a formal Laurent series with nonnegative integer coefficients. The *curvature* of $F(t)$ is

$$\mathrm{curv}\big(F(t)\big) = \limsup_{n \to \infty} \sqrt[n]{a_n}.$$

Of the following properties, parts (a) and (b) follow from the definition. For part (c), argue as in the proof of [4, (4.2.4.6)].

1.2. Let $F(t), G(t)$ be formal Laurent series with nonnegative integer coefficients.

(a) For each integer d, there is an equality $\mathrm{curv}(F(t)) = \mathrm{curv}(t^d F(t))$.
(b) A coefficientwise inequality $F(t) \preccurlyeq G(t)$ implies $\mathrm{curv}(F(t)) \leqslant \mathrm{curv}(G(t))$.
(c) There is an equality $\mathrm{curv}(F(t)G(t)) = \max\{\mathrm{curv}(F(t)), \mathrm{curv}(G(t))\}$.

1.3. Let X be a homologically degreewise finite R-complex. The *curvature* and *injective curvature* of X are

$$\mathrm{curv}_R(X) = \mathrm{curv}\big(P_X^R(t)\big) \quad \text{and} \quad \mathrm{inj\,curv}_R(X) = \mathrm{curv}\big(I_R^X(t)\big).$$

For a local homomorphism of finite flat dimension φ, the *injective curvature* of φ is

$$\mathrm{inj\,curv}(\varphi) = \mathrm{curv}\big(I_\varphi(t)\big).$$

In particular, the map φ is Gorenstein at $\mathfrak{n}$ if and only if $\mathrm{inj\,curv}(\varphi) = 0$.

1.4. Let X, Y be homologically finite complexes of R-modules.

(a) If $\varphi: R \to S$ is a local homomorphism, then $\mathrm{curv}_S(X \otimes_R^{\mathbf{L}} S) = \mathrm{curv}_R(X)$.
(b) There are inequalities $0 \leqslant \mathrm{curv}_R(X) < \infty$.
(c) The following conditions are equivalent:
  (i) $\mathrm{curv}_R(X) < 1$,
  (ii) $\mathrm{curv}_R(X) = 0$,
  (iii) $\mathrm{pd}_R(X)$ is finite.

Proof. The formal equality $P^S_{X\otimes^{\mathbf{L}}_R S}(t) = P^R_X(t)$ is by [5, (1.5.3)], and part (a) follows. Apply [4, (4.1.9), (4.2.3.5), (4.2.3.1)] to a truncation of the minimal free resolution of X to verify (b) and (c). □

Next, we turn to semidualizing complexes and their reflexive objects.

1.5. For homologically finite R-complexes K and X one has natural homothety and biduality homomorphisms, respectively:

$$\chi^R_K : R \to \mathbf{R}\mathrm{Hom}_R(K,K),$$
$$\delta^K_X : X \to \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(X,K),K\big).$$

The complex K is *semidualizing* if χ^R_K is an isomorphism; e.g., R is semidualizing.

A complex D is *dualizing* if it is semidualizing and has finite injective dimension; see [17, Chapter V]. Dualizing complexes are unique up to shift-isomorphism. Any homomorphic image of a local Gorenstein ring, e.g., any complete local ring, admits a dualizing complex by [17, (V.10.4)]. When D is dualizing for R, one has $I^D_R(t) = t^d$ for some integer d by [17, (V.3.4)].

When K is semidualizing, the complex X is *K-reflexive* if $\mathbf{R}\mathrm{Hom}_R(X,K)$ is homologically bounded and δ^K_X is an isomorphism; e.g., R and K are K-reflexive. When R admits a dualizing complex D, each homologically finite complex X is D-reflexive by [17, (V.2.1)]. A complex is R-reflexive exactly when it has finite G-dimension by [8, (2.3.8)].

The Poincaré and Bass series of a semidualizing complex are linked by [5, (1.5.3)].

1.6. When K is a semidualizing R-complex, there is a formal equality

$$P^R_K(t) I^K_R(t) = I^R_R(t).$$

Here is the fundamental object of study in this work.

1.7. The set of shift-isomorphism classes of semidualizing R-complexes is denoted $\mathfrak{S}(R)$. The class in $\mathfrak{S}(R)$ of a semidualizing complex K is denoted $[K]$. For $[K],[L] \in \mathfrak{S}(R)$ write $[K] \trianglelefteq [L]$ if L is K-reflexive; this is independent of the representatives for $[K]$ and $[L]$, and $[K] \trianglelefteq [R]$. If D is dualizing, then $[D] \trianglelefteq [L]$.

1.8. If $[K],[L] \in \mathfrak{S}(R)$ and $[K] \trianglelefteq [L]$, then $\mathbf{R}\mathrm{Hom}_R(L,K)$ is semidualizing and K-reflexive, that is, $[K] \trianglelefteq [\mathbf{R}\mathrm{Hom}_R(L,K)]$; see [9, (2.11)]. In particular, if D is dualizing for R, then the complex $L^\dagger = \mathbf{R}\mathrm{Hom}_R(L,D)$ is semidualizing; there are equalities $I^{L^\dagger}_R(t) = t^d P^R_L(t)$ and $P^R_{L^\dagger}(t) = t^{-d} I^L_R(t)$ for some $d \in \mathbb{Z}$ by [9, (1.7.7)], and so $\operatorname{inj\,curv}_R(L) = \operatorname{curv}_R(L^\dagger)$ by 1.2(a).

1.9. For semidualizing complexes K, L, M, consider the composition morphism

$$\xi_{MLK} : \mathbf{R}\mathrm{Hom}_R(M,L) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(L,K) \to \mathbf{R}\mathrm{Hom}_R(M,K).$$

This is an isomorphism when L and M are K-reflexive and M is L-reflexive by [15, (3.3)], and a formal equality of Laurent series follows from [5, (1.5.3)]

$$P^R_{\mathbf{R}\mathrm{Hom}_R(M,L)}(t) P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(M,K)}(t).$$

In particular, when $M = R$ the morphism is of the form $L \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(L, K) \to K$, and when L is K-reflexive one has $P^R_L(t) P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t) = P^R_K(t)$.

1.10. For $[K]$ in $\mathfrak{S}(R)$, the quantities $\mathrm{curv}_R(K)$ and $\mathrm{inj\,curv}_R(K)$ are well-defined. There are inequalities

$$0 \leqslant \mathrm{curv}_R(K) \leqslant \mathrm{inj\,curv}_R(R) < \infty \quad \text{and} \quad 0 \leqslant \mathrm{inj\,curv}_R(K) \leqslant \mathrm{inj\,curv}_R(R) < \infty$$

and the following conditions are equivalent:

(i) $\mathrm{curv}_R(K) < 1$,
(ii) $\mathrm{curv}_R(K) = 0$,
(iii) $[K] = [R]$.

Proof. For the first statement see 1.2(a), while the equivalence of (i)–(iii) follows from 1.4(c) and [9, (8.1)]. For the inequalities, pass to the completion of R to assume that R admits a dualizing complex D. With $(-)^{\dagger}$ as in 1.8, use 1.5 and 1.9 to verify the equality $P^R_D(t) = P^R_K(t) P^R_{K^{\dagger}}(t)$. With 1.2(c) this provides the first equality in the next sequence while the second is in 1.8:

$$\mathrm{curv}_R(K) \leqslant \max\bigl\{\mathrm{curv}_R(K), \mathrm{curv}_R\bigl(K^{\dagger}\bigr)\bigr\} = \mathrm{curv}_R(D) = \mathrm{inj\,curv}_R(R).$$

This gives one of the inequalities, while the others follow from 1.4(b) and 1.8. □

2. The metric

Here is the first step of the construction of the metric on $\mathfrak{S}(R)$.

2.1. For $[K], [L]$ in $\mathfrak{S}(R)$ with $[K] \trianglelefteq [L]$, set

$$\sigma_R\bigl([K], [L]\bigr) = \mathrm{curv}_R\bigl(\mathbf{R}\mathrm{Hom}_R(L, K)\bigr).$$

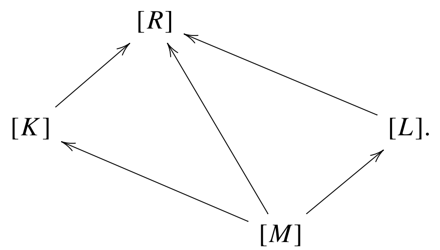
Apply 1.9 and 1.10 to $[\mathbf{R}\mathrm{Hom}_R(L, K)]$ in order to establish the following.

Lemma 2.2. *For $[K], [L] \in \mathfrak{S}(R)$ with $[K] \trianglelefteq [L]$, the quantity $\sigma_R([K], [L])$ is well defined and nonnegative. Furthermore, the following conditions are equivalent*:

(i) $\sigma_R([K], [L]) < 1$,
(ii) $\sigma_R([K], [L]) = 0$,
(iii) $[K] = [L]$.

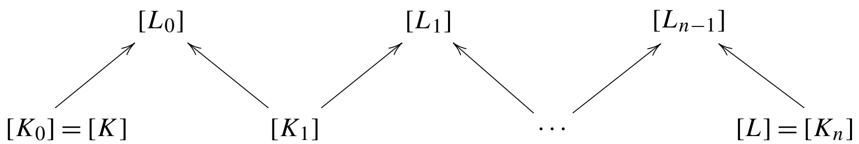
The following simple construction helps us visualize the metric.

Construction 2.3. Let $\Gamma(R)$ be the directed graph whose vertex set is $\mathfrak{S}(R)$ and whose directed edges $[K] \to [L]$ correspond exactly to the inequalities $[K] \trianglelefteq [L]$. Graphically, "smaller" semidualizing modules will be drawn below "larger" ones:



The metric will arise from the graph $\Gamma(R)$ with a "taxi-cab metric" in mind where σ_R is used to measure the length of the edges.

2.4. A *route* γ from $[K]$ to $[L]$ in $\Gamma(R)$ is a subgraph of $\Gamma(R)$ of the form



and the *length* of the route γ is the sum of the lengths of its edges

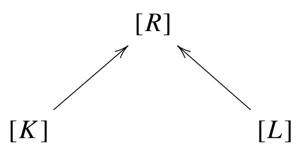
$$\operatorname{length}_R(\gamma) = \sigma_R\big([K_0],[L_0]\big) + \sigma_R\big([K_1],[L_0]\big) + \cdots + \sigma_R\big([K_n],[L_{n-1}]\big).$$

By Lemma 2.2, there is an inequality $\operatorname{length}_R(\gamma) \geqslant 0$.

Remark 2.5. The fact that $\Gamma(R)$ is a *directed* graph is only used to keep track of routes in $\Gamma(R)$. We define the metric in terms of routes instead of arbitrary paths in order to keep the notation simple. For instance, the proof of Theorem 2.11 would be even more notationally complicated without the directed structure. Note that the metric that arises by considering arbitrary paths in $\Gamma(R)$ is equal to the one we construct below. Indeed, any path in $\Gamma(R)$ from $[K]$ to $[L]$ can be expressed as a route of the same length by inserting trivial edges $[M] \to [M]$.

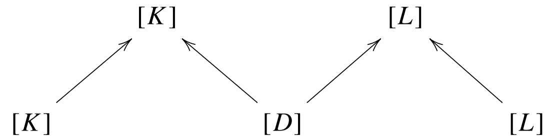
Here are some specific routes whose lengths will give rise to bounds on the metric.

2.6. Since $[K] \trianglelefteq [R]$ and $[L] \trianglelefteq [R]$, a route γ_1 from $[K]$ to $[L]$ always exists



with $\operatorname{length}_R(\gamma_1) = \operatorname{curv}_R(K) + \operatorname{curv}_R(L)$. In particular, the graph $\Gamma(R)$ is connected. We shall see in Theorem 3.5 that the graph is not complete in general.

When R admits a dualizing complex D, another route γ_2 from $[K]$ to $[L]$ is



and $\operatorname{length}_R(\gamma_2) = \operatorname{curv}_R(K^\dagger) + \operatorname{curv}_R(L^\dagger) = \operatorname{inj\,curv}_R(K) + \operatorname{inj\,curv}_R(L)$ by 1.8.

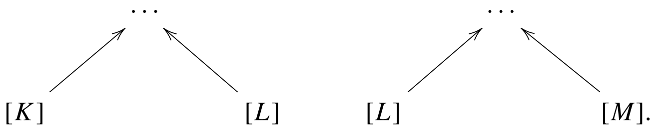
The next properties are straightforward to verify.

2.7. Fix $[K], [L], [M]$ in $\mathfrak{S}(R)$.

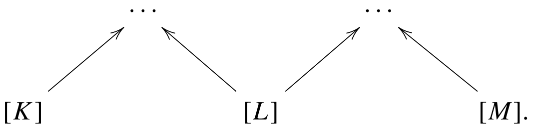
2.7.1. The set of routes from $[K]$ to $[L]$ is in length-preserving bijection with the set of routes from $[L]$ to $[K]$.

2.7.2. The diagram $[K] \to [K] \leftarrow [K]$ gives a route from $[K]$ to $[K]$ with length 0.

2.7.3. Let γ be a route from $[K]$ to $[L]$ and γ' a route from $[L]$ to $[M]$:



Let $\gamma\gamma'$ denote the concatenation of γ and γ'



It is immediate that $\operatorname{length}_R(\gamma\gamma') = \operatorname{length}_R(\gamma) + \operatorname{length}_R(\gamma')$.

Here is the definition of our metric on $\mathfrak{S}(R)$.

2.8. The *distance* from $[K]$ to $[L]$ in $\mathfrak{S}(R)$ is

$$\operatorname{dist}_R\bigl([K],[L]\bigr) = \inf\bigl\{\operatorname{length}_R(\gamma) \mid \gamma \text{ is a route from } [K] \text{ to } [L] \text{ in } \Gamma(R)\bigr\}.$$

The next result is Theorem A from the introduction.

Theorem 2.9. *The function* dist_R *is a metric on* $\mathfrak{S}(R)$.

Proof. Fix $[K], [L]$ in $\mathfrak{S}(R)$. The inequality $\operatorname{dist}_R([K],[L]) \geqslant 0$ is satisfied since $\operatorname{length}_R(\gamma) \geqslant 0$ for each route γ from $[K]$ to $[L]$ in $\Gamma(R)$ by 2.4, and at least one such route exists by 2.6. With this, the computation in 2.7.2 shows that $\operatorname{dist}_R([K],[K]) = 0$. If

$\mathrm{dist}_R([K],[L]) = 0$, then there is a route γ from $[K]$ to $[L]$ in $\Gamma(R)$ with $\mathrm{length}_R(\gamma) < 1$. Using the notation for γ as in 2.4, one has

$$\sigma_R\big([K_i],[L_j]\big) < 1 \quad \text{for } j = 0,\dots,n-1 \text{ and } i = j, j+1$$

and therefore by Lemma 2.2 there are equalities $[K_i] = [L_j]$ and so $[K] = [L]$. Thus, $\mathrm{dist}_R([K],[L]) \geqslant 0$ with equality if and only if $[K] = [L]$.

It follows from 2.7.1 that $\mathrm{dist}_R([K],[L]) = \mathrm{dist}_R([L],[K])$. To verify the triangle inequality, fix $[M]$ in $\mathfrak{S}(R)$. For each real number $\epsilon > 0$, we will verify the inequality

$$\mathrm{dist}_R\big([K],[M]\big) < \mathrm{dist}_R\big([K],[L]\big) + \mathrm{dist}_R\big([L],[M]\big) + \epsilon \tag{†}$$

and the inequality $\mathrm{dist}_R([K],[M]) \leqslant \mathrm{dist}_R([K],[L]) + \mathrm{dist}_R([L],[M])$ will follow. Fix an $\epsilon > 0$ and choose routes γ from $[K]$ to $[L]$ and γ' from $[L]$ to $[M]$ with

$$\mathrm{length}_R(\gamma) < \mathrm{dist}_R\big([K],[L]\big) + \epsilon/2 \quad \text{and} \quad \mathrm{length}_R(\gamma') < \mathrm{dist}_R\big([L],[M]\big) + \epsilon/2;$$

such routes exist by the basic properties of the infimum. The concatenation $\gamma\gamma'$ is a route from $[K]$ to $[M]$, explaining (1) in the following sequence

$$\begin{aligned}\mathrm{dist}_R\big([K],[M]\big) &\overset{(1)}{\leqslant} \mathrm{length}_R(\gamma\gamma') \\ &\overset{(2)}{=} \mathrm{length}_R(\gamma) + \mathrm{length}_R(\gamma') \\ &\overset{(3)}{<} \mathrm{dist}_R\big([K],[L]\big) + \mathrm{dist}_R\big([L],[M]\big) + \epsilon\end{aligned}$$

while (2) is by 2.7.3, and (3) is by the choice of γ and γ'. □

Remark 2.10. Given $[K], [L]$ in $\mathfrak{S}(R)$, it is not clear from the definition of the metric that there exists a route γ from $[K]$ to $[L]$ such that $\mathrm{dist}_R([K],[L]) = \mathrm{length}_R(\gamma)$. If $\mathfrak{S}(R)$ is finite, more generally, if the set $\{\mathrm{curv}_R(M) \mid [M] \in \mathfrak{S}(R)\}$ is finite, such a route would exist. (Compare this to [4, Problem 4.3.8] which asks whether the curvature function takes on finitely many values in total.) The next result gives one criterion guaranteeing that such a route exists: when $[K] \trianglelefteq [L]$, the trivial route $[K] \to [L] \leftarrow [L]$ is length-minimizing.

Theorem 2.11. *For $[K] \trianglelefteq [L]$ in $\mathfrak{S}(R)$, one has* $\mathrm{dist}_R([K],[L]) = \sigma_R([K],[L])$.

Proof. The route $[K] \to [L] \leftarrow [L]$ has length $\sigma_R([K],[L])$ giving the inequality $\mathrm{dist}_R([K],[L]) \leqslant \sigma_R([K],[L])$. Fix a route γ from $[K]$ to $[L]$ in $\Gamma(R)$. We verify the inequality $\sigma_R([K],[L]) \leqslant \mathrm{length}_R(\gamma)$; this will yield the inequality

$$\sigma_R\big([K],[L]\big) \leqslant \mathrm{dist}_R\big([K],[L]\big),$$

completing the proof. With notation for γ as in 2.4, set

$$P_{i,j}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(L_j,K_i)}(t) = P^R_{K_i}(t)/P^R_{L_j}(t) \quad \text{for } j = 0,\dots,n-1 \text{ and } i = j, j+1,$$

where the second equality is from 1.9. This gives (1) and (6) in the following sequence where the formal equalities hold in the field of fractions of the ring of formal Laurent series with integer coefficients:

$$\begin{aligned}
P_{0,0}(t)P_{1,0}(t)P_{1,1}(t)\cdots P_{n,n-1}(t) &\overset{(1)}{=} \frac{P^R_{K_0}(t)}{P^R_{L_0}(t)}\frac{P^R_{K_1}(t)}{P^R_{L_0}(t)}\frac{P^R_{K_1}(t)}{P^R_{L_1}(t)}\cdots\frac{P^R_{K_{n-1}}(t)}{P^R_{L_{n-1}}(t)}\frac{P^R_{K_n}(t)}{P^R_{L_{n-1}}(t)} \\
&\overset{(2)}{=} \frac{P^R_{K_0}(t)}{P^R_{L_0}(t)}\frac{P^R_{K_1}(t)}{P^R_{L_0}(t)}\frac{P^R_{K_1}(t)}{P^R_{L_1}(t)}\cdots\frac{P^R_{K_{n-1}}(t)}{P^R_{L_{n-1}}(t)}\frac{P^R_{K_n}(t)}{P^R_{L_{n-1}}(t)}\frac{P^R_{K_n}(t)}{P^R_{K_n}(t)} \\
&\overset{(3)}{=} P^R_{K_0}(t)\left[\frac{P^R_{K_1}(t)}{P^R_{L_0}(t)}\right]^2\left[\frac{P^R_{K_2}(t)}{P^R_{L_1}(t)}\right]^2\cdots\left[\frac{P^R_{K_n}(t)}{P^R_{L_{n-1}}(t)}\right]^2\frac{1}{P^R_{K_n}(t)} \\
&\overset{(4)}{=} \left[\frac{P^R_{K_0}(t)}{P^R_{K_n}(t)}\right]\left[\frac{P^R_{K_1}(t)}{P^R_{L_0}(t)}\right]^2\left[\frac{P^R_{K_2}(t)}{P^R_{L_1}(t)}\right]^2\cdots\left[\frac{P^R_{K_n}(t)}{P^R_{L_{n-1}}(t)}\right]^2 \\
&\overset{(5)}{=} \left[\frac{P^R_{K}(t)}{P^R_{L}(t)}\right]\left[\frac{P^R_{K_1}(t)}{P^R_{L_0}(t)}\right]^2\left[\frac{P^R_{K_2}(t)}{P^R_{L_1}(t)}\right]^2\cdots\left[\frac{P^R_{K_n}(t)}{P^R_{L_{n-1}}(t)}\right]^2 \\
&\overset{(6)}{=} P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t)\prod_{i=1}^{n}P_{i,i-1}(t)^2 \\
&\overset{(7)}{\succcurlyeq} P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t)t^d.
\end{aligned}$$

Here d is twice the sum of the orders of the Laurent series $P_{i,i-1}(t)$. Equality (2) is trivial, (3) and (4) are obtained by rearranging the factors, (5) is by the choice of K_0 and K_n, and (7) follows from the fact that the coefficients of each $P_{i,i-1}(t)$ are nonnegative integers. With 1.2(b) this explains (11) in the following sequence

$$\begin{aligned}
\mathrm{length}_R(\gamma) &\overset{(8)}{=} \mathrm{curv}\big(P_{0,0}(t)\big)+\mathrm{curv}\big(P_{1,0}(t)\big)+\mathrm{curv}\big(P_{1,1}(t)\big)+\cdots+\mathrm{curv}\big(P_{n,n-1}(t)\big) \\
&\overset{(9)}{\geqslant} \max\big\{\mathrm{curv}\big(P_{0,0}(t)\big),\mathrm{curv}\big(P_{1,0}(t)\big),\mathrm{curv}\big(P_{1,1}(t)\big),\ldots,\mathrm{curv}\big(P_{n,n-1}(t)\big)\big\} \\
&\overset{(10)}{=} \mathrm{curv}\big(P_{0,0}(t)P_{1,0}(t)P_{1,1}(t)\cdots P_{n,n-1}(t)\big) \\
&\overset{(11)}{\geqslant} \mathrm{curv}\big(P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t)\big) \\
&\overset{(12)}{=} \sigma_R\big([K],[L]\big),
\end{aligned}$$

where (8) and (12) are by definition, (9) is by the nonnegativity of each $\mathrm{curv}(P_{i,j}(t))$, and (10) is by 1.2(c). This completes the proof. □

The computations in 1.10 and 2.6 provide bounds on the metric.

Proposition 2.12. *For $[K]$ and $[L]$ in $\mathfrak{S}(R)$, there are inequalities*

$$\mathrm{dist}_R\big([K],[L]\big)\leqslant \mathrm{curv}_R(K)+\mathrm{curv}_R(L)\leqslant 2\,\mathrm{inj\,curv}_R(R).$$

In particular, the metric is completely bounded. Furthermore, there are inequalities

$$\mathrm{dist}_R\big([K],[L]\big) \leqslant \mathrm{inj\,curv}_R(K) + \mathrm{inj\,curv}_R(L) \leqslant 2\,\mathrm{inj\,curv}_R(R)$$

when R admits a dualizing complex.

Remark 2.13. The topology on $\mathfrak{S}(R)$ induced by the metric is trivial. Indeed, Lemma 2.2 and Theorem 2.11 imply that the singleton $\{[K]\}$ is exactly the open ball of radius 1 centered at the point $[K]$. Similarly, using the upper bound established in Proposition 2.12, the open ball of radius $2\,\mathrm{inj\,curv}_R(R) + 1$ is $\mathfrak{S}(R)$ itself. On the other hand, in Theorem 3.5 we show that, if $\mathfrak{S}(R)$ contains at least three elements, then $\mathfrak{S}(R)$ has nontrivial open balls.

3. Dagger duality, fixed points, and nontriviality of the metric

The first result of this section uses notation from 1.8.

Proposition 3.1. *If R admits a dualizing complex D, then the map* $\Delta:\mathfrak{S}(R)\to\mathfrak{S}(R)$ *given by sending* $[K]$ *to* $[K^\dagger]$ *is an isometric involution of* $\mathfrak{S}(R)$.

Proof. The map Δ is an involution of $\mathfrak{S}(R)$ by 1.5. To show that it is an isometry, it suffices to verify the following containment of subsets of $\mathbb{R}$:

$$\big\{\mathrm{length}_R(\gamma) \mid \gamma \text{ a route } [K] \text{ to } [L]\big\} \subseteq \big\{\mathrm{length}_R(\gamma_1) \mid \gamma_1 \text{ a route } \big[L^\dagger\big] \text{ to } \big[K^\dagger\big]\big\}. \qquad (\ddagger)$$

Indeed, this will give the inequalities in the following sequence

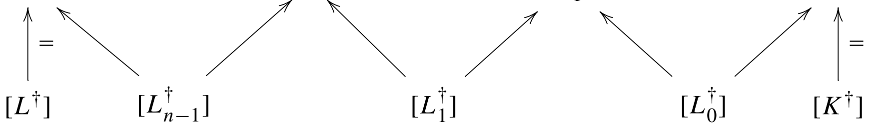
$$\mathrm{dist}_R\big([K],[L]\big) = \mathrm{dist}_R\big(\big[K^{\dagger\dagger}\big],\big[L^{\dagger\dagger}\big]\big) \leqslant \mathrm{dist}_R\big(\big[K^{\dagger}\big],\big[L^{\dagger}\big]\big) \leqslant \mathrm{dist}_R\big([K],[L]\big)$$

while 1.5 explains the equality; thus, equality is forced at each step.

When $[K] \trianglelefteq [L]$, one concludes from [13, (3.9)] that $[L^\dagger] \trianglelefteq [K^\dagger]$. Furthermore, there is a sequence of equalities where the middle equality is by the isomorphism $\mathbf{R}\mathrm{Hom}_R(K^\dagger, L^\dagger) \simeq \mathbf{R}\mathrm{Hom}_R(L, K)$ in [13, (1.7(a))]:

$$\sigma_R\big(\big[L^\dagger\big],\big[K^\dagger\big]\big) = \mathrm{curv}_R\big(\mathbf{R}\mathrm{Hom}_R\big(K^\dagger, L^\dagger\big)\big) = \mathrm{curv}_R\big(\mathbf{R}\mathrm{Hom}_R(L, K)\big) = \sigma_R\big([K],[L]\big).$$

To verify ($\ddagger$), let γ be a route from $[K]$ to $[L]$. Using the notation of 2.4, the previous paragraph shows that the following diagram



is a route $\gamma^\dagger$ from $[L^\dagger]$ to $[K^\dagger]$ with $\mathrm{length}_R(\gamma^\dagger) = \mathrm{length}_R(\gamma)$. This explains ($\ddagger$) and completes the proof. $\square$

The next result is Theorem C from the introduction. It will yield an answer to the following: In Proposition 3.1, what is implied by the existence of a fixed point for Δ? See Corollary 3.4 for the answer.

Theorem 3.2. *Let R be a local ring and K, L homologically bounded below and degreewise finite R-complexes. If $K \otimes^{\mathbf{L}}_R K \otimes^{\mathbf{L}}_R L$ is semidualizing, then $K \sim R$.*

Proof. To keep the bookkeeping simple, apply appropriate suspensions to K and L to assume $\inf(K) = 0 = \inf(L)$. Let P and Q be minimal projective resolutions of K and L, respectively; in particular, $P_0, Q_0 \neq 0$. Then $K \otimes^{\mathbf{L}}_R K \otimes^{\mathbf{L}}_R L \simeq P \otimes_R P \otimes_R Q$ is semidualizing, and so the homothety morphism is a quasi-isomorphism:

$$R \xrightarrow{\simeq} \operatorname{Hom}_R(P \otimes_R P \otimes_R Q, P \otimes_R P \otimes_R Q).$$

Here is the crucial point. For complexes X, Y, let $\theta_{XY} : X \otimes_R Y \to Y \otimes_R X$ be the natural isomorphism. This gives a cycle

$$\theta_{PP} \otimes_R Q \in \operatorname{Hom}_R(P \otimes_R P \otimes_R Q, P \otimes_R P \otimes_R Q)$$

and therefore, there exists $u \in R$ such that the homothety $\mu_u : P \otimes_R P \otimes_R Q \to P \otimes_R P \otimes_R Q$ is homotopic to $\theta_{PP} \otimes_R Q$.

Set $\overline{P} = P \otimes_R k$ and $\overline{Q} = Q \otimes_R k$. The fact that $\theta_{PP} \otimes_R Q$ and μ_u are homotopic implies that the following morphisms are also homotopic:

$$(\theta_{PP} \otimes_R Q) \otimes_R k, \mu_u \otimes_R k : (P \otimes_R P \otimes_R Q) \otimes_R k \to (P \otimes_R P \otimes_R Q) \otimes_R k.$$

Using the isomorphism $(P \otimes_R P \otimes_R Q) \otimes_R k \cong \overline{P} \otimes_k \overline{P} \otimes_k \overline{Q}$, we then deduce that the k-morphisms

$$\theta_{\overline{P}\,\overline{P}} \otimes_k \overline{Q}, \mu_{\bar{u}} : \overline{P} \otimes_k \overline{P} \otimes_k \overline{Q} \to \overline{P} \otimes_k \overline{P} \otimes_k \overline{Q}$$

are homotopic as well. The differential on $\overline{P} \otimes_k \overline{P} \otimes_k \overline{Q}$ is zero by the minimality of P and Q, and it follows that $\theta_{\overline{P}\,\overline{P}} \otimes_k \overline{Q}$ and $\mu_{\bar{u}}$ are equal.

Set $n = \operatorname{rank}_k \overline{P}_0$. We claim that $n = 1$. Suppose that $n > 1$, and fix bases $x_1, \ldots, x_n \in \overline{P}_0$ and $y_1, \ldots, y_p \in \overline{Q}_0$. The set

$$\bigl\{x_i \otimes x_j \otimes y_l \mid i, j \in \{1, \ldots, n\} \text{ and } l \in \{1, \ldots, p\}\bigr\}$$

is a basis for $\overline{P}_0 \otimes_k \overline{P}_0 \otimes_k \overline{Q}_0$. However, the equality

$$x_1 \otimes x_2 \otimes y_1 - u x_2 \otimes x_1 \otimes y_1 = 0$$

contradicts the linear independence. Thus, $n \leqslant 1$ and since $\overline{P}_0 \neq 0$ we have $n = 1$.

Next, we show that $\overline{P}_i = 0$ for $i > 0$. The equality $\theta_{\overline{P}\,\overline{P}} \otimes_k \overline{Q} = \mu_{\bar{u}}$ implies

$$0 = x \otimes x' \otimes y \pm u x' \otimes x \otimes y \in (\overline{P}_0 \otimes_k \overline{P}_i \otimes_k \overline{Q}_0) \oplus (\overline{P}_i \otimes_k \overline{P}_0 \otimes_k \overline{Q}_0)$$

for each $x \in \overline{P}_0$ and $x' \in \overline{P}_i$ and $y \in \overline{Q}_0$. Since $\overline{P}_0 \neq 0$ and

$$(\overline{P}_0 \otimes_k \overline{P}_i \otimes_k \overline{Q}_0) \cap (\overline{P}_i \otimes_k \overline{P}_0 \otimes_k \overline{Q}_0) = 0,$$

this is impossible unless $\overline{P}_i = 0$.

One concludes that there is an isomorphism $P \simeq R$, completing the proof. □

Corollary 3.3. *Let R be a local ring and $[K], [L] \in \mathfrak{S}(R)$.*

(a) *If $[K] \trianglelefteq [L]$ and $[L] \trianglelefteq [\mathbf{R}\mathrm{Hom}_R(L, K)]$, then $[L] = [K]$.*
(b) *If $[K] \trianglelefteq [L]$ and $[\mathbf{R}\mathrm{Hom}_R(L, K)] \trianglelefteq [L]$, then $[L] = [R]$.*

In particular, if $[K] \triangleleft [L] \triangleleft [R]$, then $[L]$ and $[\mathbf{R}\mathrm{Hom}_R(L, K)]$ are not comparable in the ordering on $\mathfrak{S}(R)$.

Proof. (a) If $\mathbf{R}\mathrm{Hom}_R(L, K)$ is L-reflexive and L is K-reflexive, then 1.9 provides

$$K \simeq \mathbf{R}\mathrm{Hom}_R(L, K) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(L, K), L\big) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(L, K).$$

Theorem 3.2 then yields $\mathbf{R}\mathrm{Hom}_R(L, K) \sim R$ and thus the second isomorphism in the next sequence while the first follows since $[K] \trianglelefteq [L]$ and the third is standard:

$$L \simeq \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(L, K), K\big) \sim \mathbf{R}\mathrm{Hom}_R(R, K) \simeq K.$$

(b) If L is $\mathbf{R}\mathrm{Hom}_R(L, K)$-reflexive and K-reflexive, then the isomorphism $L \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, K), K)$ with part (a) implies that $\mathbf{R}\mathrm{Hom}_R(L, K) \sim K$. The desired isomorphism then follows from an application of $\mathbf{R}\mathrm{Hom}_R(-, K)$.

The final statement follows directly from parts (a) and (b). □

In view of condition (3.4) of the next result we note the following open question: if R is a local ring, must $\mathfrak{S}(R)$ be a finite set? The answer is known in very few cases. See [10,18] for discussion of this question.

Corollary 3.4. *For a local ring R, the following conditions are equivalent*:

(i) *R is Gorenstein.*
(ii) *R admits a dualizing complex D and a semidualizing complex L such that $[\mathbf{R}\mathrm{Hom}_R(L, D)] = [L]$.*
(iii) *R admits a dualizing complex D, and $\mathfrak{S}(R)$ is finite with odd cardinality.*

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) are clear as, when R is Gorenstein, R is dualizing and $\mathfrak{S}(R) = \{[R]\}$ by [9, (8.6)].

(ii) ⇒ (i). Let D, L be as in (ii). Corollary 3.3(a) with $K = D$ provides the first and third isomorphisms in the next sequence

$$D \sim L \sim \mathbf{R}\mathrm{Hom}_R(L, D) \sim \mathbf{R}\mathrm{Hom}_R(D, D) \simeq R$$

while the others follow by hypothesis. Thus, R is Gorenstein.

(iii) $\Rightarrow$ (ii). Assume that R admits a dualizing complex D and that $\mathfrak{S}(R)$ is finite. If $[\mathbf{R}\mathrm{Hom}_R(L, D)] \neq [L]$ for all $[L] \in \mathfrak{S}(R)$, then $\mathfrak{S}(R)$ is the disjoint union of subsets of the form $\{[L], [\mathbf{R}\mathrm{Hom}_R(L, D)]\}$, each of which has two distinct elements. Thus, $\mathfrak{S}(R)$ has even cardinality, completing the proof. $\square$

Here is Theorem B from the introduction. For a specific construction of noncomparable semidualizing complexes, see Corollary 4.7. For $[K] \in \mathfrak{S}(R)$ and $\delta > 0$, set

$$B\big([K], \delta\big) = \big\{[L] \in \mathfrak{S}(R) \mid \mathrm{dist}\big([K], [L]\big) < \delta\big\}.$$

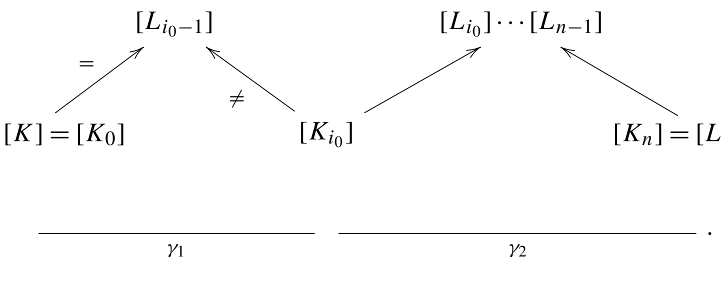
Theorem 3.5. *For a local ring R the following conditions are equivalent*:

(i) *There exist elements of $\mathfrak{S}(R)$ that are not comparable.*
(ii) *$\mathfrak{S}(R)$ has cardinality at least* 3.
(iii) *There exists $[K] \in \mathfrak{S}(R)$ and $\delta > 0$ such that the open ball $B([K], \delta)$ satisfies $\{[K]\} \subsetneq B([K], \delta) \subsetneq \mathfrak{S}(R)$.*

Proof. (ii) $\Rightarrow$ (i). Fix distinct elements $[K], [L], [M] \in \mathfrak{S}(R)$. Without loss of generality, assume that $[M] = [R]$. Suppose that every two elements in $\mathfrak{S}(R)$ are comparable. The elements $[K]$, $[L]$, $[R]$ can be reordered to assume that $[K] \lhd [L] \lhd [R]$, and we are done by Corollary 3.3.

(iii) $\Rightarrow$ (ii). Let $[K] \in \mathfrak{S}(R)$ and $\delta > 0$ be such that $\{[K]\} \subsetneq B([K], \delta) \subsetneq \mathfrak{S}(R)$. Fixing $[L] \in \mathfrak{S}(R) \smallsetminus B([K], \delta)$ and $[M] \in B([K], \delta) \smallsetminus \{[K]\}$ provides at least three distinct elements of $\mathfrak{S}(R)$: $[K], [L], [M]$.

(i) $\Rightarrow$ (iii). Fix two noncomparable elements $[K], [L] \in \mathfrak{S}(R)$, and let γ be a route in $\Gamma(R)$ from $[K]$ to $[L]$ such that $\mathrm{length}(\gamma) < \mathrm{dist}([K], [L]) + \frac{1}{2}$. Using the notation of 2.4 for γ, there exists an integer i between 0 and n such that either $[K] \neq [K_i]$ or $[K] = [K_i] \neq [L_i]$, and we let i_0 denote the smallest such integer. If $[K] \neq [K_{i_0}]$, then $[K_{i_0}] \lhd [K]$. In this event, γ can be factored as the concatenation $\gamma_1\gamma_2$ as in the following diagram:



Since $[K], [L]$ are not comparable, it follows that $[K_{i_0}] \neq [L]$ and so $\mathrm{length}(\gamma_2) > \frac{1}{2}$ by Lemma 2.2. With Theorem 2.11 this provides (1) in the following sequence

$$\mathrm{dist}\big([K], [K_{i_0}]\big) + \frac{1}{2} \overset{(1)}{<} \mathrm{length}(\gamma_1) + \mathrm{length}(\gamma_2) \overset{(2)}{=} \mathrm{length}(\gamma) \overset{(3)}{<} \mathrm{dist}\big([K], [L]\big) + \frac{1}{2}$$

while (2) is by 2.7.3 and (3) is from the choice of γ. In particular, $\mathrm{dist}([K], [K_{i_0}]) < \mathrm{dist}([K], [L])$. Fixing δ such that $\mathrm{dist}_R([K], [K_{i_0}]) < \delta < \mathrm{dist}_R([K], [L])$, one has $[K_{i_0}] \in B([K], \delta) \smallsetminus \{[K]\}$ and $[L] \in \mathfrak{S}(R) \smallsetminus B([K], \delta)$, giving the desired proper containments.

If $[K] = [K_{i_0}] \neq [L_{i_0}]$, then similar reasoning shows that, with a choice of δ such that $\mathrm{dist}_R([K], [L_{i_0}]) < \delta < \mathrm{dist}_R([K], [L])$, one has $[L_{i_0}] \in B([K], \delta) \smallsetminus \{[K]\}$ and $[L] \in \mathfrak{S}(R) \smallsetminus B([K], \delta)$. $\square$

4. Behavior of the metric under change of rings

In this section, let $\varphi : R \to S$ be a local homomorphism of finite flat dimension.

4.1. Base change

The homomorphism φ induces a well-defined injective map

$$\mathfrak{S}_\varphi : \mathfrak{S}(R) \to \mathfrak{S}(S) \quad \text{given by} \quad [K] \mapsto \big[K \otimes^{\mathbf{L}}_R S\big]$$

by [13, (4.5), (4.9)], and $[K] \trianglelefteq [L]$ if and only if $[K \otimes^{\mathbf{L}}_R S] \trianglelefteq [L \otimes^{\mathbf{L}}_R S]$ by [13, (4.8)]. When $[K] \trianglelefteq [L]$, one has $P^S_{\mathbf{R}\mathrm{Hom}_S(L\otimes^{\mathbf{L}}_R S, K\otimes^{\mathbf{L}}_R S)}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t)$ from [13, (6.15)], providing the second equality in the following sequence

$$\mathrm{dist}_S\big(\big[K \otimes^{\mathbf{L}}_R S\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) = \sigma_S\big(\big[K \otimes^{\mathbf{L}}_R S\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) = \sigma_R\big([K], [L]\big) = \mathrm{dist}_R\big([K], [L]\big)$$

while the other equalities are from Theorem 2.11.

Next we show that the metric is nonincreasing under $\mathfrak{S}_\varphi$; we do not know of an example where it decreases. For instance, equality holds in each case where $\mathfrak{S}(S)$ is completely determined in [18]. When $\mathfrak{S}_\varphi$ is surjective, the result states that $\mathfrak{S}_\varphi$ is an isometry.

Proposition 4.2. *Let $\varphi : R \to S$ be a local homomorphism of finite flat dimension. For $[K]$, $[L] \in \mathfrak{S}(R)$ there is an inequality*

$$\mathrm{dist}_S\big(\big[K \otimes^{\mathbf{L}}_R S\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) \leqslant \mathrm{dist}_R\big([K], [L]\big)$$

with equality when $[K] \trianglelefteq [L]$ or when $\mathfrak{S}_\varphi$ is surjective, e.g. if R is complete and φ is surjective with kernel generated by an R-sequence.

Proof. Fix $[K], [L]$ in $\mathfrak{S}(R)$. When $[K] \trianglelefteq [L]$, the equality is in 4.1. In general, let γ be a route from $[K]$ to $[L]$ in $\Gamma(R)$. Using the notation of 2.4, the diagram

$$\begin{array}{ccccccc} & & [L_0 \otimes^{\mathbf{L}}_R S] & & & [L_{n-1} \otimes^{\mathbf{L}}_R S] & \\ & \nearrow & & \nwarrow & \nearrow & & \nwarrow \\ [K_0 \otimes^{\mathbf{L}}_R S] & & & \cdots & & & [K_n \otimes^{\mathbf{L}}_R S] \end{array}$$

is a route $\gamma \otimes^{\mathbf{L}}_R S$ from $[K \otimes^{\mathbf{L}}_R S]$ to $[L \otimes^{\mathbf{L}}_R S]$ in $\Gamma(S)$ by 4.1 and

$$\mathrm{dist}_S\big(\big[K \otimes^{\mathbf{L}}_R S\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) \leqslant \mathrm{length}_S\big(\gamma \otimes^{\mathbf{L}}_R S\big) = \mathrm{length}_R(\gamma).$$

Since this is true for every route γ, the desired inequality now follows.

When $\mathfrak{S}_\varphi$ is surjective, the above analysis along with 4.1 implies that the routes from $[K]$ to $[L]$ are in length-preserving bijection with those from $[K \otimes^{\mathbf{L}}_R S]$ to $[L \otimes^{\mathbf{L}}_R S]$ and so $\operatorname{dist}_S([K \otimes^{\mathbf{L}}_R S], [L \otimes^{\mathbf{L}}_R S]) = \operatorname{dist}_R([K], [L])$. When R is complete and φ is surjective with kernel generated by an R-sequence, the surjectivity of $\mathfrak{S}_\varphi$ follows from [13, (4.5)] and [21, (3.2)]. □

Using [9, (2.5), (3.16)] and the inequality $P^{R_\mathfrak{p}}_{X_\mathfrak{p}}(t) \preccurlyeq P^R_X(t)$ with 1.2(b), the proof of the previous result yields the following. Example 5.3 shows that inequality may be strict or not.

Proposition 4.3. *For* $\mathfrak{p} \in \operatorname{Spec}(R)$ *and* $[K], [L] \in \mathfrak{S}(R)$*, there is an inequality*

$$\operatorname{dist}_{R_\mathfrak{p}}\big([K_\mathfrak{p}], [L_\mathfrak{p}]\big) \leqslant \operatorname{dist}_R\big([K], [L]\big).$$

4.4. Cobase change

A *Gorenstein factorization* of φ is a diagram of local homomorphisms $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ such that $\varphi = \varphi'\dot\varphi$, φ' is surjective, and $\dot\varphi$ is flat with Gorenstein closed fibre. Homomorphisms admitting Gorenstein factorizations exist in profusion, e.g., if φ is essentially of finite type or if S is complete; see [6, (1.1)].

Assume that φ admits a Gorenstein factorization as above and set $d = \operatorname{depth}(\dot\varphi)$. For each homologically finite complex of R-modules X, set

$$X(\varphi) = \Sigma^d \mathbf{R}\mathrm{Hom}_{R'}\big(S, X \otimes^{\mathbf{L}}_R R'\big).$$

It is shown in [13, (6.5), (6.12)] that this is independent of Gorenstein factorization and that the following assignment is well-defined and injective:

$$\mathfrak{S}^\varphi : \mathfrak{S}(R) \to \mathfrak{S}(S) \quad \text{given by} \quad [K] \mapsto \big[K(\varphi)\big].$$

One has $[K] \trianglelefteq [L]$ if and only if $[K(\varphi)] \trianglelefteq [L(\varphi)]$ by [13, (6.11)]. When $[K] \trianglelefteq [L]$, one has $P^S_{\mathbf{R}\mathrm{Hom}_S(L(\varphi),K(\varphi))}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t)$ from [13, (6.15)], providing the second equality in the following sequence while the others are from Theorem 2.11:

$$\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[L(\varphi)\big]\big) = \sigma_S\big(\big[K(\varphi)\big], \big[L(\varphi)\big]\big) = \sigma_R\big([K], [L]\big) = \operatorname{dist}_R\big([K], [L]\big).$$

One has $[K] \trianglelefteq [L]$ if and only if $[K(\varphi)] \trianglelefteq [L \otimes^{\mathbf{L}}_R S]$ by [13, (6.13)]. When $[K] \trianglelefteq [L]$ one has $P^S_{\mathbf{R}\mathrm{Hom}_S(L\otimes^{\mathbf{L}}_R S, K(\varphi))}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(L,K)}(t) I_\varphi(t)$ from [13, (6.15)], and using 1.2(c) and Theorem 2.11 as above there is an equality

$$\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) = \max\big\{\operatorname{dist}_R\big([K], [L]\big), \operatorname{inj\,curv}(\varphi)\big\}. \qquad (\dagger)$$

Example 5.2 shows that this formula can fail if $[K] \ntrianglelefteq [L]$. See Corollary 4.6 for the general case. The case $[K] = [L]$ yields

$$\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[K \otimes^{\mathbf{L}}_R S\big]\big) = \operatorname{inj\,curv}(\varphi) \qquad (\ddagger)$$

and thus φ is Gorenstein at $\mathfrak{n}$ if and only if $\mathfrak{S}_\varphi = \mathfrak{S}^\varphi$; see 1.3.

As with Proposition 4.2 we do not know if the next inequality can be strict.

Proposition 4.5. *Let $\varphi : R \to S$ be a local homomorphism of finite flat dimension admitting a Gorenstein factorization. For $[K], [L]$ in $\mathfrak{S}(R)$, there is an inequality*

$$\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[L(\varphi)\big]\big) \leqslant \operatorname{dist}_R\big([K], [L]\big)$$

with equality when $[K] \trianglelefteq [L]$. Equality also holds when $\mathfrak{S}^{\varphi}$ is surjective, in which case φ is Gorenstein at $\mathfrak{n}$.

Proof. When $[K] \trianglelefteq [L]$, the equality is in 4.4. As in the proof of Proposition 4.2, for arbitrary $[K], [L]$, let γ be a route from $[K]$ to $[L]$ in $\Gamma(R)$, with the notation of 2.4. The following diagram is a route $\gamma(\varphi)$ from $[K(\varphi)]$ to $[L(\varphi)]$

$$\begin{array}{ccccc} & [L_0(\varphi)] & & [L_{n-1}(\varphi)] & \\ \nearrow & & \nwarrow \qquad \nearrow & & \nwarrow \\ [K_0(\varphi)] & & \cdots & & [K_n(\varphi)] \end{array}$$

with $\operatorname{dist}_S([K(\varphi)], [L(\varphi)]) \leqslant \operatorname{length}_S(\gamma(\varphi)) = \operatorname{length}_R(\gamma)$.

If $\mathfrak{S}^{\varphi}$ is surjective, then there exists $[K] \in \mathfrak{S}(R)$ such that $[K(\varphi)] = [S]$. With $L = R$ in 4.4 Eq. (†), one has $\operatorname{inj\,curv}(\varphi) = 0$, so φ is Gorenstein at $\mathfrak{n}$ and $\mathfrak{S}_{\varphi} = \mathfrak{S}^{\varphi}$. The equality now follows from Proposition 4.2. □

By Eq. (†) of 4.4 and Example 5.2, the next inequality may be strict or not.

Corollary 4.6. *Let $\varphi : R \to S$ be a local homomorphism of finite flat dimension admitting a Gorenstein factorization. For $[K], [L]$ in $\mathfrak{S}(R)$ there is an inequality*

$$\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) \leqslant \operatorname{inj\,curv}(\varphi) + \operatorname{dist}_R\big([K], [L]\big).$$

Proof. Use the triangle inequality, 4.4 Eq. (‡), and Proposition 4.2

$$\begin{aligned}\operatorname{dist}_S\big(\big[K(\varphi)\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) &\leqslant \operatorname{dist}_S\big(\big[K(\varphi)\big], \big[K \otimes^{\mathbf{L}}_R S\big]\big) + \operatorname{dist}_S\big(\big[K \otimes^{\mathbf{L}}_R S\big], \big[L \otimes^{\mathbf{L}}_R S\big]\big) \\ &\leqslant \operatorname{inj\,curv}(\varphi) + \operatorname{dist}_R\big([K], [L]\big). \qquad \square\end{aligned}$$

See Example 5.2 for a special case of the final statement of the next result.

Corollary 4.7. *Let $\varphi : R \to S$ be a local homomorphism of finite flat dimension admitting a Gorenstein factorization and fix $[K], [L] \in \mathfrak{S}(R)$ with $[K] \trianglelefteq [L]$.*

(a) *If $[L \otimes^{\mathbf{L}}_R S] \trianglelefteq [\mathbf{R}\mathrm{Hom}_R(L, K)(\varphi)]$, then φ is Gorenstein at $\mathfrak{n}$ and $[K] = [L]$.*
(b) *If $[\mathbf{R}\mathrm{Hom}_R(L, K)(\varphi)] \trianglelefteq [L \otimes^{\mathbf{L}}_R S]$, then $[L] = [R]$.*

In particular, if $[K] \neq [R]$ and φ is not Gorenstein at $\mathfrak{n}$, then the elements $[K \otimes^{\mathbf{L}}_R S]$ and $[R(\varphi)]$ are noncomparable in the ordering on $\mathfrak{S}(S)$.

Proof. The assumption $[K] \leqslant [L]$ implies $[K(\varphi)] \leqslant [L \otimes^{\mathbf{L}}_R S]$ by 4.4, and [13, (6.9)] provides an isomorphism $\mathbf{R}\mathrm{Hom}_S(L \otimes^{\mathbf{L}}_R S, K(\varphi)) \simeq \mathbf{R}\mathrm{Hom}_R(L, K)(\varphi)$.

If $[L \otimes^{\mathbf{L}}_R S] \leqslant [\mathbf{R}\mathrm{Hom}_R(L, K)(\varphi)] = [\mathbf{R}\mathrm{Hom}_S(L \otimes^{\mathbf{L}}_R S, K(\varphi))]$, then Corollary 3.3(a) implies $[L \otimes^{\mathbf{L}}_R S] = [K(\varphi)]$. Equation (†) in 4.4 yields the conclusion for part (a). Part (b) follows similarly from Corollary 3.3(b), and the final statement is a consequence of (a) and (b) using $K = L$. □

5. Examples

This section consists of specific computations of distances in $\mathfrak{S}(R)$. We begin with a simple example upon which the others are built. It shows, in particular, that although the diameter of the metric space $\mathfrak{S}(R)$ is finite by Proposition 2.12, it can be arbitrarily large. Here, the diameter of $\mathfrak{S}(R)$ is

$$\mathrm{diam}\big(\mathfrak{S}(R)\big) = \sup\big\{\mathrm{dist}_R\big([K], [L]\big) \mid [K], [L] \in \mathfrak{S}(R)\big\}.$$

Example 5.1. Assume that $\mathfrak{m}^2 = 0$. In particular, R is Cohen–Macaulay, so each semidualizing complex is, up to shift, isomorphic to a module by [9, (3.7)]. Since R is Artinian, it admits a dualizing module D by 1.5. The set $\mathfrak{S}(R)$ contains at most two distinct elements, namely $[R]$ and $[D]$: If K is a nonfree semidualizing module, then any syzygy module from a minimal free resolution of K is a nonzero k-vector space that is K-reflexive, implying that K is dualizing by [9, (8.4)].

The elements $[R]$ and $[D]$ are distinct if and only if R is non-Gorenstein. When these conditions hold, the previous argument shows that $\mathrm{curv}_R(D) = \mathrm{curv}_R(k)$. A straightforward computation of the minimal free resolution of k shows that

$$P^R_k(t) = \sum_{n=0}^{\infty} r^n t^n = 1/(1 - rt),$$

where $r = \mathrm{edim}(R) = \mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$. In particular,

$$\mathrm{dist}_R\big([R], [D]\big) = \mathrm{curv}_R(D) = \mathrm{curv}_R(k) = r$$

and thus $\mathrm{diam}(\mathfrak{S}(R)) = r$. The trivial extension $k \ltimes k^r$ gives an explicit example.

We now give a particular example of the construction from Corollary 4.7 which has the added benefit of being an example where we can completely describe the structure of the metric space $\mathfrak{S}(S)$. Note that this process can be iterated.

Example 5.2. Fix integers $r, s \geqslant 2$ and a field k. Let $R = k \ltimes k^r$ and $S = R \ltimes R^s$. The natural map $\varphi : R \to S$ is flat and local with closed fibre $\overline{S} \cong k \ltimes k^s$. Since R is Artinian it admits a dualizing module D by 1.5. By [18, (4.7)], the set $\mathfrak{S}(S)$ consists of the four distinct elements $[S], [D \otimes^{\mathbf{L}}_R S], [R(\varphi)], [D(\varphi)]$. The next Poincaré series and curvatures are computed using 1.2(c), Example 5.1, and [13, (6.10), (6.15)]:

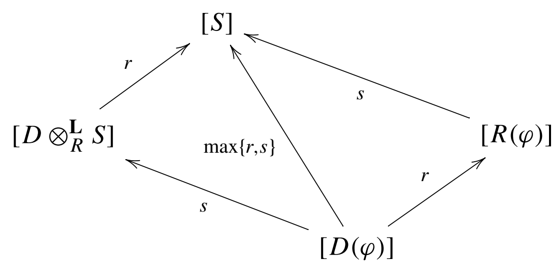
$$P^S_{D \otimes^{\mathbf{L}}_R S} = I^R_R(t), \qquad \mathrm{curv}_S\big(D \otimes^{\mathbf{L}}_R S\big) = r,$$

$$
\begin{aligned}
&P^S_{R(\varphi)}(t) = I^{\bar{S}}_{\bar{S}}(t), && \operatorname{curv}_S\big(R(\varphi)\big) = s,\\
&P^S_{D(\varphi)}(t) = I^R_R(t) I^{\bar{S}}_{\bar{S}}(t), && \operatorname{curv}_R\big(D(\varphi)\big) = \max\{r, s\},\\
&P^S_{\mathbf{R}\mathrm{Hom}_S(D\otimes^{\mathbf{L}}_R S, D(\varphi))}(t) = I^{\bar{S}}_{\bar{S}}(t), && \operatorname{curv}_S\big(\mathbf{R}\mathrm{Hom}_S\big(D\otimes^{\mathbf{L}}_R S, D(\varphi)\big)\big) = s,\\
&P^S_{\mathbf{R}\mathrm{Hom}_S(R(\varphi), D(\varphi))}(t) = I^R_R(t), && \operatorname{curv}_S\big(\mathbf{R}\mathrm{Hom}_S\big(R(\varphi), D(\varphi)\big)\big) = r.
\end{aligned}
$$

With Theorem 2.11, this gives the following distance computations.

$$
\begin{aligned}
&\operatorname{dist}_S\big([S], \big[D\otimes^{\mathbf{L}}_R S\big]\big) = r, && \operatorname{dist}_S\big([S], \big[R(\varphi)\big]\big) = s,\\
&\operatorname{dist}_R\big([S], \big[D(\varphi)\big]\big) = \max\{r, s\}, && \operatorname{dist}_S\big(\big[D\otimes^{\mathbf{L}}_R S\big], \big[D(\varphi)\big]\big) = s,\\
&\operatorname{dist}_S\big(\big[R(\varphi)\big], \big[D(\varphi)\big]\big) = r.
\end{aligned}
$$

This provides the lengths of the edges in the following sketch of $\Gamma(S)$



while [18, (4.7)] implies that this is a complete description of $\Gamma(S)$. Thus, the remaining distance is computed readily:

$$\operatorname{dist}_S\big(\big[R(\varphi)\big], \big[D\otimes^{\mathbf{L}}_R S\big]\big) = r + s.$$

In particular, the open ball in $\mathfrak{S}(S)$ of radius $r+1$ centered at $[R(\varphi)]$ contains $[D(\varphi)] \neq [R(\varphi)]$ and does not contain $[D\otimes^{\mathbf{L}}_R S]$. Furthermore, this shows that equality can hold in Corollary 4.6.

Finally, we show that the metric may or may not decrease after localizing.

Example 5.3. Let R be a non-Gorenstein ring with dualizing complex D and $\mathfrak{p}$ a prime ideal such that $R_{\mathfrak{p}}$ is Gorenstein, e.g., $R = k[\![X, Y]\!]/(X^2, XY)$ and $\mathfrak{p} = (X)R$. Then $D_{\mathfrak{p}} \sim R_{\mathfrak{p}}$, implying

$$\operatorname{dist}_{R_{\mathfrak{p}}}\big([R_{\mathfrak{p}}], [D_{\mathfrak{p}}]\big) = 0 < \operatorname{dist}_R\big([R], [D]\big).$$

On the other hand, let $S = k[\![X, Y, Z]\!]/(X, Y)^2$ with dualizing module E, and $\mathfrak{q} = (X, Y)S$; then the computations in Example 5.1 give

$$\operatorname{dist}_{S_{\mathfrak{q}}}\big([S_{\mathfrak{q}}], [E_{\mathfrak{q}}]\big) = \operatorname{curv}_{S_{\mathfrak{q}}}(E_{\mathfrak{q}}) = 2$$

while Proposition 4.2 yields the first equality in the following sequence

$$\begin{aligned}\operatorname{dist}_S\big([S],[E]\big) &= \operatorname{dist}_{S/(Z)}\big(\big[S\otimes^{\mathbf{L}}_S S/(Z)\big],\big[E\otimes^{\mathbf{L}}_S S/(Z)\big]\big)\\ &= \operatorname{curv}_{S/(Z)}\big(E\otimes^{\mathbf{L}}_S S/(Z)\big)=2\end{aligned}$$

and the last equality follows from Example 5.1 since $S/(Z)\cong k[X,Y]/(X,Y)^2$.

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