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Stability of Gorenstein categories

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Abstract

We show that an iteration of the procedure used to define the Gorenstein projective modules over a commutative ring R yields exactly the Gorenstein projective modules. Specifically, given an exact sequence of Gorenstein projective R-modules

$$G = \cdots \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \xrightarrow{\partial_0^G} \cdots$$

such that the complexes $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_R(H, G)$ are exact for each Gorenstein projective R-module H, the module $\mathrm{Coker}(\partial_1^G)$ is Gorenstein projective. The proof of this result hinges upon our analysis of Gorenstein subcategories of abelian categories.

Introduction

Let R be a commutative ring. Building from Auslander and Bridger's work [**1**, **2**] on modules of finite G-dimension, Enochs and Jenda [**8**] and Holm [**14**] introduced and studied the *Gorenstein projective* R-modules as the modules of the form $\mathrm{Coker}(\partial_1^P)$ for some exact sequence of projective R-modules

$$P = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\partial_0^P} \cdots$$

such that the complex $\mathrm{Hom}_R(P, Q)$ is exact for each projective R-module Q. The class of Gorenstein projective R-modules is denoted by $\mathcal{G}(\mathcal{P}(R))$.

In this paper, we investigate the modules that arise from an iteration of this construction. To wit, let $\mathcal{G}^2(\mathcal{P}(R))$ denote the class of R-modules M for which there exists an exact sequence of Gorenstein projective R-modules

$$G = \cdots \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \xrightarrow{\partial_0^G} \cdots$$

such that the complexes $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_R(H, G)$ are exact for each Gorenstein projective R-module H and $M \cong \mathrm{Coker}(\partial_1^G)$.

One checks readily that there is a containment $\mathcal{G}(\mathcal{P}(R)) \subseteq \mathcal{G}^2(\mathcal{P}(R))$. We answer a question from the folklore of this subject by verifying that this containment is always an equality. This is a consequence of Corollary 4.10, as is the dual version for Gorenstein injective R-modules; see Example 5.3.

Theorem A. *If R is a commutative ring, then $\mathcal{G}(\mathcal{P}(R)) = \mathcal{G}^2(\mathcal{P}(R))$.*

The proof of this result is facilitated by the consideration of a more general situation. Starting with a class of R-modules $\mathcal{W}$, we consider the associated full subcategory $\mathcal{G}(\mathcal{W})$ of the category of R-modules with objects defined as above; see Definition 4.1. Section 4 is devoted to the category-theoretic properties of $\mathcal{G}(\mathcal{W})$, those needed for the proof of Theorem A and others. For instance, the next result is contained in Proposition 4.11 and Theorem 4.12.

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THEOREM B. *Assume that* $\mathrm{Ext}^i_R(W,W') = 0$ *for all* $W, W' \in \mathcal{W}$ *and all* $i \geqslant 1$. *The Gorenstein subcategory* $\mathcal{G}(\mathcal{W})$ *is an exact category, and it is closed under kernels of epimorphisms* (*or cokernels of monomorphisms*) *if* $\mathcal{W}$ *is so.*

Most of the paper focuses on subcategories of an abelian category $\mathcal{A}$. The reader is encouraged to keep certain module-categories in mind. Specific examples are provided in Section 3, and we apply our results to these examples in Section 5.

1. *Categories and resolutions*

Here we set some notation and terminology for use throughout this paper.

DEFINITION 1.1. In this work $\mathcal{A}$ is an abelian category. We use the term 'subcategory' for a 'full additive subcategory that is closed under isomorphisms'. Write $\mathcal{P} = \mathcal{P}(\mathcal{A})$ and $\mathcal{I} = \mathcal{I}(\mathcal{A})$ for the subcategories of projective and injective objects in $\mathcal{A}$, respectively. A subcategory $\mathcal{X}$ of $\mathcal{A}$ is *exact* if it is closed under direct summands and extensions.

We fix subcategories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$, and $\mathcal{V}$ of $\mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$. Write $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}^{\geqslant 1}_{\mathcal{A}}(X,Y) = 0$ for each object X in $\mathcal{X}$ and each object Y in $\mathcal{Y}$. For an object A in $\mathcal{A}$, write $A \perp \mathcal{Y}$ if $\mathrm{Ext}^{\geqslant 1}_{\mathcal{A}}(A,Y) = 0$ for each object Y in $\mathcal{Y}$ (and $\mathcal{X} \perp A$ if $\mathrm{Ext}^{\geqslant 1}_{\mathcal{A}}(X,A) = 0$ for each object X in $\mathcal{X}$). We say that $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each object X in $\mathcal{X}$, there exists an exact sequence in $\mathcal{X}$

$$0 \longrightarrow X \longrightarrow W \longrightarrow X' \longrightarrow 0$$

such that W is an object in $\mathcal{W}$. The subcategory $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$. We say that $\mathcal{V}$ is a *generator* for $\mathcal{Y}$ if, for each object Y in $\mathcal{Y}$, there exists an exact sequence in $\mathcal{Y}$

$$0 \longrightarrow Y' \longrightarrow V \longrightarrow Y \longrightarrow 0$$

such that V is an object in $\mathcal{V}$. The subcategory $\mathcal{V}$ is a *projective generator* for $\mathcal{Y}$ if $\mathcal{V}$ is a generator for $\mathcal{Y}$ and $\mathcal{V} \perp \mathcal{Y}$.

DEFINITION 1.2. An *$\mathcal{A}$-complex* is a sequence of homomorphisms in $\mathcal{A}$

$$A = \cdots \xrightarrow{\partial^A_{n+1}} A_n \xrightarrow{\partial^A_n} A_{n-1} \xrightarrow{\partial^A_{n-1}} \cdots$$

such that $\partial^A_{n-1}\partial^A_n = 0$ for each integer n; the nth *homology object* of A is

$$\mathrm{H}_n(A) = \mathrm{Ker}(\partial^A_n)/\,\mathrm{Im}(\partial^A_{n+1}).$$

We frequently identify objects in $\mathcal{A}$ with complexes concentrated in degree 0.

Fix an integer i. The ith *suspension* of a complex A, denoted by $\Sigma^i A$, is the complex with

$$(\Sigma^i A)_n = A_{n-i} \quad \text{and} \quad \partial^{\Sigma^i A}_n = (-1)^i \partial^A_{n-i}.$$

The *hard truncation* $A_{\geqslant i}$ is the complex

$$A_{\geqslant i} = \cdots \xrightarrow{\partial^A_{i+2}} A_{i+1} \xrightarrow{\partial^A_{i+1}} A_i \longrightarrow 0$$

and the hard truncations $A_{>i}$, $A_{\leqslant i}$, and $A_{<i}$ are defined similarly.

The complex A is *$\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact* if the complex $\mathrm{Hom}_{\mathcal{A}}(X,A)$ is exact for each object X in $\mathcal{X}$. Dually, it is *$\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact* if the complex $\mathrm{Hom}_{\mathcal{A}}(A,X)$ is exact for each object X in $\mathcal{X}$.

Definition 1.3. Let A, A' be $\mathcal{A}$-complexes. The Hom-complex $\mathrm{Hom}_{\mathcal{A}}(A, A')$ is the complex of abelian groups defined as $\mathrm{Hom}_{\mathcal{A}}(A, A')_n = \prod_p \mathrm{Hom}_{\mathcal{A}}(A_p, A'_{p+n})$ with nth differential $\partial_n^{\mathrm{Hom}_{\mathcal{A}}(A,A')}$ given by $\{f_p\} \mapsto \{\partial_{p+n}^{A'} f_p - (-1)^n f_{n-1}\partial_p^A\}$. A *morphism* is an element of $\mathrm{Ker}(\partial_0^{\mathrm{Hom}_{\mathcal{A}}(A,A')})$ and α is *null-homotopic* if it is in $\mathrm{Im}(\partial_1^{\mathrm{Hom}_{\mathcal{A}}(A,A')})$. Given a second morphism $\alpha' \colon A \to A'$ we say that α and α' are *homotopic* if the difference $\alpha - \alpha'$ is null-homotopic. The morphism α is a *homotopy equivalence* if there is a morphism $\beta\colon A' \to A$ such that $\beta\alpha$ is homotopic to id_A and $\alpha\beta$ is homotopic to $\mathrm{id}_{A'}$. The complex A is *contractible* if the identity morphism id_A is null-homotopic. When A is contractible, it is exact, as is each of the complexes $\mathrm{Hom}_{\mathcal{A}}(A, N)$ and $\mathrm{Hom}_{\mathcal{A}}(M, A)$ for all objects M and N in $\mathcal{A}$.

A morphism of complexes $\alpha\colon A \to A'$ induces homomorphisms $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(A) \to \mathrm{H}_n(A')$, and α is a *quasi-isomorphism* when each $\mathrm{H}_n(\alpha)$ is an isomorphism. The *mapping cone* of α is the complex $\mathrm{Cone}(\alpha)$ defined as $\mathrm{Cone}(\alpha)_n = A'_n \oplus A_{n-1}$ with nth differential

$$\partial_n^{\mathrm{Cone}(\alpha)} = \begin{pmatrix} \partial_n^{A'} & \alpha_{n-1} \\ 0 & -\partial_{n-1}^{A} \end{pmatrix}.$$

This definition gives a degreewise split exact sequence $0 \to A' \to \mathrm{Cone}(\alpha) \to \Sigma A \to 0$. Further, the morphism α is a quasi-isomorphism if and only if $\mathrm{Cone}(\alpha)$ is exact. Finally, if id_A is the identity morphism for A, then $\mathrm{Cone}(\mathrm{id}_A)$ is contractible.

Definition 1.4. A complex X is *bounded* if $X_n = 0$ for $|n| \gg 0$. When $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$, the natural morphism $X \to M = \mathrm{H}_0(X)$ is a quasi-isomorphism. In this event, X is an *$\mathcal{X}$-resolution* of M if each X_n is an object in $\mathcal{X}$, and the following exact sequence is the *augmented $\mathcal{X}$-resolution* of M associated to X:

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow M \longrightarrow 0.$$

Instead of writing '$\mathcal{P}$-resolution' we will write 'projective resolution'. The *$\mathcal{X}$-projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\mathrm{pd}(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

The objects of $\mathcal{X}$-projective dimension 0 are exactly the objects of $\mathcal{X}$. We set

$$\mathrm{res}\,\widehat{\mathcal{X}} = \text{the subcategory of objects } M \text{ of } \mathcal{A} \text{ with } \mathcal{X}\text{-}\mathrm{pd}(M) < \infty.$$

We define *$\mathcal{Y}$-coresolutions* and *$\mathcal{Y}$-injective dimension* dually. The *augmented $\mathcal{Y}$-coresolution* associated to a $\mathcal{Y}$-coresolution Y is denoted ${}^+Y$, and the $\mathcal{Y}$-injective dimension of M is denoted $\mathcal{Y}\text{-}\mathrm{id}(M)$. We set

$$\mathrm{cores}\,\widehat{\mathcal{Y}} = \text{the subcategory of objects } N \text{ of } \mathcal{A} \text{ with } \mathcal{Y}\text{-}\mathrm{id}(N) < \infty.$$

Definition 1.5. An $\mathcal{X}$-resolution X is *$\mathcal{X}$-proper* (or simply *proper*) if the augmented resolution X^+ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact. We set

$$\mathrm{res}\,\widetilde{\mathcal{X}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{X}\text{-resolution}.$$

One checks readily that $\mathrm{res}\,\widetilde{\mathcal{X}}$ is additive. If X' is an object in $\mathcal{X}$, then the complex $0 \to X' \to 0$ is a proper $\mathcal{X}$-resolution of X'; hence X' is in $\mathrm{res}\,\widetilde{\mathcal{X}}$ and so $\mathcal{X} \subseteq \mathrm{res}\,\widetilde{\mathcal{X}}$.

Projective resolutions are always $\mathcal{P}$-proper, and so $\mathcal{A}$ has enough projectives if and only if $\mathrm{res}\,\widetilde{\mathcal{P}} = \mathcal{A}$. If M is an object in $\mathcal{A}$ that admits an $\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$ and a projective resolution $P \xrightarrow{\simeq} M$, then there exists a quasi-isomorphism $P \xrightarrow{\simeq} X$.

Proper coresolutions are defined dually, and we set

$$\mathrm{cores}\,\widetilde{\mathcal{Y}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{Y}\text{-coresolution}.$$

Again, $\operatorname{cores}\widetilde{\mathcal{Y}}$ is additive and $\mathcal{Y} \subseteq \operatorname{cores}\widetilde{\mathcal{Y}}$. Injective coresolutions are always $\mathcal{I}$-proper, and so $\mathcal{A}$ has enough injectives if and only if $\operatorname{cores}\widetilde{\mathcal{I}} = \mathcal{A}$. If N is an object in $\mathcal{A}$ that admits a $\mathcal{Y}$-coresolution $N \xrightarrow{\simeq} Y$ and an injective resolution $N \xrightarrow{\simeq} I$, then there exists a quasi-isomorphism $Y \xrightarrow{\simeq} I$.

The next lemmata are standard or have standard proofs; for Lemma 1.6 see the proof of [3, Theorem 2.3], for Lemma 1.7 see the proof of [3, Proposition 2.1], for Lemma 1.8 repeatedly apply Definition 1.1, and for the 'horseshoe lemma', Lemma 1.9, see [9, proof of Lemma 8.2.1].

LEMMA 1.6. *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence in $\mathcal{A}$.*

(a) *If $A_3 \perp \mathcal{W}$, then $A_1 \perp \mathcal{W}$ if and only if $A_2 \perp \mathcal{W}$. If $A_1 \perp \mathcal{W}$ and $A_2 \perp \mathcal{W}$, then $A_3 \perp \mathcal{W}$ if and only if the given sequence is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{W})$ exact.*

(b) *If $\mathcal{V} \perp A_1$, then $\mathcal{V} \perp A_2$ if and only if $\mathcal{V} \perp A_3$. If $\mathcal{V} \perp A_2$ and $\mathcal{V} \perp A_3$, then $\mathcal{V} \perp A_1$ if and only if the given sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V}, -)$ exact.*

LEMMA 1.7. *If $\mathcal{X} \perp \mathcal{Y}$, then $\mathcal{X} \perp \operatorname{res}\widehat{\mathcal{Y}}$ and $\operatorname{cores}\widehat{\mathcal{X}} \perp \mathcal{Y}$.*

LEMMA 1.8. *If $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, then every object X in $\mathcal{X}$ admits a proper $\mathcal{W}$-coresolution, and so $\mathcal{X} \subseteq \operatorname{cores}\widetilde{\mathcal{W}}$. If $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, then every object Y in $\mathcal{Y}$ admits a proper $\mathcal{V}$-resolution, and so $\mathcal{Y} \subseteq \operatorname{res}\widetilde{\mathcal{V}}$.*

LEMMA 1.9. *Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$.*

(a) *Assume that A' and A'' admit proper $\mathcal{X}$-resolutions $X' \xrightarrow{\simeq} A'$ and $X'' \xrightarrow{\simeq} A''$. If the given sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact, then A is in $\operatorname{res}\widetilde{\mathcal{X}}$ with proper $\mathcal{X}$-resolution $X \xrightarrow{\simeq} A$ such that there exists a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "\binom{\mathrm{id}_{X'}}{0}"] \arrow[d, "\simeq"'] & X \arrow[r, "(0\ \mathrm{id}_{X''})"] \arrow[d, "\simeq"'] & X'' \arrow[r] \arrow[d, "\simeq"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

in which the top row is degreewise split exact and

$$\partial_n^X = \begin{pmatrix} \partial_n^{X'} & f_n \\ 0 & \partial_n^{X''} \end{pmatrix}.$$

(b) *Assume that A' and A'' admit proper $\mathcal{Y}$-coresolutions $A' \xrightarrow{\simeq} Y'$ and $A'' \xrightarrow{\simeq} Y''$. If the given sequence is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{Y})$-exact, then A is in $\operatorname{cores}\widetilde{\mathcal{Y}}$ with proper $\mathcal{Y}$-coresolution $A \xrightarrow{\simeq} Y$ such that there exists a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "\simeq"'] & A \arrow[r] \arrow[d, "\simeq"'] & A'' \arrow[r] \arrow[d, "\simeq"'] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "\binom{\mathrm{id}_{Y'}}{0}"] & Y \arrow[r, "(0\ \mathrm{id}_{Y''})"] & Y'' \arrow[r] & 0 \end{tikzcd}$$

in which the bottom row is degreewise split exact and

$$\partial_n^Y = \begin{pmatrix} \partial_n^{Y'} & g_n \\ 0 & \partial_n^{Y''} \end{pmatrix}.$$

2. *Technical results*

This section consists of three lemmata which the reader may wish to skip during the first reading. The first result is for use in the proof of Lemma 2.2.

LEMMA 2.1. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in $\mathcal{A}$. Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and $\mathcal{V}$ is a projective generator for $\mathcal{Y}$.*

(a) *Assume that N' is an object in $\operatorname{cores}\widetilde{\mathcal{X}}$ and N'' is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$. If $N' \perp \mathcal{W}$ and $N'' \perp \mathcal{W}$, then N is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$.*

(b) *Assume that N'' is an object in $\operatorname{res}\widetilde{\mathcal{Y}}$, and N' is an object in $\operatorname{res}\widetilde{\mathcal{V}}$. If $\mathcal{V} \perp N'$ and $\mathcal{V} \perp N''$, then N is an object in $\operatorname{res}\widetilde{\mathcal{V}}$.*

Proof. We prove part (a); the proof of part (b) is dual. Let X' be a proper $\mathcal{X}$-coresolution of N' and set $N'_i = \operatorname{Ker}(\partial^{X'}_{-i})$ for $i \geqslant 0$, which yields an exact sequence

$$0 \longrightarrow N'_i \longrightarrow X'_{-i} \longrightarrow N'_{i+1} \longrightarrow 0. \qquad (*_i)$$

Each N'_i is an object in $\operatorname{cores}\widetilde{\mathcal{X}}$. By induction on i, Lemma 1.6 (a) implies that $N'_i \perp \mathcal{W}$.

We will construct exact sequences

$$0 \longrightarrow N \longrightarrow W_0 \longrightarrow N_1 \longrightarrow 0, \qquad (\circledast_0)$$
$$0 \longrightarrow N'_1 \longrightarrow N_1 \longrightarrow N''_1 \longrightarrow 0, \qquad (\dagger_0)$$

such that ($\circledast_0$) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, W_0 is an object in $\mathcal{W}$, and N''_1 is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$ such that $N''_1 \perp \mathcal{W}$. Inducting on $i \geqslant 0$, this will yield exact sequences

$$0 \longrightarrow N_i \longrightarrow W_{-i} \longrightarrow N_{i+1} \longrightarrow 0, \qquad (\circledast_i)$$
$$0 \longrightarrow N'_{i+1} \longrightarrow N_{i+1} \longrightarrow N''_{i+1} \longrightarrow 0, \qquad (\dagger_i)$$

such that ($\circledast_i$) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, W_{-i} is an object in $\mathcal{W}$, and N''_{i+1} is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$ such that $N''_{i+1} \perp \mathcal{W}$. Splicing together the sequences ($\circledast_i$) will then yield a proper $\mathcal{W}$-coresolution of N.

Consider the following pushout diagram, with the given exact sequence as the top row and with ($*_0$) as the leftmost column.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & X'_0 \arrow[r] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & V \arrow[r] \arrow[d] & N'' \arrow[r] & 0 \\
 & N'_1 \arrow[r, "\cong"] \arrow[d] & N'_1 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \qquad (1)$$

Since $N'' \perp \mathcal{W}$ and $X_0' \perp \mathcal{W}$, the middle row of (1) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact and $V \perp \mathcal{W}$ by Lemma 1.6 (a). Note that X_0' is in $\operatorname{cores}\widetilde{\mathcal{W}}$ by Lemma 1.8. Since N'' is in $\operatorname{cores}\widetilde{\mathcal{W}}$, Lemma 1.9(a) implies that V is in $\operatorname{cores}\widetilde{\mathcal{W}}$. Hence, a proper $\mathcal{W}$-coresolution of V provides an exact sequence

$$0 \longrightarrow V \longrightarrow W_0 \longrightarrow N_1'' \longrightarrow 0$$

that is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact with objects W_0 in $\mathcal{W}$ and N_1'' in $\operatorname{cores}\widetilde{\mathcal{W}}$. By assumption, we have $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{X}$, and so $W_0 \perp \mathcal{W}$ because W_0 is in $\mathcal{W}$. Since $V \perp \mathcal{W}$, Lemma 1.6(a) implies that $N_1'' \perp \mathcal{W}$. With the center column of (1) this yields another pushout diagram, and we shall show that the middle row and the rightmost column satisfy the conditions for ($\circledast_0$) and ($\dagger_0$), respectively.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"'] & V \arrow[r] \arrow[d] & N_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & W_0 \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & 0 \\
 & & N_1'' \arrow[r, "\cong"] \arrow[d] & N_1'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{2}$$

We have already seen that W_0 is in $\mathcal{W}$ and N_1'' is in $\operatorname{cores}\widetilde{\mathcal{W}}$. Since $N_1'' \perp \mathcal{W}$ and $N_1' \perp \mathcal{W}$, the rightmost column of (2) with Lemma 1.6(a) yields $N_1 \perp \mathcal{W}$, and so the middle row of (2) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. □

Next is a key lemma for both Theorems A and B from the introduction.

LEMMA 2.2. *Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, and that $\mathcal{V}$ is a projective generator for $\mathcal{Y}$.*

(a) *If M is an object in $\operatorname{cores}\widetilde{\mathcal{X}}$ and $M \perp \mathcal{W}$, then M is in $\operatorname{cores}\widetilde{\mathcal{W}}$.*
(b) *If N is an object in $\operatorname{res}\widetilde{\mathcal{Y}}$ and $\mathcal{V} \perp N$, then N is in $\operatorname{res}\widetilde{\mathcal{V}}$.*

Proof. We prove part (a); the proof of part (b) is dual. Let $M \xrightarrow{\simeq} X$ be a proper $\mathcal{X}$-coresolution of M. Setting $M' = \operatorname{Im}(\partial_0^X)$ yields an exact sequence

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow M' \longrightarrow 0 \tag{3}$$

that is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact. Since $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ there is an exact sequence with objects W_0 in $\mathcal{W}$ and X' in $\mathcal{X}$.

$$0 \longrightarrow X_0 \longrightarrow W_0 \longrightarrow X' \longrightarrow 0 \tag{4}$$

Consider the pushout diagram with (3) as the top row and with (4) as the middle column.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\cong"'] & X_0 \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & W_0 \arrow[r] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & U \arrow[r] \arrow[d] & 0 \\
 & & X' \arrow[r, "\cong"] \arrow[d] & X' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{5}
$$

We will show that U is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$ and that the middle row of (5) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. It will then follow that a proper $\mathcal{W}$-coresolution for M can be obtained by splicing the middle row of (5) with a proper $\mathcal{W}$-coresolution of U.

The object M' is in $\operatorname{cores}\widetilde{\mathcal{X}}$ by construction, and X' and X_0 are in $\mathcal{X}$. Thus, X' is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$ by Lemma 1.8, and $X' \perp \mathcal{W}$ and $X_0 \perp \mathcal{W}$ by hypothesis. Since the top row of (5) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact, the assumption $\mathcal{W} \subseteq \mathcal{X}$ implies that the top row of (5) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Hence, the assumption that $M \perp \mathcal{W}$ yields $M' \perp \mathcal{W}$ by Lemma 1.6(a). With the rightmost column of (5), Lemma 2.1(a) implies that U is an object in $\operatorname{cores}\widetilde{\mathcal{W}}$. Since $X' \perp \mathcal{W}$ and $M' \perp \mathcal{W}$, Lemma 1.6(a) yields $U \perp \mathcal{W}$, and so the middle row of (5) is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. □

The last result in this section is a tool for Proposition 4.6.

LEMMA 2.3. *For $n = 0, 1, 2, \dots, t$, let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be subcategories of $\mathcal{A}$.*

(a) *Assume that $\mathcal{X}_n$ is a cogenerator for $\mathcal{X}_{n+1}$ for each $n \geqslant 0$ and $\mathcal{X}_t \perp \mathcal{X}_0$. If $\mathcal{X}_t$ is closed under extensions, then $\mathcal{X}_0$ is an injective cogenerator for $\mathcal{X}_t$.*

(b) *Assume that $\mathcal{Y}_n$ is a generator for $\mathcal{Y}_{n+1}$ for each $n \geqslant 0$ and $\mathcal{Y}_0 \perp \mathcal{Y}_t$. If $\mathcal{Y}_t$ is closed under extensions, then $\mathcal{Y}_0$ is a projective generator for $\mathcal{Y}_t$.*

Proof. We prove part (a); the proof of part (b) is dual. Since $\mathcal{X}_t \perp \mathcal{X}_0$ by assumption, it remains to show that $\mathcal{X}_0$ is a cogenerator for $\mathcal{X}_t$. Fix an object X_t in $\mathcal{X}_t$. By reverse induction on $i < t$, we will construct exact sequences

$$0 \longrightarrow X_t \longrightarrow X_i \longrightarrow X_t^{(i)} \longrightarrow 0 \tag{$*_i$}$$

with objects X_i in $\mathcal{X}_i$ and $X_t^{(i)}$ in $\mathcal{X}_t$. Since $\mathcal{X}_{t-1}$ is a cogenerator for $\mathcal{X}_t$, the sequence $(*_{t-1})$ is known to exist. By induction, we assume that $(*_i)$ has been constructed and construct $(*_{i-1})$ from it. From $(*_i)$ we have the object X_i in $\mathcal{X}_i$. Since $\mathcal{X}_{i-1}$ is a cogenerator for $\mathcal{X}_i$, there is an exact sequence

$$0 \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow X_i' \longrightarrow 0 \tag{$\circledast_i$}$$

with X_{i-1} in $\mathcal{X}_{i-1}$ and X_i' in $\mathcal{X}_i$.

Consider the pushout diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X_t \arrow[r, "\cong"] \arrow[d] & X_t \arrow[d] & & \\
0 \arrow[r] & X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near end] & X_{i-1} \arrow[r] \arrow[d] & X_i' \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & X_t^{(i)} \arrow[r] \arrow[d] & X_t^{(i-1)} \arrow[r] \arrow[d] & X_i' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

with $(*_i)$ as the leftmost column and $(\circledast_i)$ as the middle row. The object X_i' is in $\mathcal{X}_i$, and hence in $\mathcal{X}_t$. Since $X_t^{(i)}$ is also in $\mathcal{X}_t$, the exactness of the bottom row, with the fact that $\mathcal{X}_t$ is closed under extensions, implies that $X_t^{(i-1)}$ is in $\mathcal{X}_t$, so the center column of the diagram is the desired sequence $(*_{i-1})$. $\square$

3. *Categories of interest*

Much of the motivation for this work comes from module categories. In reading this paper, the reader may find it helpful to keep in mind the examples outlined in the next few paragraphs, wherein R is a commutative ring.

Definition 3.1. Let $\mathcal{M}(R)$ denote the category of R-modules. To be clear, we write $\mathcal{P}(R)$ for the subcategory of projective R-modules and $\mathcal{I}(R)$ for the subcategory of injective R-modules. If $\mathcal{X}(R)$ is a subcategory of $\mathcal{M}(R)$, then $\mathcal{X}^{\mathrm{f}}(R)$ is the subcategory of finitely generated modules in $\mathcal{X}(R)$. Also set $\mathcal{A}b = \mathcal{M}(\mathbb{Z})$, the category of abelian groups.

The study of semidualizing modules, defined next, was initiated independently (with different names) by Foxby [**10**], Golod [**13**], and Vasconcelos [**19**].

Definition 3.2. An R-module C is *semidualizing* if it satisfies the following.

(1) The R-module C admits a (possibly unbounded) resolution by finite rank free R-modules.
(2) The natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.
(3) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A finitely generated projective R-module of rank 1 is semidualizing. If R is Cohen–Macaulay, then C is *dualizing* if it is semidualizing and $\mathrm{id}_R(C)$ is finite.

Based on the work of Enochs and Jenda [**8**], the following notions were introduced and studied in this generality by Holm and Jørgensen [**15**] and White [**20**].

Definition 3.3. Let C be a semidualizing R-module, and set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } P \otimes_R C \text{ where } P \text{ is } R\text{-projective}$$
$$\mathcal{I}_C(R) = \text{the subcategory of modules } \operatorname{Hom}_R(C, I) \text{ where } I \text{ is } R\text{-injective.}$$

Modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective* and *C-injective*, respectively. A *complete $\mathcal{P}\mathcal{P}_C$-resolution* is a complex X of R-modules satisfying the following.

(1) The complex X is exact and $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact.
(2) The R-module X_i is projective if $i \geqslant 0$ and X_i is C-projective if $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution X such that $M \cong \operatorname{Coker}(\partial_1^X)$, in which case X is a *complete $\mathcal{P}\mathcal{P}_C$-resolution of M*. We set

$$\mathcal{G}\mathcal{P}_C(R) = \text{the subcategory of } G_C\text{-projective } R\text{-modules.}$$

Projective R-modules and C-projective R-modules are G_C-projective, and $\mathcal{P}_C(R)$ is an injective cogenerator for $\mathcal{G}\mathcal{P}_C(R)$ by [**15**, Lemma 2.5 and Proposition 2.13] and [**20**, Propositions 3.2 and 3.9].

A *complete $\mathcal{I}_C\mathcal{I}$-coresolution* is a complex Y of R-modules such that:

(1) The complex Y is exact and $\operatorname{Hom}_R(\mathcal{I}_C(R), -)$-exact.
(2) The R-module Y_i is injective if $i \leqslant 0$ and Y_i is C-injective if $i > 0$.

An R-module N is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-coresolution Y such that $N \cong \operatorname{Ker}(\partial_0^Y)$, in which case Y is a *complete $\mathcal{I}_C\mathcal{I}$-coresolution of N*. We set

$$\mathcal{G}\mathcal{I}_C(R) = \text{the subcategory of } G_C\text{-injective } R\text{-modules.}$$

An R-module that is injective or C-injective is G_C-injective, and $\mathcal{I}_C(R)$ is a projective generator for $\mathcal{G}\mathcal{I}_C(R)$ by [**15**, Lemma 2.6, and Proposition 2.13] and results dual to [**20**, Propositions 3.2 and 3.9].

The next definition was first introduced by Auslander and Bridger [**1**, **2**] in the case $C = R$, and in this generality by Golod [**13**] and Vasconcelos [**19**].

Definition 3.4. Assume that R is noetherian and C is a semidualizing R-module. A finitely generated R-module H is *totally C-reflexive* if:

(1) $\operatorname{Ext}_R^{\geqslant 1}(H, C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(H, C), C)$, and
(2) the natural biduality map $H \to \operatorname{Hom}_R(\operatorname{Hom}_R(H, C), C)$ is an isomorphism.

Each finitely generated R-module that is either projective or C-projective is totally C-reflexive. We set

$$\mathcal{G}_C(R) = \text{the subcategory of totally } C\text{-reflexive } R\text{-modules}$$

and $\mathcal{G}(R) = \mathcal{G}_R(R)$. The equality $\mathcal{G}_C(R) = \mathcal{G}\mathcal{P}_C^{\mathrm{f}}(R)$ holds by [**20**, Theorem 5.4], and $\mathcal{P}_C^{\mathrm{f}}(R)$ is an injective cogenerator for $\mathcal{G}_C(R)$ by [**20**, Proposition 3.9, Lemma 5.3 and Theorem 5.4].

Over a noetherian ring, the next categories were introduced by Avramov and Foxby [**4**] when C is dualizing, and by Christensen [**7**] for arbitrary C. (Note that these works (and others) use the notation $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ for certain categories of complexes, while our categories consist precisely of the modules in these other categories.) In the non-noetherian setting, these definitions are taken from [**16**, **20**].

Definition 3.5. Let C be a semidualizing R-module. The *Auslander class* of C is the subcategory $\mathcal{A}_C(R)$ of R-modules M such that:

(1) $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0 = \operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$, and
(2) the natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the subcategory $\mathcal{B}_C(R)$ of R-modules N such that:
(1) $\mathrm{Ext}_R^{\geqslant 1}(C, M) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, M))$, and
(2) the natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

For a discussion of the next subcategory, consult [9, Section 5.3].

DEFINITION 3.6. The category of *flat cotorsion* R-modules is the subcategory $\mathcal{F}'(R)$ of flat R-modules F such that $\mathrm{Ext}_R^{\geqslant 1}(F', F) = 0$ for each flat R-module F'.

Gerko [12] introduced our final subcategory of interest.

DEFINITION 3.7. Assume that $(R, \mathfrak{m}, k)$ is local and noetherian. The *complexity* of a finitely generated R-module M is

$$\mathrm{cx}_R(M) = \inf\{d \in \mathbb{N} \mid \text{there exists a } c > 0 \text{ such that } \beta_n^R(M) \leqslant cn^{d-1} \text{ for } n \gg 0\},$$

where $\beta_n^R(M) = \mathrm{rank}_k(\mathrm{Tor}_i^R(M, k))$ is the nth *Betti number* of M. Let $\mathcal{G}'(R)$ denote the subcategory of modules in $\mathcal{G}(R)$ with finite complexity.

4. *Gorenstein subcategories*

In this section, we introduce and study the Gorenstein subcategory $\mathcal{G}(\mathcal{X})$.

DEFINITION 4.1. An exact complex in $\mathcal{X}$ is *totally $\mathcal{X}$-acyclic* if it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Let $\mathcal{G}(\mathcal{X})$ denote the subcategory of $\mathcal{A}$ with objects of the form $M \cong \mathrm{Coker}(\partial_1^X)$ for some totally $\mathcal{X}$-acyclic complex X; we say that X is a *complete $\mathcal{X}$-resolution of* M. Note that the isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X' \oplus X'', X) \cong \mathrm{Hom}_{\mathcal{A}}(X', X) \oplus \mathrm{Hom}_{\mathcal{A}}(X'', X),$$
$$\mathrm{Hom}_{\mathcal{A}}(X, X' \oplus X'') \cong \mathrm{Hom}_{\mathcal{A}}(X, X') \oplus \mathrm{Hom}_{\mathcal{A}}(X, X'')$$

show that the direct sum of totally $\mathcal{X}$-acyclic $\mathcal{X}$-complexes is totally $\mathcal{X}$-acyclic, and hence $\mathcal{G}(\mathcal{X})$ is additive. Set $\mathcal{G}^0(\mathcal{X}) = \mathcal{X}$ and $\mathcal{G}^1(\mathcal{X}) = \mathcal{G}(\mathcal{X})$, and inductively set $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$ for $n \geqslant 1$.

REMARK 4.2. Any contractible $\mathcal{X}$-complex is totally $\mathcal{X}$-acyclic; see Definition 1.3. In particular, for any object X in $\mathcal{X}$, the complex

$$0 \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0$$

is a complete $\mathcal{X}$-resolution, and so X is an object in $\mathcal{G}(\mathcal{X})$. Hence $\mathcal{X} \subseteq \mathcal{G}(\mathcal{X})$, and inductively $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{G}^{n+1}(\mathcal{X})$ for each $n \geqslant 0$.

There is a containment $\mathcal{G}(\mathcal{X}) \subseteq \mathrm{res}\,\widetilde{\mathcal{X}} \cap \mathrm{cores}\,\widetilde{\mathcal{X}}$. Indeed, If M is an object in $\mathcal{G}(\mathcal{X})$ with complete $\mathcal{X}$-resolution X, then the hard truncation $X_{\geqslant 0}$ is a proper $\mathcal{X}$-resolution of M and $X_{<0}$ is a proper $\mathcal{X}$-coresolution of M.

The orthogonality properties documented next will be very useful in the sequel; compare to [6, Proposition 4.2.5].

Proposition 4.3. *If $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{Y}$, then $\mathcal{G}^n(\mathcal{X}) \perp \operatorname{res}\widehat{\mathcal{W}}$ and $\operatorname{cores}\widehat{\mathcal{V}} \perp \mathcal{G}^n(\mathcal{Y})$ for each $n \geqslant 1$. In particular, if $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}^n(\mathcal{W}) \perp \operatorname{res}\widehat{\mathcal{W}}$ and $\operatorname{cores}\widehat{\mathcal{W}} \perp \mathcal{G}^n(\mathcal{W})$ for each $n \geqslant 1$.*

Proof. Assuming that $\mathcal{X} \perp \mathcal{W}$, we will show that $\mathcal{G}(\mathcal{X}) \perp \mathcal{W}$; the conclusion that $\mathcal{G}^n(\mathcal{X}) \perp \mathcal{W}$ will follow by induction, and $\mathcal{G}^n(\mathcal{X}) \perp \operatorname{res}\widehat{\mathcal{W}}$ will then follow from Lemma 1.7. The other conclusion is verified dually. Let M be an object in $\mathcal{G}(\mathcal{X})$ with complete $\mathcal{X}$-resolution X, and let W be an object in $\mathcal{W}$. For each integer i set $M_i = \operatorname{Coker}(\partial_i^X)$. Note that $M \cong M_1$. The exact sequence

$$0 \longrightarrow M_{i+1} \xrightarrow{\epsilon_i} X_{i-1} \longrightarrow M_i \longrightarrow 0$$

is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, and so it is $\operatorname{Hom}_{\mathcal{A}}(-, W)$-exact. In particular, the map $\operatorname{Hom}_{\mathcal{A}}(\epsilon_i, W)$ is surjective. Since $\mathcal{X} \perp \mathcal{W}$, part of the beginning of the associated long exact sequence in $\operatorname{Ext}_{\mathcal{A}}(-, W)$ is

$$\operatorname{Hom}_{\mathcal{A}}(X_{i-1}, W) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(\epsilon_i, W)} \operatorname{Hom}_{\mathcal{A}}(M_{i+1}, W) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(M_i, W) \longrightarrow 0$$

so the surjectivity of $\operatorname{Hom}_{\mathcal{A}}(\epsilon_i, W)$ implies that $\operatorname{Ext}^1_{\mathcal{A}}(M_i, W) = 0$. In particular,

$$\operatorname{Ext}^1_{\mathcal{A}}(M, W) \cong \operatorname{Ext}^1_{\mathcal{A}}(M_1, W) = 0.$$

If $j \geqslant 2$, then the remainder of the long exact sequence yields isomorphisms

$$\operatorname{Ext}^j_{\mathcal{A}}(M_i, W) \cong \operatorname{Ext}^{j-1}_{\mathcal{A}}(M_{i+1}, W).$$

Inductively, this yields the second isomorphism in the next sequence and the desired vanishing

$$\operatorname{Ext}^j_{\mathcal{A}}(M, W) \cong \operatorname{Ext}^j_{\mathcal{A}}(M_1, W) \cong \operatorname{Ext}^1_{\mathcal{A}}(M_j, W) = 0. \qquad \square$$

We next present a 'horseshoe lemma' for complete $\mathcal{X}$-resolutions; compare to [6, Corollary 4.3.5(a)].

Proposition 4.4. *Consider an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

that is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. If M' and M'' are objects in $\mathcal{G}(X)$, then so is M. Furthermore, given complete $\mathcal{X}$-resolutions X' and X'' of M' and M'', respectively, there is a degreewise split exact sequence of complexes

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

such that X is a complete $\mathcal{X}$-resolution of M, the induced sequence

$$0 \longrightarrow \operatorname{Coker}(\partial_1^{X'}) \longrightarrow \operatorname{Coker}(\partial_1^{X}) \longrightarrow \operatorname{Coker}(\partial_1^{X''}) \longrightarrow 0$$

is equivalent to the original sequence, and

$$\partial_n^X = \begin{pmatrix} \partial_n^{X'} & f_n \\ 0 & \partial_n^{X''} \end{pmatrix}$$

for each $n \in \mathbb{Z}$.

Proof. Let X' and X'' be complete $\mathcal{X}$-resolutions for M' and M'', respectively. Lemma 1.9 yields a degreewise split exact sequence of complexes $0 \to X' \to X \to X'' \to 0$ such that $M \cong \operatorname{Coker}(\partial_1^X)$. Note that each $X_i \cong X_i' \oplus X_i''$ is in $\mathcal{X}$. Since the complexes X' and X'' are both $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact, the same is true of X. Therefore, X is a complete $\mathcal{X}$-resolution of M. $\square$

Corollary 4.5. *If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{W})$ is closed under extensions.*

Proof. Proposition 4.3 implies that $\mathcal{G}(\mathcal{W}) \perp \mathcal{W}$ and $\mathcal{W} \perp \mathcal{G}(\mathcal{W})$. Hence, any exact sequence $0 \to M' \to M \to M'' \to 0$ with M' and M'' objects in $\mathcal{G}(\mathcal{W})$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Now apply Proposition 4.4. □

It is unclear in general whether $\mathcal{W}$ is an injective cogenerator for $\mathcal{G}^n(\mathcal{X})$ without the extra hypotheses in our next result.

Proposition 4.6. *Fix an integer $n \geqslant 1$.*

(a) *If $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and $\mathcal{G}^n(\mathcal{X})$ is closed under extensions, then $\mathcal{W}$ is an injective cogenerator for $\mathcal{G}^n(\mathcal{X})$.*

(b) *If $\mathcal{V}$ is a projective generator for $\mathcal{Y}$ and $\mathcal{G}^n(\mathcal{Y})$ is closed under extensions, then $\mathcal{V}$ is a projective generator for $\mathcal{G}^n(\mathcal{Y})$.*

Proof. We prove part (a); the proof of part (b) is dual. Set $\mathcal{X}_0 = \mathcal{W}$, $\mathcal{X}_1 = \mathcal{X}$, and $\mathcal{X}_t = \mathcal{G}^{t-1}(\mathcal{X})$ for $t = 2, \ldots, n+1$. By Proposition 4.3 we know that $\mathcal{G}^n(\mathcal{X}) \perp \mathcal{W}$, so the desired conclusion follows from Lemma 2.3. □

In Section 5 we document the consequences of the following result for the examples of Section 3.

Corollary 4.7. *If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{W})$.*

Proof. This follows from Corollary 4.5 and Proposition 4.6. □

The next result extends part of Remark 4.2 and represents a first step in the proof of Theorem A from the introduction.

Theorem 4.8. *Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and that $\mathcal{V}$ is a projective generator for $\mathcal{Y}$.*

(a) *If $\mathcal{X}$ is closed under extensions, then $\mathcal{G}^n(\mathcal{X}) \subseteq \mathrm{cores}\,\widetilde{\mathcal{W}}$ for each $n \geqslant 0$.*

(b) *If $\mathcal{Y}$ is closed under extensions, then $\mathcal{G}^n(\mathcal{Y}) \subseteq \mathrm{res}\,\widetilde{\mathcal{V}}$ for each $n \geqslant 0$.*

Proof. We prove part (a) by induction on n; the proof of part (b) is dual. The case $n = 0$ is given in Lemma 1.8. When $n = 1$, note that an object M in $\mathcal{G}(\mathcal{X})$ is in $\mathrm{cores}\,\widetilde{\mathcal{X}}$ by Remark 4.2, and one has $M \perp \mathcal{W}$ by Proposition 4.3; now apply Lemma 2.2.

Assume that $n > 1$ and $\mathcal{G}^{n-1}(\mathcal{X}) \subseteq \mathrm{cores}\,\widetilde{\mathcal{W}}$. Fix an object M in $\mathcal{G}^n(\mathcal{X})$ and a complete $\mathcal{G}^{n-1}(\mathcal{X})$-resolution G of M. By definition, the complex G is exact and there is an isomorphism $M \cong \mathrm{Ker}(\partial^G_{-1})$. For each integer j, set $M_j = \mathrm{Ker}(\partial^G_j)$ and observe that each M_j is an object in $\mathcal{G}^n(\mathcal{X})$. Since each object G_j is in $\mathcal{G}^{n-1}(\mathcal{X})$, Proposition 4.3 implies that $M_j \perp \mathcal{W}$ and $G_j \perp \mathcal{W}$ for each integer j, and we consider the exact sequences

$$0 \longrightarrow M_j \longrightarrow G_j \longrightarrow M_{j-1} \longrightarrow 0. \qquad (\maltese_j)$$

Our induction assumption implies that each object G_j is in $\mathrm{cores}\,\widetilde{\mathcal{W}}$.

By induction on $i \geqslant 0$, we construct exact sequences in $\mathcal{A}$

$$0 \longrightarrow M \longrightarrow W_0 \longrightarrow W_{-1} \longrightarrow \cdots \longrightarrow W_{-i} \longrightarrow U_{-i} \longrightarrow 0 \qquad (\dagger_i)$$

$$0 \longrightarrow M_{-(i+2)} \longrightarrow U_{-i} \longrightarrow V_{-i} \longrightarrow 0 \qquad (*_i)$$

$$0 \longrightarrow U_{-i} \longrightarrow W_{-(i+1)} \longrightarrow U_{-(i+1)} \longrightarrow 0 \qquad (\circledast_i)$$

$$0 \longrightarrow M_{-(i+3)} \longrightarrow U_{-(i+1)} \longrightarrow V_{-(i+1)} \longrightarrow 0 \qquad (\ddagger_i)$$

satisfying the following properties:

(a_i) the objects $W_0, \dots, W_{-(i+1)}$ are in $\mathcal{W}$;
(b_i) the sequence ($\dagger_i$) is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact;
(c_i) one has $U_{-i} \perp \mathcal{W}$ and $U_{-(i+1)} \perp \mathcal{W}$;
(d_i) one has $V_{-i} \perp \mathcal{W}$ and $V_{-(i+1)} \perp \mathcal{W}$;
(e_i) one has V_{-i} and $V_{-(i+1)}$ in $\operatorname{cores}\widetilde{\mathcal{W}}$.

The sequence ($\dagger_i$) is obtained by splicing the sequences ($\dagger_0$), ($\circledast_0$), ..., ($\circledast_{i-1}$). Continuing to splice inductively, conditions (a_i)–(c_i) show that this process yields a proper $\mathcal{W}$-coresolution of M, as desired.

We begin with the base case $i = 0$. The membership $G_{-1} \in \operatorname{cores}\widetilde{\mathcal{W}}$ yields a proper $\mathcal{W}$-coresolution of G_{-1} and hence an exact sequence

$$0 \longrightarrow G_{-1} \longrightarrow W_0 \longrightarrow V_0 \longrightarrow 0 \qquad (6)$$

that is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact and with objects $W_0 \in \mathcal{W}$ and $V_0 \in \operatorname{cores}\widetilde{\mathcal{W}}$. Using the conditions $G_{-1} \perp \mathcal{W}$ and $W_0 \perp \mathcal{W}$, Lemma 1.6(a) implies that $V_0 \perp \mathcal{W}$. Consider the pushout diagram with ($\maltese_{-1}$) as the top row and (6) as the middle column.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\cong"'] & G_{-1} \arrow[r] \arrow[d] & M_{-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & W_0 \arrow[r] \arrow[d] & U_0 \arrow[r] \arrow[d] & 0 \\
 & & V_0 \arrow[r, "\cong"] \arrow[d] & V_0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \qquad (7)$$

Applying Lemma 1.6(a) to the rightmost column of this diagram, the conditions $M_{-2} \perp \mathcal{W}$ and $V_0 \perp \mathcal{W}$ imply that $U_0 \perp \mathcal{W}$. For each object $W' \in \mathcal{W}$, use the condition $U_0 \perp W'$ with the long exact sequence in $\mathrm{Ext}_{\mathcal{A}}(-, W')$ asociated to the middle row of this diagram to conclude that this row is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Set ($\dagger_0$) equal to the middle row of (7), and set ($*_0$) equal to the rightmost column of (7). Construct the next pushout diagram using ($\maltese_{-2}$) in the top row and the rightmost column of (7) in the left column.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_{-2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & G_{-2} \arrow[r] \arrow[d] & M_{-3} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & U_0 \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & M_{-3} \arrow[r] & 0 \\
 & V_0 \arrow[r, "\cong"] \arrow[d] & V_0 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{8}
$$

As in the above discussion, the condition $V_0 \perp \mathcal{W}$ implies that the middle column of (8) is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Using Lemma 1.6(a) with this column, the conditions $G_{-2} \perp \mathcal{W}$ and $V_0 \perp \mathcal{W}$ yield $Z_0 \perp \mathcal{W}$. We know that G_{-2} and V_0 are in $\operatorname{cores}\widetilde{\mathcal{W}}$, so an application of Lemma 1.9(b) to this column implies that $Z_0 \in \operatorname{cores}\widetilde{\mathcal{W}}$. A proper $\mathcal{W}$-coresolution of Z_0 provides an exact sequence

$$0 \longrightarrow Z_0 \longrightarrow W_{-1} \longrightarrow V_{-1} \longrightarrow 0 \tag{9}$$

that is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact and with objects $W_{-1} \in \mathcal{W}$ and $V_{-1} \in \operatorname{cores}\widetilde{\mathcal{W}}$. Again using Lemma 1.6(a), the conditions $Z_0 \perp \mathcal{W}$ and $W_{-1} \perp \mathcal{W}$ conspire with the $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exactness of (9) to imply that $V_{-1} \perp \mathcal{W}$. Consider the next pushout diagram, with the middle row of (8) as the top row and with (9) as the middle column.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U_0 \arrow[r] \arrow[d, "\cong"'] & Z_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & M_{-3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U_0 \arrow[r] & W_{-1} \arrow[r] \arrow[d] & U_{-1} \arrow[r] \arrow[d] & 0 \\
 & & V_{-1} \arrow[r, "\cong"] \arrow[d] & V_{-1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{10}
$$

Set $(\circledast_0)$ equal to the middle row of this diagram, and $(\ddagger_0)$ equal to the rightmost column. Due to Lemma 1.6(a), the conditions $M_{-3} \perp \mathcal{W}$ and $V_{-1} \perp \mathcal{W}$ imply that $U_{-1} \perp \mathcal{W}$. Thus, conditions (a_0)–(e_0) are satisfied, establishing the base case.

For the induction step, assume that the exact sequences $(\dagger_i)$, $(*_i)$, $(\circledast_i)$, and $(\ddagger_i)$ have been constructed satisfying conditions (a_i)–(e_i). Note that condition (c_i) implies that the sequence $(\circledast_i)$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Thus, we may splice together the sequences $(\dagger_i)$ and $(\circledast_i)$ to construct the sequence $(\dagger_{i+1})$ which is exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact and such that $W_0, \ldots, W_{-(i+1)} \in \mathcal{W}$. Also, set $(*_{i+1}) = (\ddagger_i)$.

The next pushout diagram has $(\maltese_{i+3})$ in the top row and $(\ddagger_i)$ in the left column.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_{-(i+3)} \arrow[r] \arrow[d] & G_{-(i+3)} \arrow[r] \arrow[d] & M_{-(i+4)} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & U_{-(i+1)} \arrow[r] \arrow[d] & Z_{-(i+1)} \arrow[r] \arrow[d] \arrow[ul, phantom, "\lrcorner", very near start] & M_{-(i+4)} \arrow[r] & 0 \\
& V_{-(i+1)} \arrow[r, "\cong"] \arrow[d] & V_{-(i+1)} \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd} \tag{11}$$

With the long exact sequence in $\mathrm{Ext}_{\mathcal{A}}(-,-)$, the condition $V_{-(i+1)} \perp \mathcal{W}$ implies that the middle column of (11) is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Using Lemma 1.6(a) with this column, the conditions $G_{-(i+3)} \perp \mathcal{W}$ and $V_{-(i+1)} \perp \mathcal{W}$ yield $Z_{-(i+1)} \perp \mathcal{W}$. As $G_{-(i+3)}$ and $V_{-(i+1)}$ are in $\operatorname{cores}\widetilde{\mathcal{W}}$, apply Lemma 1.9(b) to this column to conclude that $Z_{-(i+1)} \in \operatorname{cores}\widetilde{\mathcal{W}}$. A proper $\mathcal{W}$-coresolution of $Z_{-(i+1)}$ provides an exact sequence

$$0 \longrightarrow Z_{-(i+1)} \longrightarrow W_{-(i+2)} \longrightarrow V_{-(i+2)} \longrightarrow 0 \tag{12}$$

that is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact and with objects $W_{-(i+2)} \in \mathcal{W}$ and $V_{-(i+2)} \in \operatorname{cores}\widetilde{\mathcal{W}}$. Again using Lemma 1.6(a) and the $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exactness of (12), the conditions $Z_{-(i+1)} \perp \mathcal{W}$ and $W_{-(i+2)} \perp \mathcal{W}$ imply that $V_{-(i+2)} \perp \mathcal{W}$. Consider the next pushout diagram, with the middle row of (11) as the top row and with (12) as the middle column.

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U_{-(i+1)} \arrow[r] \arrow[d, "\cong"'] & Z_{-(i+1)} \arrow[r] \arrow[d] & M_{-(i+4)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U_{-(i+1)} \arrow[r] & W_{-(i+2)} \arrow[r] \arrow[d] & U_{-(i+2)} \arrow[r] \arrow[d] \arrow[ul, phantom, "\lrcorner", very near start] & 0 \\
& & V_{-(i+2)} \arrow[r, "\cong"] \arrow[d] & V_{-(i+2)} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd} \tag{13}$$

Set $(\circledast_{i+1})$ equal to the middle row of this diagram, and set $(\ddagger_{i+1})$ equal to the rightmost column. With Lemma 1.6(a), the conditions $M_{-(i+4)} \perp \mathcal{W}$ and $V_{-(i+2)} \perp \mathcal{W}$ imply that $U_{-(i+2)} \perp \mathcal{W}$. Thus, conditions (a_{i+1})–(e_{i+1}) are satisfied, establishing the induction step. $\square$

What follows is the second step in the proof of Theorem A from the introduction. See Example 5.9 for the necessity of the cogeneration hypothesis.

THEOREM 4.9. *If $\mathcal{X}$ is closed under extensions and $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$, then $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{W})$ for each $n \geqslant 1$.*

Proof. Let N be an object in $\mathcal{G}^n(\mathcal{X})$. By Theorem 4.8 we know that N admits a proper $\mathcal{W}$-resolution $W' \xrightarrow{\simeq} N$ and a proper $\mathcal{W}$-coresolution $N \xrightarrow{\simeq} W''$. We will show that ${}^+W'$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact and that ${}^+W''$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Since we already know that ${}^+W'$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and that ${}^+W''$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact, this will show that the concatenated complex

$$\cdots \longrightarrow W_1' \longrightarrow W_0' \longrightarrow W_0'' \longrightarrow W_{-1}' \longrightarrow \cdots$$

is a complete $\mathcal{W}$-resolution of N, completing the proof.

We will show that ${}^+W'$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact; the proof of the other fact is dual. For each $i \geqslant 0$, there is an exact sequence

$$0 \longrightarrow N_{i+1} \longrightarrow W_i' \longrightarrow N_i \longrightarrow 0. \qquad (*_i)$$

We have $N_0 = N$ and so $N_0 \perp \mathcal{W}$ is true by Proposition 4.3, and $W_i' \perp \mathcal{W}$ by assumption. Using Lemma 1.6(a), an induction argument implies that $N_i \perp \mathcal{W}$ for each i. It follows that $(*_i)$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact, and it follows that ${}^+W'$ is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact, as desired. □

Theorem A from the introduction follows from the next result; see Example 5.3.

COROLLARY 4.10. *If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}^n(\mathcal{W}) = \mathcal{G}(\mathcal{W})$ for each $n \geqslant 1$.*

Proof. Note that Corollaries 4.5 and 4.7 imply that $\mathcal{G}(\mathcal{W})$ is closed under extensions and that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{W})$.

We argue by induction on n, the case $n = 1$ being trivial. For $n > 1$, if $\mathcal{G}^{n-1}(\mathcal{W}) = \mathcal{G}(\mathcal{W})$, then setting $\mathcal{X} = \mathcal{G}(\mathcal{W})$ in Theorem 4.9 yields the final containment in the next sequence

$$\mathcal{G}(\mathcal{W}) \subseteq \mathcal{G}^n(\mathcal{W}) = \mathcal{G}(\mathcal{G}^{n-1}(\mathcal{W})) = \mathcal{G}(\mathcal{G}(\mathcal{W})) \subseteq \mathcal{G}(\mathcal{W}),$$

and hence the desired conclusion. □

With Corollary 4.5, the final two results of this section contain Theorem B from the introduction; compare to [**6**, Corollary 4.3.5].

PROPOSITION 4.11. *If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{W})$ is closed under direct summands.*

Proof. Let A' and A'' be objects in $\mathcal{A}$ such that $A' \oplus A''$ is in $\mathcal{G}(\mathcal{W})$. We construct proper $\mathcal{W}$-resolutions $W' \xrightarrow{\simeq} A'$ and $W'' \xrightarrow{\simeq} A''$ such that $(W')^+$ and $(W'')^+$ are $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Dually, one constructs proper $\mathcal{W}$-coresolutions $A' \xrightarrow{\simeq} V'$ and $A'' \xrightarrow{\simeq} V''$ such that ${}^+V'$ and ${}^+V''$ are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact, and this shows that A' and A'' are in $\mathcal{G}(\mathcal{W})$.

Observe first that A' and A'' both admit (augmented) proper $\mathcal{G}(\mathcal{W})$-resolutions

$$X' = \cdots \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)} A' \oplus A'' \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)} A' \oplus A'' \xrightarrow{(\mathrm{id}\ 0)} A' \longrightarrow 0,$$
$$X'' = \cdots \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)} A' \oplus A'' \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)} A' \oplus A'' \xrightarrow{(0\ \mathrm{id})} A'' \longrightarrow 0,$$

where properness follows from the contractibility of X' and X''. From Proposition 4.3 we know that $\mathcal{W} \perp (A' \oplus A'')$, so the additivity of $\operatorname{Ext}_{\mathcal{A}}$ implies that $\mathcal{W} \perp A'$ and $\mathcal{W} \perp A''$. Lemma 2.2(b) and Corollary 4.7 imply that A' and A'' admit proper $\mathcal{W}$-resolutions $W' \xrightarrow{\simeq} A'$ and $W'' \xrightarrow{\simeq} A''$ because

$$\mathcal{W} \subseteq \mathcal{G}(\mathcal{W}) \quad \text{and} \quad A', A'' \in \operatorname{res}\widetilde{\mathcal{G}(\mathcal{W})}$$

by the preceeding argument. Hence, $W' \oplus W'' \xrightarrow{\simeq} A' \oplus A''$ is a proper $\mathcal{W}$-resolution. We show that $(W')^+$ and $(W'')^+$ are $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. As $A' \oplus A''$ is in $\mathcal{G}(\mathcal{W})$, it admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\simeq} A' \oplus A''$ such that W^+ is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Hence, the resolutions W and $W' \oplus W''$ are homotopy equivalent. Because W^+ is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, we know that $(W' \oplus W'')^+$ is also $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, and so $(W')^+$ and $(W'')^+$ are $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. □

THEOREM 4.12. *Assume that $\mathcal{W} \perp \mathcal{W}$.*
(a) *If $\mathcal{W}$ is closed under kernels of epimorphisms, then so is $\mathcal{G}(\mathcal{W})$.*
(b) *If $\mathcal{W}$ is closed under cokernels of monomorphisms, then so is $\mathcal{G}(\mathcal{W})$.*

Proof. We prove part (a); the proof of part (b) is dual. Consider an exact sequence in $\mathcal{A}$ with objects N and N'' in $\mathcal{G}(\mathcal{W})$:

$$0 \longrightarrow N' \longrightarrow N \xrightarrow{\tau} N'' \longrightarrow 0. \tag{14}$$

Let W and W'' be complete $\mathcal{W}$-resolutions of N and N'', respectively.

We first construct a commutative diagram of the following form.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2^W"] & W_1 \arrow[r, "\partial_1^W"] \arrow[d, dashed, "\widetilde{\tau}_1"'] & W_0 \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\tau}_0"'] & N \arrow[r, "\epsilon"] \arrow[d, "\tau"'] & W_{-1} \arrow[r, "\partial_{-1}^W"] \arrow[d, dashed, "\widetilde{\tau}_{-1}"'] & W_{-2} \arrow[r, "\partial_{-2}^W"] \arrow[d, dashed, "\widetilde{\tau}_{-2}"'] & \cdots \\
\cdots \arrow[r, "\partial_2^{W''}"] & W''_1 \arrow[r, "\partial_1^{W''}"] & W''_0 \arrow[r, "\pi''"] & N'' \arrow[r, "\epsilon''"] & W''_{-1} \arrow[r, "\partial_{-1}^{W''}"] & W''_{-2} \arrow[r, "\partial_{-2}^{W''}"] & \cdots
\end{tikzcd} \tag{15}$$

Here $\epsilon\pi = \partial_0^W$ and $\epsilon''\pi'' = \partial_0^{W''}$. Since $(W_{\geqslant 0})^+$ is a chain complex and $(W''_{\geqslant 0})^+$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$ exact, one can successively lift τ to the left as in the diagram; argue as in [**14**, Proposition 1.8]. Dually, since ${}^+W''_{<0}$ is a chain complex and ${}^+W_{<0}$ is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$ exact, one can successively lift τ to the right as in the diagram.

Thus, we have constructed a morphism of chain complexes $\widetilde{\tau}\colon W \to W''$ such that the induced map $\operatorname{Coker}(\partial_1^W) \to \operatorname{Coker}(\partial_1^{W''})$ is equivalent to τ.

Next, we show that there exist a complex $\widetilde{W}$ and a morphism $\tau'\colon W \oplus \widetilde{W} \to W''$ satisfying the following properties.

(i) The complex $\widetilde{W}$ is contractible and $\widetilde{W}_n$ is in $\mathcal{W}$ for each $n \in \mathbb{Z}$;
(ii) The morphism τ'_n is an epimorphism for each $n \in \mathbb{Z}$;
(iii) The natural monomorphism $W \xrightarrow{\epsilon} W \oplus \widetilde{W}$ satisfies $\widetilde{\tau} = \tau'\epsilon$.

The complex $\widetilde{W} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W''})$ is contractible; see Definition 1.3. Let $f\colon \widetilde{W} \to W''$ denote the composition of the natural morphisms

$$\widetilde{W} = \Sigma^{-1}\operatorname{Cone}(\mathrm{id}_{W''}) \longrightarrow W''.$$

Note that each f_n is a split epimorphism. It follows that the homomorphisms

$$\tau'_n = (\widetilde{\tau}_n \ f_n)\colon W_n \oplus \widetilde{W}_n \longrightarrow W''_n$$

describe a morphism of complexes satisfying the desired properties.

Because of property (i) the complex $\widetilde{W}$ is a complete $\mathcal{W}$-resolution; see Remark 4.2. Set $\widetilde{N} = \operatorname{Coker}(\partial_1^{\widetilde{W}})$, which is an object in $\mathcal{G}(\mathcal{W})$ with complete resolution $\widetilde{W}$. Furthermore, one has

$$\operatorname{Coker}(\partial_1^{W \oplus \widetilde{W}}) \cong N \oplus \widetilde{N},$$

and the morphism τ' induces a homomorphism

$$N \oplus \widetilde{N} \xrightarrow{f=(\tau\ \pi)} N''.$$

Because τ is surjective, the map f is also surjective. We will show that $\mathrm{Ker}(f)$ is in $\mathcal{G}(\mathcal{W})$, and then we will show that $N' = \mathrm{Ker}(\tau)$ is in $\mathcal{G}(\mathcal{W})$.

The morphism τ' is degreewise surjective. As $\mathcal{W}$ is closed under kernels of epimorphisms, it follows that the complex $W' = \mathrm{Ker}(\widetilde{\tau})$ consists of objects in $\mathcal{W}$. The next exact sequence shows that W' is exact because W, $\widetilde{W}$, and W'' are so:

$$0 \longrightarrow W' \longrightarrow W \oplus \widetilde{W} \xrightarrow{\tau'} W'' \longrightarrow 0.$$

This sequence induces a second exact sequence

$$0 \longrightarrow W'_{\geqslant 0} \longrightarrow W_{\geqslant 0} \oplus \widetilde{W}_{\geqslant 0} \longrightarrow W''_{\geqslant 0} \longrightarrow 0$$

with associated long exact sequence of the form

$$0 \longrightarrow \mathrm{Coker}(\partial_1^{W'}) \longrightarrow N \xrightarrow{f} N'' \longrightarrow 0.$$

Thus, we have $\mathrm{Ker}(f) \cong \mathrm{Coker}(\partial_1^{W})$. To show that $\mathrm{Ker}(f)$ is in $\mathcal{G}(\mathcal{W})$, it suffices to show that W' is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. For each object U in $\mathcal{W}$, the next sequence of complexes is exact as $\mathcal{W} \perp \mathcal{W}$:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(U, W') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(U, W) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(U, W'') \longrightarrow 0.$$

Since W and W'' are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact, the associated long exact sequence shows that W' is also $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Dually, one shows that W' is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact, thus showing that $\mathrm{Ker}(f)$ is in $\mathcal{G}(\mathcal{W})$.

To see that N' is in $\mathcal{G}(\mathcal{W})$, consider the following pullback diagram, with (14) as the rightmost column and with the middle row that is the natural split exact sequence.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{N} \arrow[r, "\gamma"] \arrow[d, "\alpha", "\cong"'] & \mathrm{Ker}(f) \arrow[r] \arrow[d, "\delta"] \arrow[dr, phantom, "\ulcorner", very near start] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{N} \arrow[r, "\beta"] & N \oplus \widetilde{N} \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, "\tau"] & 0 \\
 & & N'' \arrow[r, "\cong"] \arrow[d] & N'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{16}$$

Let $\sigma\colon N \oplus \widetilde{N} \to \widetilde{N}$ denote the natural surjection. It follows that $\sigma\beta = \mathrm{id}_{\widetilde{N}}$. Since α is an isomorphism, the equality $\delta\gamma = \beta\alpha$ implies that

$$(\alpha^{-1}\sigma\delta)\gamma = \alpha^{-1}\sigma\beta\alpha = \alpha^{-1}\alpha = \mathrm{id}_{\widetilde{N}},$$

and so the top row of (16) is split exact. Hence, the object N' is a direct summand of $\mathrm{Ker}(f)$. We have shown that $\mathrm{Ker}(f)$ is in $\mathcal{G}(\mathcal{W})$. The category $\mathcal{G}(\mathcal{W})$ is closed under direct summands by Proposition 4.11, and so N' is in $\mathcal{G}(\mathcal{W})$ as desired. □

5. *Consequences for categories of interest*

Let R be a commutative ring, and let C be a semidualizing R-module. We now apply the results of Section 4 to the examples in Section 3. We begin with some computations.

EXAMPLE 5.1. The relevant definitions yield equalities $\mathcal{G}(\mathcal{P}(R)) = \mathcal{GP}(R)$ and $\mathcal{G}(\mathcal{I}(R)) = \mathcal{GI}(R)$. If R is noetherian, then $\mathcal{G}(\mathcal{P}^{\mathrm{f}}(R)) = \mathcal{G}(R)$.

The next result generalizes the previous example.

PROPOSITION 5.2. *Let R be a commutative ring. If C is R-semidualizing, then*

$$\mathcal{G}(\mathcal{P}_C(R)) = \mathcal{GP}_C(R) \cap \mathcal{B}_C(R) \quad \text{and} \quad \mathcal{G}(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R).$$

If, further, R is noetherian, then $\mathcal{G}(\mathcal{P}^{\mathrm{f}}_C(R)) = \mathcal{G}_C(R) \cap \mathcal{B}_C(R)$.

Proof. We will prove the first equality; the others are proved similarly. For one containment, let M be an object in $\mathcal{G}(\mathcal{P}_C(R))$. To show that M is an object in $\mathcal{GP}_C(R)$, we use [20, Proposition 3.2]: it suffices to show that M admits a proper $\mathcal{P}_C(R)$-coresolution and $M \perp \mathcal{P}_C(R)$.

The first of these is given in Remark 4.2 which says that M is in $\operatorname{cores}\widetilde{\mathcal{P}_C(R)}$; the second one is given in Proposition 4.3, which implies that $\mathcal{G}(\mathcal{P}_C) \perp \mathcal{P}_C$. To show that M is an object in $\mathcal{B}_C(R)$, we need to verify $\operatorname{Ext}_R^{\geqslant 1}(C, M) = 0$ and $\operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C, M)) = 0$ and $M \cong C \otimes_R \operatorname{Hom}_R(C, M)$. The first of these is given in Proposition 4.3, which implies that $\mathcal{P}_C \perp \mathcal{G}(\mathcal{P}_C)$, and the others are in [18, Proposition 2.2(a)].

For the reverse containment, fix an object N in $\mathcal{GP}_C(R) \cap \mathcal{B}_C(R)$. Since N is in $\mathcal{GP}_C(R)$, it admits a complete $\mathcal{PP}_C$-resolution Y, so the complex $Y_{<0}$ is a proper $\mathcal{P}_C(R)$-coresolution of N. Also, N admits a proper $\mathcal{P}_C(R)$-resolution Z by [18, Corollary 2.4(a)] as N is in $\mathcal{B}_C(R)$. Once it is shown that $Y^+_{<0}$ is $\operatorname{Hom}_R(\mathcal{P}_C(R), -)$-exact and Z^+ is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, a complete $\mathcal{PP}_C$-resolution of N will be obtained by splicing Z and $Y_{<0}$.

To see that $Y^+_{<0}$ is $\operatorname{Hom}_R(\mathcal{P}_C(R), -)$-exact, set $N^{(0)} = N$ and $N^{(i)} = \operatorname{Coker}(\partial^Y_{i-1})$ for each $i \leqslant -1$. From [16, Lemma 5.1(a)], we know that Y_i is in $\mathcal{B}_C(R)$ for each $i \leqslant -1$. Since N is also in $\mathcal{B}_C(R)$, an induction argument using the exact sequence

$$0 \longrightarrow N^{(i-1)} \longrightarrow Y_i \longrightarrow N^{(i)} \longrightarrow 0 \qquad (*_i)$$

implies that $N^{(i)}$ is in $\mathcal{B}_C(R)$ for each $i \leqslant -1$. For each projective R-module P, this yields the vanishing in the next sequence

$$\operatorname{Ext}^1_R(P \otimes_R C, N^{(i)}) \cong \operatorname{Hom}_R(P, \operatorname{Ext}^1_R(C, N^{(i)})) = 0,$$

while the isomorphism is from Hom–tensor adjointness. It follows that each sequence $(*_i)$ is $\operatorname{Hom}_R(\mathcal{P}_C(R), -)$-exact, and thus so is $Y^+_{<0}$.

To see that Z^+ is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, it suffices to let P be projective and to justify the following sequence for $i \geqslant 1$:

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(Z^+, P \otimes_R C)) = \mathrm{H}_{-i}(\operatorname{Hom}_R(Z, P \otimes_R C)) \cong \operatorname{Ext}^i_R(N, P \otimes_R C) = 0.$$

The isomorphism is derived from [18, Corollary 4.2(a)] because N and $P \otimes_R C$ are in $\mathcal{B}_C(R)$. The vanishing follows because N is in $\mathcal{GP}_C(R)$ and $\mathcal{GP}_C(R) \perp \mathcal{P}_C(R)$; see [20, Proposition 3.2]. □

We now outline the consequences of Corollaries 4.7 and 4.10 for the examples of Section 3. The first example below contains Theorem A from the introduction.

EXAMPLE 5.3. The category $\mathcal{P}_C(R)$ is an injective cogenerator and a projective generator for $\mathcal{GP}_C(R) \cap \mathcal{B}_C(R)$, and $\mathcal{G}^n(\mathcal{P}_C(R)) = \mathcal{GP}_C(R) \cap \mathcal{B}_C(R)$ for each $n \geqslant 1$. Hence, $\mathcal{P}(R)$ is an injective cogenerator and a projective generator for $\mathcal{GP}(R)$, and $\mathcal{G}^n(\mathcal{P}(R)) = \mathcal{GP}(R)$. If $\mathcal{A}$ has enough projectives, then $\mathcal{P}(\mathcal{A})$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{P}(\mathcal{A}))$, and $\mathcal{G}^n(\mathcal{P}(\mathcal{A})) = \mathcal{G}(\mathcal{P}(\mathcal{A}))$.

EXAMPLE 5.4. The category $\mathcal{I}_C(R)$ is an injective cogenerator and a projective generator for $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$, and $\mathcal{G}^n(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ for each $n \geqslant 1$. Hence, $\mathcal{I}(R)$ is an injective cogenerator and a projective generator for $\mathcal{GI}(R)$, and $\mathcal{G}^n(\mathcal{I}(R)) = \mathcal{GI}(R)$. If $\mathcal{A}$ has enough injectives, then $\mathcal{I}(\mathcal{A})$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{I}(\mathcal{A}))$, and $\mathcal{G}^n(\mathcal{I}(\mathcal{A})) = \mathcal{G}(\mathcal{I}(\mathcal{A}))$.

EXAMPLE 5.5. Assume that R is noetherian. Then $\mathcal{P}^{\mathrm{f}}_C(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}_C(R) \cap \mathcal{B}_C(R)$, and $\mathcal{G}^n(\mathcal{P}^{\mathrm{f}}_C(R)) = \mathcal{G}_C(R) \cap \mathcal{B}_C(R)$. In particular, $\mathcal{P}^{\mathrm{f}}(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}(R)$, and $\mathcal{G}^n(\mathcal{P}^{\mathrm{f}}(R)) = \mathcal{G}(R)$.

EXAMPLE 5.6. The category $\mathcal{F}'(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{F}'(R))$, and $\mathcal{G}^n(\mathcal{F}'(R)) = \mathcal{G}(\mathcal{F}'(R))$ for each $n \geqslant 1$.

With Proposition 4.3 and Corollary 4.5 in mind, we now show that $\mathcal{W} \perp \mathcal{W}$ need not imply that $\mathcal{G}(\mathcal{W}) \perp \mathcal{G}(\mathcal{W})$.

EXAMPLE 5.7. Let $(R, \mathfrak{m}, k)$ be a local, nonregular, Gorenstein, artinian ring. With $\mathcal{W} = \mathcal{P}^{\mathrm{f}}(R)$, we have $\mathcal{G}(\mathcal{W}) = \mathcal{G}(R) = \mathcal{M}^{\mathrm{f}}(R)$ where the last equality holds because R is artinian and Gorenstein; see [**6**, Theorems 1.4.8 and 1.4.9]. We know that $\mathrm{Ext}^{\geqslant 1}_R(k,k) \neq 0$ since R is nonregular, and so $\mathcal{G}(\mathcal{W}) \not\perp \mathcal{G}(\mathcal{W})$.

We conclude with some questions and final observations.

QUESTIONS 5.8. Must there be an equality $\mathcal{G}^n(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ for each $n \geqslant 1$? Is $\mathcal{G}(\mathcal{X})$ always exact? Is $\mathcal{G}(\mathcal{X})$ always closed under kernels of epimorphisms or cokernels of monomorphisms? Must $\mathcal{G}(\mathcal{W})$ be contained in $\mathcal{G}(\mathcal{X})$? Can $\mathcal{G}(\mathcal{F}(R))$ or $\mathcal{G}(\mathcal{F}'(R))$ or $\mathcal{G}(\mathcal{G}'(R))$ be identified as in Proposition 5.2?

The final examples are presented with an eye toward the last question of Questions 5.8.

EXAMPLE 5.9. If $(R, \mathfrak{m})$ is a noetherian local ring and $\dim(R) \geqslant 1$, then $\mathcal{G}(\mathcal{F}(R)) \nsubseteq \mathcal{G}(\mathcal{P}(R))$. Indeed, the ring of formal power series $R[\![X]\!]$ is a flat R-module, so it is in $\mathcal{G}(\mathcal{F}(R))$. Suppose by way of contradiction that $R[\![X]\!]$ is in $\mathcal{G}(\mathcal{P}(R))$. First note that [**17**, Proposition 6] yields $\mathrm{pd}_R(R[\![X]\!]) < \infty$, and so [**14**, Proposition 2.7] implies that $\mathrm{pd}_R(R[\![X]\!]) = \mathcal{G}\text{-}\mathrm{pd}_R(R[\![X]\!]) = 0$. It follows that $R[\![X]\!]$ is projective, contradicting [**5**, Theorem 2.1].

From this it follows that the conclusion of Theorem 4.9 need not hold if $\mathcal{W}$ is not a cogenerator for $\mathcal{X}$. To see this, assume that R is $\mathfrak{m}$-adically complete. Standard results combine to show that $\mathcal{P}(R)$ is a projective generator for $\mathcal{F}(R)$ and that $\mathcal{F}(R)$ is closed under extensions. Furthermore, one has $\mathcal{F}(R) \perp \mathcal{P}(R)$ by [**9**, Theorem 5.3.28].

With Theorem 4.9, the previous example provides the next result.

Corollary 5.10. *If R is a complete local notherian ring and $\dim(R) \geqslant 1$, then $\mathcal{P}(R)$ is not a cogenerator for $\mathcal{F}(R)$.*

Example 5.11. Let $(R, \mathfrak{m})$ be a noetherian local ring. If R is not $\mathfrak{m}$-adically complete, then $\mathcal{G}(\mathcal{F}'(R)) \nsubseteq \mathcal{G}(\mathcal{P}(R))$. The $\mathfrak{m}$-adic completion $\widehat{R}$ is flat and cotorsion; see, for example, [**9**, Theorem 5.3.28]. Arguing as in Example 5.9, it then suffices to note that $\widehat{R}$ is not projective by [**11**, Theorem A].

Example 5.12. Assume that R is local and noetherian. Using Example 5.3 and [**12**, Proposition 2.8], it is straightforward to show that $\mathcal{P}^{\mathrm{f}}(R)$ is an injective cogenerator and a projective generator for $\mathcal{G}'(R)$ and that $\mathcal{G}'(R)$ is closed under extensions. Theorem 4.9 now implies that

$$\mathcal{G}^n(\mathcal{G}'(R)) \subseteq \mathcal{G}(R) \quad \text{for each } n \geqslant 1.$$

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