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# Characterizing local rings via homological dimensions and regular sequences

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**Abstract**

Let $(R, \mathfrak{m})$ be a Noetherian local ring of depth $d$ and $C$ a semidualizing $R$-complex. Let $M$ be a finite $R$-module and $t$ an integer between 0 and $d$. If the $G_C$-dimension of $M/\mathfrak{a}M$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d - t$ then either the $G_C$-dimension of $M$ is at most $t$ or $C$ is a dualizing complex. Analogous results for other homological dimensions are also given.

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## 0. Introduction

Throughout this work, $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring with identity and all modules are unitary.

The characterization of a local ring by the homological properties of its finite modules begins with the classical theorem of Auslander, Buchsbaum, and Serre; cf., [6, (2.2.7)]. Here pd and id denote projective and injective dimensions, respectively.

**Theorem A.** *The following conditions are equivalent:*

(i) *$R$ is regular;*
(ii) $\mathrm{pd}_R k < \infty$*;*
(iii) $\mathrm{pd}_R M < \infty$ *for all finite $R$-modules $M$;*
(iv) $\mathrm{id}_R k < \infty$*;*
(v) $\mathrm{id}_R M < \infty$ *for all finite $R$-modules $M$.*

Other criteria for the regularity of $R$ have been given since the appearance of this result. For example, Foxby [14] offers the following.

**Theorem A1.** *If $M$ is a finite $R$-module such that $\mathrm{pd}_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ of finite projective dimension, then $M$ is free or $R$ is regular.*

Jothilingam and Mangayarcarassy [22] are responsible for the next result in this style. They prove that the conclusion of Foxby's result holds if the hypothesis is satisfied by ideals generated by regular sequences, the ideal generated by the empty sequences being the zero ideal. This result provides a central motivation for our work in this paper.

**Theorem A2.** *If $M$ is a finite $R$-module such that $\mathrm{pd}_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by $R$-regular sequences, then $M$ is free or $R$ is regular.*

Beginning in the late 1960's, several homological dimensions have appeared that can be used to detect ring-theoretic properties of $R$. The Gorenstein dimension for finite modules was introduced by Auslander [1] and developed by Auslander and Bridger [2]; see also the monograph of Christensen [7]. This was extended to the $\mathrm{G}_C$-dimension by Foxby [13] and Golod [19], and was studied extensively by Christensen [8] and Gerko [18]. Other homological dimensions include Avramov's virtual projective dimension [4], the complete intersection dimension of Avramov, Gasharov, and Peeva [3], Veliche's upper G-dimension [26], the Gorenstein injective dimension of Enochs and Jenda [10,11], and Gerko's lower complete intersection dimension and Cohen–Macaulay dimension [18]. See Section 1 for definitions.

Each of these homological dimensions exhibits a theorem characterizing the appropriate ring-theoretic property as in Theorem A. The goal of this paper is to give analogues of Theorem A for each one. This is done in Section 2 after the relevant background is given in Section 1.

## 1. Homological dimensions

We shall employ a small amount of technology from the derived category of $R$-modules $\mathcal{D}(R)$. We refer the reader to [20,27], or [16] for the appropriate background.

A complex of $R$-modules $X$ is a sequence of $R$-module homomorphisms

$$\cdots \xrightarrow{\partial_{i+1}} X_i \xrightarrow{\partial_i} X_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_{i-1}\partial_i = 0$ for each $i \in \mathbb{Z}$. The symbol $\simeq$ denotes an isomorphism in $\mathcal{D}(R)$. The $i$th homology of $X$ is denoted $\mathrm{H}_i(X)$. The complex $X$ is *homologically bounded* if $\mathrm{H}_i(X) = 0$ for almost all integers $i$; it is *homologically finite* if it is homologically bounded and each $\mathrm{H}_i(X)$ is a finite $R$-module. The *infimum of* $X$ is

$$\inf(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\};$$

this is finite when $X \not\simeq 0$ is homologically bounded and is $\infty$ for $X \simeq 0$.

For complexes $X, Y$ the derived tensor product of $X$ and $Y$ is denoted $X \otimes^{\mathbf{L}}_R Y$ while $\mathbf{R}\mathrm{Hom}_R(X, Y)$ denotes the derived homomorphisms from $X$ to $Y$.

A homologically finite complex of $R$-modules $C$ is *semidualizing* if the natural homothety morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. When $C$ is a finite $R$-module, it is semidualizing if and only if the natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for each integer $i \neq 0$. The $R$-modules $R$ is always semidualizing. A semidualizing complex is *dualizing* if and only if it has finite injective dimension.

Semidualizing modules are studied extensively in [13,18,19] where they are called "suitable" modules. Semidualizing complexes are investigated in [8], whence comes our treatment of the $\mathrm{G}_C$-dimension, as well as in [15,17].

Let $C$ be a semidualizing $R$-complex. A homologically finite complex of $R$-modules $X$ is *$C$-reflexive* if $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded and the natural biduality morphism $X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ is an isomorphism in $\mathcal{D}(R)$. The *$\mathrm{G}_C$-dimension of* $X$ is

$$\mathrm{G}_C\text{-}\dim_R(X) = \begin{cases} \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) & \text{when } X \text{ is } C\text{-reflexive} \\ \infty & \text{otherwise.} \end{cases}$$

Note that this definition provides $\mathrm{G}_C\text{-}\dim_R(0) = -\infty$. When $C$ and $X$ are both modules, we have the following alternate description of the $\mathrm{G}_C$-dimension of $X$ in terms of resolutions by appropriate modules.

The *$G_C$-class*, denoted $\mathrm{G}_C(R)$, is the collection of finite $R$-modules $M$ such that

(1) $\mathrm{Ext}^{\ell}_R(M, C) = 0$ for all $\ell > 0$;
(2) $\mathrm{Ext}^{\ell}_R(\mathrm{Hom}_R(M, C), C) = 0$ for $\ell > 0$; and
(3) the biduality morphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.

The $\mathrm{G}_C$-dimension of a nonzero finite $R$-module $X$ is then the infimum of the set of all nonnegative integers $r$ such that there exists an exact sequence

$$0 \to M_r \to M_{r-1} \to \cdots \to M_0 \to X \to 0$$

with each $M_i$ in $G_C(R)$. In particular, $\mathrm{G}_C\text{-}\dim_R(M) \geq 0$.

When $C = R$ one writes G-$\dim_R(X)$ in place of $\mathrm{G}_R\text{-}\dim_R(X)$. When $X$ is a nonzero finite $R$-module, this is the G-dimension of [1,2]; in general, it is the G-dimension of [7,28].

For each semidualizing complex, the $\mathrm{G}_C$-dimension is a refinement of the projective dimension and satisfies an analogue of Theorem A. Furthermore, a finite $R$-module of finite $\mathrm{G}_C$-dimension satisfies an "AB formula"; see [8, (3.14),(3.15),(8.4)].

**Theorem B.** *Let $C$ be a semidualizing $R$-complex and $N$ a finite $R$-module.*

(1) *There is an inequality* $\mathrm{G}_C\text{-}\dim_R(N) \le \mathrm{pd}_R(N)$ *with equality when* $\mathrm{pd}_R(N)$ *is finite.*
(2) *If* $\mathrm{G}_C\text{-}\dim_R(N)$ *is finite, then* $\mathrm{G}_C\text{-}\dim_R(N) = \mathrm{depth}(R) - \mathrm{depth}_R(N)$.
(3) *The following conditions are equivalent:*
 (i) *$C$ is dualizing;*
 (ii) $\mathrm{G}_C\text{-}\dim_R k < \infty$*;*
 (iii) $\mathrm{G}_C\text{-}\dim_R M < \infty$ *for all finite $R$-modules $M$.*

Some important ring-theoretic properties are implied by the existence and behavior of dualizing modules.

**Theorem C.** *If $R$ possesses a dualizing module, then $R$ is Cohen–Macaulay. If the $R$-module $R$ is dualizing, then $R$ is Gorenstein.*

We shall make use of the following properties of the $\mathrm{G}_C$-dimension contained in [8, (5.10),(6.5)].

**Proposition D.** *Let $C$ be a semidualizing complex, $M$ a finite $R$-module and $x \in \mathfrak{m}$ an $R$-regular element. Set $R' = R/xR$.*

(1) *The $R'$-complex $C' = C \otimes^{\mathbf{L}}_R R'$ is semidualizing.*
(2) *If $x$ is $M$-regular and* $\mathrm{G}_C\text{-}\dim_R(M) < \infty$*, then* $\mathrm{G}_{C'}\text{-}\dim_{R'}(M/xM) = \mathrm{G}_C\text{-}\dim_R(M)$.

The following result is well-known for the G-dimension. We do not know of a reference for it in this generality, so we include a proof here.

**Lemma 1.** *Let $C$ be a semidualizing $R$-complex and $X \to Y \to Z \to \Sigma X$ a distinguished triangle in $\mathcal{D}(R)$. If two of the complexes $X, Y, Z$ are $C$-reflexive, then so is the third.*

It follows that, given an exact sequence of finite $R$-modules

$$0 \to M_r \to M_{r-1} \to \cdots \to M_1 \to 0$$

with $\mathrm{G}_C\text{-}\dim_R(M_i) < \infty$ for $i \neq j$, one also has $\mathrm{G}_C\text{-}\dim_R(M_j) < \infty$. Furthermore, it is straightforward to show that $\mathrm{G}_C\text{-}\dim_R(M_1 \oplus M_2) < \infty$ implies $\mathrm{G}_C\text{-}\dim_R(M_i) < \infty$ for $i = 1, 2$.

**Proof.** Since $R$ is Noetherian, an analysis of the long exact sequence coming from the distinguished triangle $X \to Y \to Z \to \Sigma X$ shows that, if two of the complexes $X, Y, Z$ are homologically finite, so is the third. Similarly for the distinguished triangle

$\mathbf{R}\mathrm{Hom}_R(Z, C) \to \mathbf{R}\mathrm{Hom}_R(Y, C) \to \mathbf{R}\mathrm{Hom}_R(X, C) \to \Sigma\mathbf{R}\mathrm{Hom}_R(Z, C)$. The naturality of the biduality morphism gives rise to a commuting diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C) \arrow[d] \\
Y \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Y, C), C) \arrow[d] \\
Z \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Z, C), C) \arrow[d] \\
\Sigma X \arrow[r] & \Sigma\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)
\end{tikzcd}$$

where each column is a distinguished triangle. Thus, if the biduality morphism is an isomorphism for two of the complexes $X$, $Y$, $Z$, then so is the third. □

Three homological dimensions of note have been introduced that characterize the complete intersection property like projective dimension does for regularity: virtual projective dimension, complete intersection dimension, and lower complete intersection dimension. Here we consider the last of these, as it is simultaneously the least restrictive and most flexible of the three. The interested reader is encouraged to consult the original sources [4,3] for information on the other two. The original treatment of lower complete intersection dimension is in [18]. The (equivalent and slightly simpler) description we give here is from Sather-Wagstaff [25].

Quite simply, a finite $R$-module $M$ has finite lower complete intersection dimension, denoted $\mathrm{CI}_*\text{-}\dim_R(M) < \infty$, if it has finite G-dimension and the Betti numbers $\beta_R^n(M)$ are bounded above by a polynomial in $n$. When $\mathrm{CI}_*\text{-}\dim_R(M) < \infty$ we set $\mathrm{CI}_*\text{-}\dim_R(M) = \text{G-}\dim_R(M)$. Basic properties of the $\mathrm{CI}_*$-dimension are taken from [18] and summarized in the following.

**Theorem E.** *Let $M_1, \ldots, M_r$ be finite $R$-modules.*

(1) *The following conditions on $R$ are equivalent.*
   (i) *$R$ is complete intersection;*
   (ii) $\mathrm{CI}_*\text{-}\dim_R k < \infty$*;*
   (iii) $\mathrm{CI}_*\text{-}\dim_R M < \infty$ *for all finite $R$-modules $M$.*

(2) *Given an exact sequence*

$$0 \to M_r \to M_{r-1} \to \cdots \to M_1 \to 0$$

*with* $\mathrm{CI}_*\text{-}\dim_R(M_i) < \infty$ *for $i \neq j$, one also has* $\mathrm{CI}_*\text{-}\dim_R(M_j) < \infty$.

(3) *If* $\mathrm{CI}_*\text{-}\dim_R(M_1 \oplus M_2) < \infty$*, then* $\mathrm{CI}_*\text{-}\dim_R(M_i) < \infty$ *for $i = 1, 2$.*

(4) *When $x \in \mathfrak{m}$ is $R$- and $M_1$-regular, set $R' = R/xR$ and $M_1' = M_1/xM_1$. If* $\mathrm{CI}_*\text{-}\dim_R(M_1) < \infty$*, then* $\mathrm{CI}_*\text{-}\dim_{R'}(M_1') = \mathrm{CI}_*\text{-}\dim_R(M_1)$.

The final homological dimension we shall employ is the Gorenstein injective dimension of Enochs and Jenda [10,11].

An $R$-module $L$ is *Gorenstein injective* if there exists an exact complex of injective $R$-modules $I$ such that the complex $\mathrm{Hom}_R(I, J)$ is exact for each injective $R$-module $J$ and $L \cong \mathrm{Coker}(\partial_1^I)$. A *Gorenstein injective resolution* of a module $M$ is a complex of Gorenstein injective modules $L$ with $L_i = 0$ for all $i > 0$, $\mathrm{H}_i(L) = 0$ for all $i \neq 0$, and $\mathrm{H}_0(L) \cong M$. The *Gorenstein injective dimension* of $M$ is

$$\mathrm{Gid}_R(M) = \inf\{\sup\{n \in \mathbb{Z} \mid L_n \neq 0\} \mid L \text{ a Gorenstein injective resolution of } M\}.$$

The finiteness of Gorenstein injective dimension characterizes Gorenstein rings like the finiteness of injective dimension does for regular rings; see [23, (2.7)].

**Theorem F.** *The following conditions are equivalent:*

(i) *$R$ is Gorenstein;*
(ii) $\mathrm{Gid}_R k < \infty$*;*
(iii) $\mathrm{Gid}_R M < \infty$ *for all finite $R$-modules $M$.*

The behavior of Gorenstein injective dimension with respect to exact sequences is described in the work of Holm [21].

**Proposition G.** *Let $M_1, \ldots, M_r$ be finite $R$-modules.*

(1) *Given an exact sequence*

$$0 \to M_r \to M_{r-1} \to \cdots \to M_1 \to 0$$

*with* $\mathrm{Gid}_R(M_i) < \infty$ *for $i \neq j$, one also has* $\mathrm{Gid}_R(M_j) < \infty$.
(2) *If* $\mathrm{Gid}_R(M_1 \oplus M_2) < \infty$*, then* $\mathrm{Gid}_R(M_i) < \infty$ *for $i = 1, 2$.*

Gorenstein injective dimension also behaves well with respect to killing a regular element. When $R$ possesses a dualizing complex, this is contained in the work of Christensen, Frankild, and Holm [9, (5.5)].

**Lemma 2.** *Let $M$ be a finite $R$-module and $x \in \mathfrak{m}$ an $R$- and $M$-regular element. Set $R' = R/xR$ and $M' = M/xM$. If* $\mathrm{Gid}_R(M) < \infty$*, then* $\mathrm{Gid}_{R'}(M') < \infty$.

**Proof.** For a Gorenstein injective module $L$, the $R'$-module $\mathrm{Hom}_R(R', L)$ is Gorenstein injective by [12, (3.1)]. We argue as in [9, (3.10)]. Let $L$ be a bounded Gorenstein injective resolution of $M$, and $I$ an injective resolution of $M$ that is bounded above. Using [5, (1.1.I),(1.4.I)] there is a quasi-isomorphism $L \xrightarrow{\simeq} I$ From [9, (3.10)], the induced homomorphism $\mathrm{Hom}_R(R', L) \to \mathrm{Hom}_R(R', I)$ is a quasi-isomorphism as well. This morphism preserves the $R'$-structures, and $\mathrm{Hom}_R(R', L)$ is a bounded complex of Gorenstein injective modules. Thus, the isomorphism $\mathbf{R}\mathrm{Hom}_R(R', M) \simeq \mathbf{R}\mathrm{Hom}_R(R', L)$ in $\mathcal{D}(R)$ is also an isomorphism in $\mathcal{D}(R')$ and $\mathrm{Gid}_{R'}(\mathbf{R}\mathrm{Hom}_R(R', M)) < \infty$. Since $x$ is $M$-regular, there are $R'$-isomorphisms

$$M' \simeq R' \otimes^{\mathbf{L}}_R M \simeq \Sigma \mathbf{R}\mathrm{Hom}_R(R', M)$$

and it follows that $\mathrm{Gid}_{R'}(M') < \infty$. $\square$

## 2. Characterizations of rings

The main result of this paper is the following.

**Theorem 3.** *Let $C$ be a semidualizing $R$-complex and $M$ a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between 0 and $d$. If $\mathrm{G}_C\text{-}\dim_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d-t$ then either $\mathrm{G}_C\text{-}\dim_R(M) \leq t$ or $C$ is a dualizing complex.*

**Proof.** Assume without loss of generality that $M$ is nonzero and argue by induction on $d$. If $d = 0$, then $t = 0$ and $\mathrm{G}_C\text{-}\dim_R M = 0$ by the AB-formula and we are done. Similar reasoning allows us to assume that $t < d$ for the rest of the proof.

Next, consider the case $d \geq 1$ and $\operatorname{depth}_R M = 0$. Fix an $R$-regular element $x$ such that $x \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p}\in S} \mathfrak{p}$, where $S = \operatorname{Ass} M \setminus \{\mathfrak{m}\}$. Since $M$ is Noetherian there exists an integer $\alpha$ such that $0 :_M x^\alpha = 0 :_M x^{\alpha+i}$ for all $i \geq 0$. The assumption $\operatorname{depth}_R M = 0$ yields $0 :_M \mathfrak{m} \neq 0$ and hence $0 :_M x^\alpha \neq 0$. Set $N = 0 :_M x^\alpha$ and use the exact sequence

$$0 \to N \to M \xrightarrow{x^\alpha} M \to M/x^\alpha M \to 0$$

to deduce that $\mathrm{G}_C\text{-}\dim_R N < \infty$.

The definition of $N$ implies that every associated prime of $N$ contains $x$. On the other hand, $x$ is not in any prime in $S$, so the containment $\operatorname{Ass} N \subseteq \operatorname{Ass} M$ gives $\operatorname{Ass} N = \{\mathfrak{m}\}$. Thus, there exists an integer $n \geq 1$ such that $\mathfrak{m}^n N = 0$ and $\mathfrak{m}^{n-1} N \neq 0$. If $n = 1$ then $k$ is a direct summand of $N$ so that $\mathrm{G}_C\text{-}\dim_R k < \infty$ and $C$ is a dualizing complex. If $n > 1$ we can assume that $\mathfrak{m}^n N = 0$ and $\mathfrak{m}^{n-1} N \neq 0$. It is straightforward to verify that $N = \mathrm{H}^0_{\mathfrak{m}}(M)$, and therefore $\operatorname{depth}_R(M/N) > 0$. Fix $y \in \mathfrak{m}^{n-1} \setminus ((\bigcup_{\mathfrak{p}\in S} \mathfrak{p}) \cup (\bigcup_{\mathfrak{p}\in \operatorname{Ass}(M/N)} \mathfrak{p}) \cup \operatorname{Ann} N)$. It follows that $yN \neq 0$ and $y\mathfrak{m}N = 0$. Let $\beta$ be an integer such that $0 :_M y^\beta = 0 :_M y^{\beta+i}$ for all $i \geq 0$. Arguing as above yields $0 :_M y^\beta = \mathrm{H}^0_{\mathfrak{m}}(M) = N$.

Applying the functor $R/y \otimes_R (-)$ to the exact sequence

$$0 \to N \xrightarrow{\varphi} M \to M/N \to 0 \tag{*}$$

induces the sequence

$$0 \to R/y \otimes_R N \xrightarrow{\overline{\varphi}} R/y \otimes_R M \to R/y \otimes_R (M/N) \to 0 \tag{**}$$

which is exact because $\ker(\overline{\varphi}) = \operatorname{Tor}_1^R(R/y, M/N) = 0$; cf., [24, (16.5.i)]. From (*) we have $\mathrm{G}_C\text{-}\dim_R(M/N) < \infty$ and hence the exact sequence

$$0 \to M/N \xrightarrow{\cdot y} M/N \to R/y \otimes_R (M/N) \to 0$$

yields $\mathrm{G}_C\text{-}\dim_R(R/y \otimes_R (M/N)) < \infty$. Furthermore, $\mathrm{G}_C\text{-}\dim_R(R/y \otimes_R M)$ is finite and so (**) gives $\mathrm{G}_C\text{-}\dim_R(R/y \otimes_R N) < \infty$. Similarly, the exact sequence

$$0 \to yN \to N \to N/yN \to 0$$

yields $\mathrm{G}_C\text{-}\dim_R(yN) < \infty$. Since $\mathfrak{m}(yN) = 0$, the $R$-module $k$ is a direct summand of $yN$ and so $\mathrm{G}_C\text{-}\dim_R(k) < \infty$. Therefore $C$ is dualizing.

Now suppose that depth $R \geq 1$ and $\operatorname{depth}_R M \geq 1$. Fix an element $z \in \mathfrak{m}$ that is both $R$- and $M$-regular, and set $\bar{R} = R/xR$ and $\bar{M} = M/xM$. The $\bar{R}$-complex $\bar{C} = C \otimes^{\mathbf{L}}_R \bar{R}$ is semidualizing, and it follows easily from Proposition D that $\mathrm{G}_{\bar{C}}\text{-}\dim_{\bar{R}}(\bar{M}/\mathfrak{b}\bar{M})$ is finite for each ideal $\mathfrak{b}$ generated by an $\bar{R}$-regular sequence of length at most $d - 1 - t$. Thus, by induction, either $\bar{C}$ is dualizing for $\bar{R}$ or $\mathrm{G}_{\bar{C}}\text{-}\dim_{\bar{R}}(\bar{M}) \leq t$. If $\bar{C}$ is dualizing for $\bar{R}$, then $C$ is dualizing for $R$. If $\mathrm{G}_{\bar{C}}\text{-}\dim_{\bar{R}}(\bar{M}) \leq t$, then $\mathrm{G}_C\text{-}\dim_R(M) \leq t$. $\square$

Applying Theorem 3 to a semidualizing *module* yields a criterion for the Cohen–Macaulay property that is parallel to Theorem A2.

**Corollary 4.** *Let $C$ be a semidualizing $R$-module and $M$ a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\mathrm{G}_C\text{-}\dim_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d-t$ then either $\mathrm{G}_C\text{-}\dim_R(M) \leq t$ or $R$ is Cohen–Macaulay.*

Using $C = R$ gives a criterion for the Gorenstein property. A similar result in terms of upper G-dimension follows immediately from this one.

**Corollary 5.** *Let $M$ be a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\text{G-}\dim_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d - t$ then either $\text{G-}\dim_R(M) \leq t$ or $R$ is Gorenstein.*

Modifying the proof of Theorem 3 appropriately, one obtains analogous results using virtual projective dimension, complete intersection dimension, or lower complete intersection dimension to describe complete intersection rings. We state here the version for lower complete intersection dimension, as the others can be derived immediately from it.

**Corollary 6.** *Let $M$ be a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\mathrm{CI}_*\text{-}\dim_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d - t$ then either $\mathrm{CI}_*\text{-}\dim_R(M) \leq t$ or $R$ is complete intersection.*

A similar modification of the proof of Theorem 3 produces a criterion for the regularity property that generalizes Theorem A2.

**Corollary 7.** *Let $M$ be a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\operatorname{pd}_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d - t$ then either $\operatorname{pd}_R(M) \leq t$ or $R$ is regular.*

The analogous statements for injective dimension and Gorenstein injective dimension differ slightly from the previous results because of the Bass formula. The proof is the same, though, modulo easy arguments to deal with the first cases.

**Corollary 8.** *Let $M$ be a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\operatorname{id}_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d - t$ then either $\operatorname{depth}_R M \geq d - t$ or $R$ is regular.*

**Corollary 9.** *Let $M$ be a finite $R$-module. Set $d = \operatorname{depth}(R)$ and fix an integer $t$ between $0$ and $d$. If $\operatorname{Gid}_R(M/\mathfrak{a}M)$ is finite for all ideals $\mathfrak{a}$ generated by an $R$-regular sequence of length at most $d-t$ then either $\operatorname{depth}_R M \geq d-t$ or $R$ is Gorenstein.*

Our final variation on this theme has a similar proof, but considers ideals generated by parts of a system of parameters instead of regular sequences.

**Corollary 10.** *Let $C$ be a semidualizing $R$-module. The following conditions on $R$ are equivalent.*

(i) *$R$ is Cohen–Macaulay.*
(ii) *There exists a finite $R$-module $M$ such that for every ideal $\mathfrak{a}$ generated by part of a system of parameters for $R$, one has $\mathrm{G}_C\text{-}\dim \dim_R(M/\mathfrak{a}M)$ finite.*
(iii) *For every ideal $\mathfrak{a}$ generated by part of a system of parameters for $R$, one has $\mathrm{G}_C\text{-}\dim \dim_R(M/\mathfrak{a}M)$ finite.*

It would be interesting to know whether the parallel result using Cohen–Macaulay dimension to characterize Cohen–Macaulay rings holds. The only obstruction is the current lack of understanding of the behavior of this homological dimension with respect to exact sequences.

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