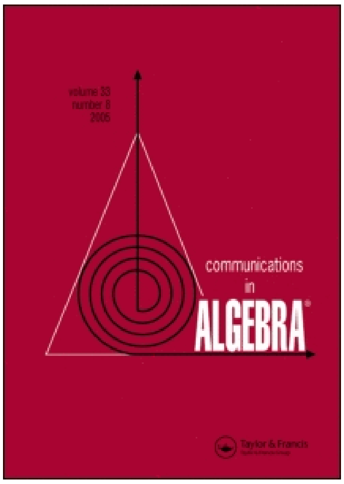


Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Rings that are Homologically of Minimal Multiplicity

Keivan Borna[a]; Sean Sather-Wagstaff[b]; Siamak Yassemi[c]
[a] Faculty of Mathematical Sciences and Computer, Tarbiat Moallem University, Tehran, Iran [b] Department of Mathematics, North Dakota State University, Fargo, North Dakota, USA [c] Department of Mathematics, University of Tehran, Tehran, Iran

Online publication date: 16 March 2011

To cite this Article Borna, Keivan , Sather-Wagstaff, Sean and Yassemi, Siamak(2011) 'Rings that are Homologically of Minimal Multiplicity', Communications in Algebra, 39: 3, 782 — 807
To link to this Article: DOI: 10.1080/00927871003596214
URL: http://dx.doi.org/10.1080/00927871003596214

Communications in Algebra®, 39: 782–807, 2011

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927871003596214

RINGS THAT ARE HOMOLOGICALLY OF MINIMAL MULTIPLICITY

Keivan Borna[1], Sean Sather-Wagstaff[2], and Siamak Yassemi[3]
[1]*Faculty of Mathematical Sciences and Computer, Tarbiat Moallem University, Tehran, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran*
[2]*Department of Mathematics, North Dakota State University, Fargo, North Dakota, USA*
[3]*Department of Mathematics, University of Tehran, Tehran, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran*

Let R be a local Cohen–Macaulay ring with canonical module ω_R. We investigate the following question of Huneke: If the sequence of Betti numbers $\{\beta_i^R(\omega_R)\}$ has polynomial growth, must R be Gorenstein? This question is well understood when R has minimal multiplicity. We investigate this question for a more general class of rings which we say are homologically of minimal multiplicity. We provide several characterizations of the rings in this class and establish a general ascent and descent result.

Key Words: Betti numbers; Canonical module; Gorenstein rings; Minimal multiplicity.

2000 Mathematics Subject Classification: 13D07; 13D02; 13H10.

1. INTRODUCTION

Throughout this article $(R, \mathfrak{m}, k)$ is a commutative local noetherian ring. Recall that a finitely generated R-module ω_R is a *canonical module* for R if

$$\mathrm{Ext}_R^i(k, \omega_R) \cong \begin{cases} k & \text{if } i = \dim(R) \\ 0 & \text{if } i \neq \dim(R). \end{cases}$$

In some of the literature, canonical modules are also called dualizing modules. They were introduced by Grothendieck [13] for the study of local cohomology. Foxby [9], Reiten [17], and Sharp [18] prove that R admits a canonical module if and only if R is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring. In particular, if R is complete and Cohen–Macaulay, then it admits a canonical module.

Received September 22, 2009; Revised December 9, 2009. Communicated by I. Swanson.
Address correspondence to Prof. Sean Sather-Wagstaff, Department of Mathematics, NDSU Dept. # 2750, P.O. Box 6050, Fargo, ND 58108-6050, USA; E-mail: Sean.Sather-Wagstaff@ndsu.edu

One useful property is the following: The ring R is Gorenstein if and only if R is its own canonical module. This leads to the following question of Huneke.[1]

Question 1.1. Assume that R is Cohen–Macaulay with canonical module ω_R. If the sequence of Betti numbers $\{\beta_i^R(\omega_R)\}$ is bounded above by a polynomial in i, must R be Gorenstein?

For rings of minimal multiplicity, it is straightforward to answer this question. Reduce to the case where $\mathfrak{m}^2 = 0$ and show that $\beta_i^R(\omega_R) = (r^2 - 1)r^{i-1}$ for all $i \geq 1$; here r is the Cohen–Macaulay type of R. (See Example 2.4 below.) This question has been answered in the affirmative for other classes of rings by Jorgensen and Leuschke [15] and Christensen et al. [8]. These classes include the classes of Golod rings, rings with codimension at most 3, rings that are one link from a complete intersection, rings with $\mathfrak{m}^3 = 0$, Teter rings, and nontrivial fiber product rings.

In this article, we investigate Question 1.1 for the following classes of rings which contain the rings of minimal multiplicity.

Definition 1.2. Let m, n and t be integers with $m, t \geq 1$ and $n \geq 0$. The ring R is *homologically of minimal multiplicity of type* (m, n, t) if there exists a local ring homomorphism $\varphi: (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $M \neq 0$ such that:

(1) The ring S has a canonical module ω_S;
(2) The map φ is flat with Gorenstein closed fibre $S/\mathfrak{m}S$;
(3) One has $\operatorname{Tor}_i^S(\omega_S, M) = 0$ for $i \geq t$; and
(4) One has $\mathfrak{n}^2 M = 0$ and $m = \beta_0^S(M)$ and $n = \beta_0^S(\mathfrak{n}M)$.

The ring R is *strongly homologically of minimal multiplicity of type* (m, n) if there exists a local ring homomorphism $\varphi: (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $M \neq 0$ satisfying conditions (1), (2), (4), and the following one:

(3′) The S-module M is in the Auslander class $\mathcal{A}(S)$.

(Consult Section 2 for background information on Auslander classes.)

The following facts are proved in Section 2. If R is Cohen–Macaulay and has minimal multiplicity, then it is strongly homologically of minimal multiplicity of type $(1, e(R) - 1)$; here $e(R)$ is the Hilbert–Samuel multiplicity of R with respect to $\mathfrak{m}$. If R is Gorenstein, then it is strongly homologically of minimal multiplicity of type (m, n) for all integers $m, n \geq 1$. Also, if R is homologically of minimal multiplicity, then it is Cohen–Macaulay.

We provide an affirmative answer to Question 1.1 for rings that are strongly homologically of minimal multiplicity in the following result, which is contained in Theorems 3.5 and 3.13.

Theorem 1.3. *Assume that R is homologically of minimal multiplicity of type (m, n, t) and with canonical module ω_R.*

[1]To the best of our knowledge, Huneke has only posed this question in conversations and talks, not in print.

(a) *One has* $\beta^R_{t+s}(\omega_R) = (n/m)^s \cdot \beta^R_t(\omega_R)$ *for all* $s \geq 0$.
(b) *If* $n > m$ *and R is not Gorenstein, then the sequence* $\{\beta^R_i(\omega_R)\}$ *grows exponentially.*
(c) *If* $n = m$, *then the sequence* $\{\beta^R_i(\omega_R)\}$ *is eventually constant.*
(d) *If* $n < m$, *then R is Gorenstein.*
(e) *If R is not Gorenstein, then* $m \mid n$.
(f) *If R is strongly homologically of minimal multiplicity of type* (m, n) *and* $n = m$, *then R is Gorenstein.*

Section 3 also contains further analysis of the behavior of the Betti numbers under various hypotheses. While this investigation is motivated by questions about the Betti numbers of canonical modules, our methods yield results about Betti numbers of arbitrary modules. For instance, Theorem 1.3(f) is essentially a special case of Theorem 3.10(b). Accordingly, we state and prove these more general results, and periodically give explicit specializations to the case of rings that are (strongly) homologically of minimal multiplicity.

Section 4 contains three alternate characterizations of the rings that are homologically of minimal multiplicity. One of them, Theorem 4.5, states that, if R is homologically of minimal multiplicity, then one can assume in Definition 1.2 that the homomorphism φ is flat with regular closed fibre and that the ring S is complete with algebraically closed residue field. The second one, Theorem 4.9, shows that R is homologically of minimal multiplicity whenever there is a "quasi-Gorenstein" homomorphism $R \to S$ satisfying conditions (1), (3), and (4) of Definition 1.2. (Definition 4.7 contains background information on quasi-Gorenstein homomorphisms.) The third characterization is dual to the original definition, using Ext-vanishing in place of Tor-vanishing; see Remark 3.3 and Proposition 4.11. Similar characterizations are given for rings that are strongly homologically of minimal multiplicity.

Finally, Section 5 documents ascent and descent behavior for these classes of rings. The most general statements are contained in Corollaries 5.15 and 5.16. The result for flat maps is given here; see Theorems 5.7 and 5.8.

Theorem 1.4. *Assume that* $\psi\colon R \to R'$ *is a flat local ring homomorphism with Gorenstein closed fibre* $R'/\mathfrak{m}R'$. *If* R' *is (strongly) homologically of minimal multiplicity, then so is R. The converse holds when k is perfect and* $R'/\mathfrak{m}R'$ *is regular.*

Example 5.14 shows that the converse statement can fail when $R'/\mathfrak{m}R'$ is only assumed to be of minimal multiplicity. It also shows that, in general, the localized tensor product of rings that are strongly homologically of minimal multiplicity need not be homologically of minimal multiplicity. On the other hand, we do not know at this time whether this class of rings is closed under localization. See Section 5 for other open problems.

2. BASIC PROPERTIES

In this section we make some observations about rings that are (strongly) homologically of minimal multiplicity. We begin with a definition that is due to Foxby.

Definition 2.1. Let S be a Cohen–Macaulay local ring with canonical module ω_S. The *Auslander class* of S is the class $\mathscr{A}(S)$ consisting of all R-modules M satisfying the following conditions:

(1) The natural map $\xi_M\colon M \to \mathrm{Hom}_S(\omega_S, \omega_S \otimes_S M)$ given by $\xi_M(m)(x) = x \otimes m$ is an isomorphism; and
(2) One has $\mathrm{Tor}_i^S(\omega_S, M) = 0 = \mathrm{Ext}_S^i(\omega_S, \omega_S \otimes_S M)$ for all $i \geq 1$.

The *Bass class* of S is the class $\mathscr{B}(S)$ consisting of all R-modules M satisfying the following conditions:

(1) The natural map $\gamma_M\colon \omega_S \otimes_S \mathrm{Hom}_S(\omega_S, M) \to M$ given by $\gamma_M(x \otimes \psi) = \psi(x)$ is an isomorphism; and
(2) One has $\mathrm{Ext}_S^i(\omega_S, M) = 0 = \mathrm{Tor}_i^S(\omega_S, \mathrm{Hom}_S(\omega_S, M))$ for all $i \geq 1$.

Here are some straightforward facts about Auslander classes.

Remark 2.2. Let S be a Cohen–Macaulay local ring with canonical module ω_S. The Auslander class $\mathscr{A}(S)$ contains every projective S-module. Furthermore, if two modules in a short exact sequence are in $\mathscr{A}(S)$, then so is the third module. It follows that $\mathscr{A}(S)$ contains every S-module of finite projective dimension.

From the definitions, we conclude that rings that are strongly homologically of minimal multiplicity of type (m, n) are homologically of minimal multiplicity of type $(m, n, 1)$. Also, with φ and M as in Definition 1.2, the condition $n \geq 1$ implies that $\mathfrak{n}M \neq 0$.

For the sake of clarity, we recall the definition of minimal multiplicity, first studied by Abhyankar [1].

Definition 2.3. Let $(R, \mathfrak{m})$ be a local ring. The *Hilbert–Samuel multiplicity* of R, denoted $e(R)$, is the normalized leading coefficient of the polynomial that agrees with the function $\mathrm{length}_R(R/\mathfrak{m}^n)$ for $n \gg 0$. If R is Cohen–Macaulay, then there is an inequality $e(R) \geq \beta_0^R(\mathfrak{m}) - \dim(R) + 1$, and R has *minimal multiplicity* when $e(R) = \beta_0^R(\mathfrak{m}) - \dim(R) + 1$.

Example 2.4. Let k be a field, let r be a positive integer, and consider the ring $R = k[X_1, \ldots, X_r]/(X_1, \ldots, X_r)^2$. This is a local artinian ring of minimal multiplicity, with multiplicity $e(R) = r + 1$ and type r. (In particular, R is Gorenstein if and only if $r = 1$.) Hence, the canonical module ω_R has $\beta_0^R(\omega_R) = r$. Furthermore, the exact sequence

$$0 \to k^{r^2-1} \to R^r \to \omega_R \to 0$$

(obtained by truncating a minimal free resolution of ω_R) can be used to show that $\beta_i^R(\omega_R) = (r^2 - 1)r^{i-1}$ for all $i \geq 1$.

We will have several opportunities to use the following fact from [12, 0.(10.3.1)].

Remark 2.5. Let $(R, \mathfrak{m}, k)$ be a local ring, and let φ_0: $k \to l$ be a field extension. Then there is a flat local ring homomorphism φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ such that S is complete, the extension $k \to l$ induced by φ is precisely φ_0, and $\mathfrak{n} = \mathfrak{m}S$.

The next three results explain the location of rings homologically of minimal multiplicity in the heierarchy of rings.

Proposition 2.6. *If R is a local Cohen–Macaulay ring with minimal multiplicity, then it is strongly homologically of minimal multiplicity of type* $(1, e(R) - 1)$.

Proof. Remark 2.5 provides a flat local ring homomorphism ψ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ such that S is complete, l is the algebraic closure of k, and $\mathfrak{n} = \mathfrak{m}S$. It follows readily that S is Cohen–Macaulay and has a canonical module ω_S. Furthermore, we have $e(S) = e(R)$ and $\beta_0^S(\mathfrak{n}) = \beta_0^R(\mathfrak{m})$ and $\dim(S) = \dim(R)$. In particular, the ring S has minimal multiplicity.

The fact that S is Cohen–Macaulay and has infinite residue field implies that there exists an S-regular sequence $\mathbf{x} \in \mathfrak{n}\backslash\mathfrak{n}^2$ such that $\text{length}_S(S/(\mathbf{x})S) = e(S)$. (The sequence $\mathbf{x}$ generates a minimal reduction of $\mathfrak{n}$.) This explains the second equality in the following sequence:

$$\begin{aligned}\beta_0^S(\mathfrak{n}) - \dim(S) + 1 &= e(S)\\ &= \text{length}_S(S/(\mathbf{x})S)\\ &= 1 + \beta_0^S(\mathfrak{n}/(\mathbf{x})S) + \text{length}_S(\mathfrak{n}^2(S/(\mathbf{x})S))\\ &= 1 + \beta_0^S(\mathfrak{n}) - \dim(S) + \text{length}_S(\mathfrak{n}^2(S/(\mathbf{x})S)).\end{aligned}$$

The first equality is from the minimal multiplicity condition. The third equality is explained by the filtration $\mathfrak{n}^2(S/(\mathbf{x})S) \subseteq \mathfrak{n}(S/(\mathbf{x})S) \subseteq S/(\mathbf{x})S$. The fourth equality is from the fact that $\mathbf{x}$ is a maximal S-regular sequence in $\mathfrak{n}\smallsetminus\mathfrak{n}^2$. From this sequence, it follows that $\mathfrak{n}^2(S/(\mathbf{x})S) = 0$. (See also the proof of [1, (1)].)

Since the sequence $\mathbf{x}$ is S-regular, the S-module $M = S/(\mathbf{x})S$ has finite projective dimension. Remark 2.2 then implies that $M \in \mathscr{A}(S)$. It follows that R is strongly homologically of minimal multiplicity of type (m, n) where $m = \beta_0^S(M) = 1$ and $n = \beta_0^S(\mathfrak{n}M) = e(R) - 1$. □

Proposition 2.7. *If R is a local Gorenstein ring, then it is strongly homologically of minimal multiplicity of type (m, n) for all integers $m \geq 1$ and $n \geq 0$.*

Proof. Fix integers $m \geq 1$ and $n \geq 0$. The ring $S = R[\![X_1, \ldots, X_n]\!]$ is local with maximal ideal $\mathfrak{n} = (\mathfrak{m}, X_1, \ldots, X_n)S$ and residue field k. The natural inclusion φ: $R \to S$ is flat with Gorenstein closed fibre $S/\mathfrak{m}S \cong k[\![X_1, \ldots, X_n]\!]$. Since R is Gorenstein, the same is true of S. Thus S has canonical module $\omega_S = S$. It follows readily from the definition that every S-module is in $\mathscr{A}(S)$. In particular, the S-module

$$M = k^{m-1} \oplus S/(\mathfrak{m}S + ((X_1, \ldots, X_n)S)^2)$$

is in $\mathscr{A}(S)$. It is straightforward to show that $\mathfrak{n}^2M = 0$ and $\beta_0^S(M) = m$. To complete the proof, use the isomorphisms

$$\begin{aligned}\mathfrak{n}M &\cong \mathfrak{n}S/(\mathfrak{m}S + ((X_1, \ldots, X_n)S)^2)\\ &\cong (X_1, \ldots, X_n)k[\![X_1, \ldots, X_n]\!]/((X_1, \ldots, X_n)k[\![X_1, \ldots, X_n]\!])^2\\ &\cong k^n\end{aligned}$$

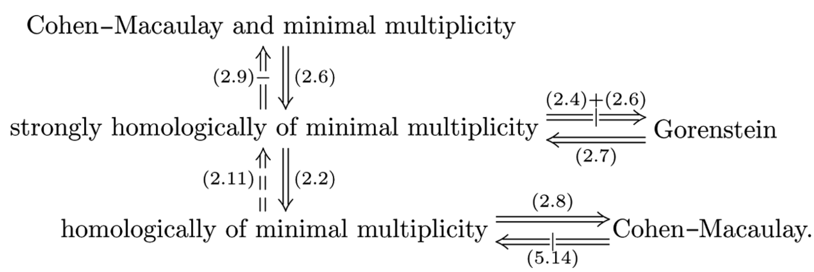
to see that $\beta_0^S(\mathfrak{n}M) = n$. □

Proposition 2.8. *If R is homologically of minimal multiplicity, then it is Cohen–Macaulay.*

Proof. Definition 1.2, the ring S has a canonical module, so it is Cohen–Macaulay. The homomorphism φ is flat and local, and it follows that R is Cohen–Macaulay. □

Remark 2.9. If R is strongly homologically of minimal multiplicity, then R need not have minimal multiplicity. To see this, let R be a local Gorenstein ring that is not of minimal multiplicity. (For example, it is straightforward to show that the ring $R = k[X]/(X^3)$ satisfies these conditions.) Proposition 2.7 shows that R is strongly homologically of minimal multiplicity.

Remark 2.10. The following diagram summarizes the relations between the classes of rings under consideration:



At this time we do not know whether the vertical implication marked (2.11) holds. We pose this explicitly as a question next.

Question 2.11. If R is homologically of minimal multiplicity, must it be strongly homologically of minimal multiplicity?

We end this section with a natural result to be used later.

Lemma 2.12. *Let $\mathfrak{a}$ be an ideal of R with $\mathfrak{a}$-adic completion $\widehat{R}^{\mathfrak{a}}$.*

(a) *Then R is homologically of minimal multiplicity of type (m, n, t) if and only if $\widehat{R}^{\mathfrak{a}}$ is homologically of minimal multiplicity of type (m, n, t).*

(b) *Then R is strongly homologically of minimal multiplicity of type (m, n) if and only if $\widehat{R}^{\mathfrak{a}}$ is strongly homologically of minimal multiplicity of type (m, n).*

Proof. We prove part (a), and the proof of part (b) is similar.

Assume that $\widehat{R}^{\mathfrak{a}}$ is homologically of minimal multiplicity of type (m, n, t). Let $\varphi_1\colon \widehat{R}^{\mathfrak{a}} \to S_1$ be a ring homomorphism, and let M_1 be an S_1-module as in Definition 1.2. It is straightforward to verify that the composition $\varphi_1\psi\colon R \to S_1$ and the S_1-module M_1 satisfy the axioms to show that R is homologically of minimal multiplicity of type (m, n, t).

Assume next that R is homologically of minimal multiplicity of type (m, n, t). Let $\varphi_2\colon R \to S_2$ be a ring homomorphism, and let M_2 be an S_2-module as in Definition 1.2. Then the induced map $\widehat{\varphi_2}^{\mathfrak{a}}\colon \widehat{R}^{\mathfrak{a}} \to \widehat{S_2}^{\mathfrak{a}}$ and the $\widehat{S_2}^{\mathfrak{a}}$ module $M_2 \cong \widehat{M_2}^{\mathfrak{a}}$ show that $\widehat{R}^{\mathfrak{a}}$ is homologically of minimal multiplicity of type (m, n, t). □

3. PATTERNS IN BETTI NUMBERS

This section contains the proof of Theorem 1.3 from the introduction.

Theorem 3.1. *Let $(S, \mathfrak{n}, l)$ be a local ring, and let M and N be finitely generated S-modules. Let m, n, and t be integers, and assume that there is an exact sequence of S-module homomorphisms $0 \to l^n \to M \to l^m \to 0$.*

(a) *If $\operatorname{Tor}_i^S(N, M) = 0$ for $i = t, t+1$, then $m\beta_{t+1}^S(N) = n\beta_t^S(N)$.*
(b) *If $m \geq 1$ and $\operatorname{Tor}_i^S(N, M) = 0$ for $i = t, \ldots, t+s$ for some positive integer s, then $\beta_{t+s}^S(N) = (n/m)^s \cdot \beta_t^S(N)$.*

Proof. For each integer i, we have $\operatorname{Tor}_i^S(N, l^m) \cong l^{m\beta_i^S(N)}$ and $\operatorname{Tor}_i^S(N, l^n) \cong l^{n\beta_i^S(N)}$. Thus, a piece of the long exact sequence in $\operatorname{Tor}^S(N, -)$ associated to the given sequence has the form

$$\operatorname{Tor}_{i+1}^S(N, M) \to l^{m\beta_{i+1}^S(N)} \to l^{n\beta_i^S(N)} \to \operatorname{Tor}_i^S(N, M).$$

If $\operatorname{Tor}_t^S(N, M) = 0 = \operatorname{Tor}_{t+1}^S(N, M)$, then the sequence yields an isomorphism $l^{m\beta_{t+1}^S(N)} \cong l^{n\beta_t^S(N)}$ and hence the equality $m\beta_{t+1}^S(N) = n\beta_t^S(N)$. Recursively, if $\operatorname{Tor}_i^S(N, M) = 0$ for $i = t, \ldots, t+s$, then $\beta_{t+s}^S(N) = (n/m)^s \cdot \beta_t^S(N)$. □

The next result is dual to the previous one. It can be proved using the ideas from Theorem 3.1 with $\operatorname{Ext}_S^i(N, -)$ in place of $\operatorname{Tor}_i^S(N, -)$. See also Remark 3.3.

Theorem 3.2. *Let $(S, \mathfrak{n}, l)$ be a local ring, and let M and N be finitely generated S-modules. Let m, n, and t be integers, and assume that there is an exact sequence of S-module homomorphisms $0 \to l^m \to M \to l^n \to 0$.*

(a) *If $\operatorname{Ext}_S^i(N, M) = 0$ for $i = t, t+1$, then $n\beta_t^S(N) = m\beta_{t+1}^S(N)$.*
(b) *If $m \geq 1$ and $\operatorname{Ext}_S^i(N, M) = 0$ for $i = t, \ldots, t+s$ for some positive integer s, then $\beta_{t+s}^S(N) = (n/m)^s \cdot \beta_t^S(N)$.*

Remark 3.3. Theorem 3.2 is more than just dual to Theorem 3.1; it is equivalent to Theorem 3.1. To show this, we require a few facts from Matlis duality. Let $(S, \mathfrak{n}, l)$

be a local ring. Let $E_S(l)$ denote the injective hull of the residue field l, and consider the Matlis duality functor $(-)^\vee = \mathrm{Hom}_S(-, E_S(l))$.

First, recall that a module M has finite length if and only if its Matlis dual $M^\vee$ has finite length. When M has finite length, the natural biduality map $M \to M^{\vee\vee}$ is an isomorphism, and $\mathfrak{n}^2 M = 0$ if and only if $\mathfrak{n}^2 M^\vee = 0$. Using this, it is straightforward to show that there is an exact sequence $0 \to l^m \to M \to l^n \to 0$ if and only if there is an exact sequence $0 \to l^n \to M^\vee \to l^m \to 0$.

Second, if M has finite length, then there are isomorphisms

$$\mathrm{Tor}_i^S(N, M^\vee)^\vee \cong \mathrm{Ext}_S^i(N, M^{\vee\vee}) \cong \mathrm{Ext}_S^i(N, M).$$

for each integer i and each S-module N. The first is a version of Hom-tensor adjointness, and the second comes from the biduality isomorphism $M \to M^{\vee\vee}$. Thus, we have $\mathrm{Tor}_i^S(N, M^\vee) = 0$ if and only if $\mathrm{Ext}_S^i(N, M) = 0$. (Furthermore, it is readily shown that $M \in \mathcal{A}(S)$ if and only if $M^\vee \in \mathcal{B}(S)$.) The equivalence of Theorems 3.1 and 3.2 now follows readily.

Each of the remaining results of this article has a dual version that is equivalent via a similar argument. Because the results are equivalent, and not just similar, we only state the "Tor-version" and leave the "Ext-version" for the reader.

Corollary 3.4. *Let $(S, \mathfrak{n}, l)$ be a local ring, and let M and N be finitely generated S-modules. Let m, n and t be integers with $m \geq 1$, and assume that there is an exact sequence of S-module homomorphisms $0 \to l^n \to M \to l^m \to 0$. Assume that $\mathrm{Tor}_i^S(N, M) = 0$ for all $i \geq t$.*

(a) *If $n > m$ and $\beta_t^S(N) \neq 0$, then the sequence $\{\beta_i^S(N)\}$ grows exponentially.*
(b) *If $n = m$, then the sequence $\{\beta_i^S(N)\}$ is eventually constant.*
(c) *If $n < m$, then the sequence $\{\beta_i^S(N)\}$ is eventually zero, that is, the module N has finite projective dimension.*
(d) *If N has infinite projective dimension, then $\mathfrak{n}M \neq 0$ and the number n/m is a positive integer.*

Proof. Theorem 3.1(b) implies that $\beta_i^S(N) = (n/m)^{i-t} \cdot \beta_t^S(N)$ for all $i \geq t$. The conclusions (a)–(c) now follow immediately in this case, recalling that N has finite projective dimension if and only if $\beta_i^S(N) = 0$ for $i \gg 0$.

For part (d), assume that N has infinite projective dimension. It follows that $\beta_i^S(N) = (n/m)^{i-t} \cdot \beta_t^S(N)$ is a positive integer for all $i \geq t$. We conclude that n/m is a positive integer. If $\mathfrak{n}M = 0$, then $M \cong k^{m+n}$. Since $m + n \geq 1$, our Tor-vanishing assumption implies that $\mathrm{Tor}_t^S(N, k) = 0$, contradicting the infinitude of $\mathrm{pd}_S(N)$. □

The next result contains parts (a)–(e) of Theorem 1.3 from the introduction.

Theorem 3.5. *Assume that R is homologically of minimal multiplicity of type (m, n, t), and set $r = n/m$.*

(a) *If R has a canonical module ω_R, then $\beta_{t+s}^R(\omega_R) = r^s \cdot \beta_t^R(\omega_R)$ for all $s \geq 0$.*
(b) *Assume that $n > m$ and R has a canonical module ω_R. If R is not Gorenstein, then the sequence $\{\beta_i^R(\omega_R)\}$ grows exponentially.*

(c) *If* $n = m$ *and* R *has a canonical module* ω_R, *then the sequence* $\{\beta_i^R(\omega_R)\}$ *is eventually constant.*
(d) *If* $n < m$, *then* R *is Gorenstein.*
(e) *If* R *is not Gorenstein, then* $m \mid n$ *and* $n \geq 1$.

Proof. Using Lemma 2.12(a) we may assume that R is complete, so R has a canonical module ω_R in (d)–(e). Let φ: $R \to S$ be as in Definition 1.2. The fact that φ is flat with Gorenstein closed fibre implies that $\omega_S \cong S \otimes_R \omega_R$ and $\mathrm{Tor}_i^R(S, \omega_R) = 0$ for all $i \geq 1$. It follows that $\beta_i^R(\omega_R) = \beta_i^S(\omega_S)$ for all i. The desired conclusions now follow from Theorem 3.1(b) and Corollary 3.4, using the fact that R is Gorenstein if and only if $\beta_i^R(\omega_R) = 0$ for some $i \geq 1$, equivalently, for all $i \gg 0$. □

The following question is motivated by Theorem 3.5(e).

Question 3.6. Assume that R is not Gorenstein. If R is homologically of minimal multiplicity of type (r, rm, t), must R be homologically of minimal multiplicity of type $(1, m, t)$? If R is strongly homologically of minimal multiplicity of type (r, rm), must R be strongly homologically of minimal multiplicity of type $(1, m)$?

The next result gives two criteria that yield affirmative answers for Question 3.6.

Proposition 3.7. *Let* $(S, \mathfrak{n}, l)$ *be a local ring, and let* M *and* N *be finitely generated* S*-modules. Let* m, n, *and* t *be integers with* $m \geq 1$, *and assume that there is an exact sequence of* S*-module homomorphisms*

$$0 \to l^n \to M \xrightarrow{\tau} l^m \to 0 \tag{3.7.1}$$

and that $\mathrm{Tor}_i^S(N, M) = 0$ *for* $i \geq t$. *Assume that* $\mathrm{pd}_S(N)$ *is infinite, and set* $r = n/m$ *and* $e = \mathrm{edim}(S) = \beta_0^S(\mathfrak{n})$.

(a) *There is an equality* $\beta_0^S(M) = m$.
(b) *There are inequalities* $r \leq \mathrm{length}_S(S/\mathrm{Ann}_S(M)) - 1 \leq e$.
(c) *One has* $r = \mathrm{length}_S(S/\mathrm{Ann}_S(M)) - 1$ *if and only if* $M \cong (S/\mathrm{Ann}_S(M))^m$.
(d) *One has* $r = e$ *if and only if* $M \cong (S/\mathfrak{n}^2)^m$.

Proof. Set $J = \mathrm{Ann}_S(M)$ and $a = \mathrm{length}_S(S/J)$.

(a) The surjection τ: $M \twoheadrightarrow l^m$ implies that $\beta_0^S(M) \geq m$. Suppose that $\beta_0^S(M) > m$. It follows that $\mathrm{Ker}(\tau) \cong l^n$ contains a minimal generator for M. We conclude that $M \cong l \oplus M'$ for some submodule $M' \subseteq M$. (To see this, let $x_1 \in M$ be a minimal generator in l^n, and complete this to a minimal generating sequence $x_1, \ldots, x_p$ for M. The module $M/(x_2, \ldots, x_p)$ is cyclic and nonzero, generated by the residue of x_1, which we denote $\overline{x_1}$. Since $\mathfrak{n}x_1 = 0$, it follows that $M/(x_2, \ldots, x_p) \cong l\overline{x_1}$. The composition $lx_1 \subseteq M \to M/(x_2, \ldots, x_p) \cong l\overline{x_1}$ is an isomorphism, so the surjection $M \to l$ splits.) The condition $0 = \mathrm{Tor}_i^S(N, M) \cong \mathrm{Tor}_i^S(N, M') \oplus \mathrm{Tor}_i^S(N, l)$ for $i \geq t$ implies that $\mathrm{Tor}_t^S(N, l) = 0$, contradicting the infinitude of $\mathrm{pd}_S(N)$.

(b) Since $\mathfrak{n}^2 M = 0$, we have $\mathfrak{n}^2 \subseteq J \subseteq \mathfrak{n}$ and hence

$$a - 1 \leq \text{length}(S/\mathfrak{n}^2) - 1 = e.$$

This is the second desired inequality.

There is an S-module epimorphism π: $(S/J)^m \twoheadrightarrow M$. Since $m = \beta_0^S(M)$, we conclude that $\text{Ker}(\pi) \subseteq \mathfrak{n}(S/J) = \mathfrak{n}/J$. Since $\mathfrak{n}^2 \subseteq J$, we see that $\mathfrak{n}\,\text{Ker}(\pi) = 0$, so $\text{Ker}(\pi) \cong l^s$ for some integer s. Using the exact sequence

$$0 \to l^s \to (S/J)^m \xrightarrow{\pi} M \to 0,$$

we have the first equality in the sequence

$$s = am - \text{length}_S(M) = am - (m + n) = m(a - 1 - r).$$

The second equality is from the sequence (3.7.1). The third equality is from the definition $r = n/m$. Since $s \geq 0$ and $m > 0$, we have $a - 1 - r \geq 0$, that is, $r \leq a - 1$. This completes the proof of (b).

For the rest of the proof, we continue with the notation from the proof of part (b).

(c) We have $M \cong (S/J)^m$ if and only if π is an isomorphism, that is, if and only if $l^s \cong \text{Ker}(\pi) = 0$. Since $s = m(a - 1 - r)$ and $m > 0$, we conclude that $s = 0$ if and only if $r = a - 1$.

(d) Assume first that $r = e$. Part (b) implies that $r \leq a - 1 \leq e = r$ and thus $r = a - 1$. Hence, part (c) yields an isomorphism $M \cong (S/J)^m$. The surjection $S/\mathfrak{n}^2 \twoheadrightarrow S/J$ yields the inequality in the sequence

$$a = \text{length}_S(S/J) \leq \text{length}_S(S/\mathfrak{n}^2) = e + 1 = r + 1 = a.$$

It follows that $\text{length}_S(S/J) = \text{length}_S(S/\mathfrak{n}^2)$, so the surjection $S/\mathfrak{n}^2 \twoheadrightarrow S/J$ is an isomorphism. Hence, we have $J = \mathfrak{n}^2$, and thus $M \cong (S/J)^m \cong (S/\mathfrak{n}^2)^m$.

For the converse, assume that $M \cong (S/\mathfrak{n}^2)^m$. It follows that in the exact sequence (3.7.1) we have $l^{rm} \cong l^n \cong \mathfrak{n}M \cong (\mathfrak{n}/\mathfrak{n}^2)^m \cong l^{em}$ and hence $r = e$. □

The following results describe relations between m, n, $\beta_1^S(N)$, and $\beta_0^S(N)$.

Proposition 3.8. *Let $(S, \mathfrak{n}, l)$ be a local ring, and let M and N be finitely generated S-modules such that $\text{pd}_S(N)$ is infinite. Let m and n be integers with $m \geq 1$, and assume that there is an exact sequence of S-module homomorphisms*

$$0 \to l^n \xrightarrow{\alpha} M \xrightarrow{\tau} l^m \to 0 \qquad (3.8.1)$$

and that $\text{Tor}_i^S(N, M) = 0$ for $i \geq 1$. Set $r = n/m$.

(a) *There is an inequality $\beta_1^S(N) \leq \beta_0^S(N)r$.*
(b) *There is an equality $\text{Ker}(N \otimes_S \tau) = \mathfrak{n}(N \otimes_S M)$.*
(c) *One has $\beta_1^S(N) = \beta_0^S(N)r$ if and only if $\mathfrak{n}(N \otimes_S M) = 0$.*

Proof. For each index i, set $b_i = \beta_i^S(N)$.

(a) Apply $N \otimes_S -$ to the sequence (3.8.1) to obtain the exact sequence

$$0 \to \mathrm{Tor}_1^S(N, l)^m \xrightarrow{\gamma} N \otimes_S l^{mr} \xrightarrow{N \otimes_S \alpha} N \otimes_S M \xrightarrow{N \otimes_S \tau} N \otimes_S l^m \to 0. \tag{3.8.2}$$

Notice that we have

$$\mathrm{Tor}_1^S(N, l)^m \cong (l^{b_1})^m \cong l^{b_1 m} \quad \text{and} \quad N \otimes_S l^{mr} \cong l^{b_0 mr}.$$

The sequence (3.8.2) implies that $\mathrm{Tor}_1^S(N, l)^m \subseteq N \otimes_S l^{mr}$, so we have $b_1 m \leq b_0 mr$. Since $m \geq 1$, this implies $b_1 \leq b_0 r$.

(b) Proposition 3.7(a) shows that $m = \beta_0^S(M)$ and moreover, the surjection τ is naturally identified with the natural surjection $M \to M \otimes_S l$. Accordingly, we have $l^n \cong \mathfrak{n}M$, so the sequence (3.8.1) has the form

$$0 \to \mathfrak{n}M \xrightarrow{\alpha} M \xrightarrow{\tau} l^m \to 0.$$

Thus, the sequence (3.8.2) has the form

$$0 \to \mathrm{Tor}_1^S(N, l)^m \xrightarrow{\gamma} N \otimes_S \mathfrak{n}M \xrightarrow{N \otimes_S \alpha} N \otimes_S M \xrightarrow{N \otimes_S \tau} N \otimes_S l^m \to 0.$$

It follows that $\mathfrak{n}(N \otimes_S M) = \mathrm{Im}(N \otimes_S \alpha) = \mathrm{Ker}(N \otimes_S \tau)$.

(c) We have

$$\mathfrak{n}(N \otimes_S M) = \mathrm{Ker}(N \otimes_S \tau) \cong \mathrm{Coker}(\gamma) \cong l^{m(b_0 r - b_1)}.$$

Hence, we have $\mathfrak{n}(N \otimes_S M) = 0$ if and only if $m(b_0 r - b_1) = 0$, that is, if and only if $b_1 = b_0 r$. □

Corollary 3.9. *Let R be a local ring with a canonical module ω_R. If R is homologically of minimal multiplicity of type $(m, n, 1)$, then $\beta_1^R(\omega_R) \leq \beta_0^R(\omega_R)n/m$.*

Proof. If R is Gorenstein, then $\beta_1^R(\omega_R) = 0 \leq \beta_0^R(\omega_R)n/m$. When R is not Gorenstein, argue as in the proof of Theorem 3.5 to derive the desired inequality from Proposition 3.8(a). □

Note that the hypotheses of parts (a) and (b) of the next result hold automatically when $N = \omega_S$ and M is in the Auslander class $\mathcal{A}(S)$.

Theorem 3.10. *Let $(S, \mathfrak{n}, l)$ be a local ring, and let M and N be finitely generated S-modules such that $\mathrm{pd}_S(N)$ is infinite. Let m and n be integers with $m \geq 1$, and assume that there is an exact sequence of S-module homomorphisms*

$$0 \to l^n \to M \xrightarrow{\tau} l^m \to 0 \tag{3.10.1}$$

and that $\mathrm{Tor}_i^S(N, M) = 0$ for $i \geq 1$. Set $r = n/m$.

(a) *If* $M \cong \mathrm{Hom}_S(N, N \otimes_S M)$, *then* $\beta_1^S(N) < \beta_0^S(N)r$.
(b) *Assume that* $\mathrm{Ext}_S^1(N, N \otimes_S M) = 0$ *and* $\mathrm{length}_S(\mathrm{Hom}_S(N, N \otimes_S M)) = \mathrm{length}_S(M)$. *Then there are equalities*

$$\begin{aligned}\beta_1^S(N) &= \frac{1}{2}\left[\beta_0^S(N)(r+1) \pm \sqrt{\beta_0^S(N)^2(r+1)^2 - 4(\beta_0^S(N)^2 - 1)(r+1)}\right]\\ &= \frac{1}{2}\left[\beta_0^S(N)(r+1) \pm \sqrt{(r+1)[\beta_0^S(N)^2(r-3)+4]}\right].\end{aligned}$$

In particular, the integer

$$\beta_0^S(N)^2(r+1)^2 - 4(\beta_0^S(N)^2 - 1)(r+1) = (r+1)[\beta_0^S(N)^2(r-3)+4]$$

is a perfect square.
(c) *If* $N = \omega_S \not\cong S$ *and* $M \in \mathscr{A}(S)$, *then*

$$\beta_1^S(\omega_S) = \beta_0^S(\omega_S)(r^2-1)/r \quad \text{and} \quad r = \frac{\beta_1^S(\omega_S) + \sqrt{\beta_1^S(\omega_S)^2 + 4\beta_0^S(\omega_S)^2}}{2\beta_0^S(\omega_S)}.$$

Proof. For each index i, set $b_i = \beta_i^S(N)$. Note that Corollary 3.4(c) implies that $r \geq 1$ and $\mathfrak{n}M \neq 0$.

(a) Since we have $\mathfrak{n}M \neq 0$, the isomorphism $M \cong \mathrm{Hom}_S(N, N \otimes_S M)$ implies that $\mathfrak{n}(N \otimes_S M) \neq 0$. The conclusion $b_1 < b_0 r$ follows from parts (a) and (c) of Proposition 3.8.

(b) By definition, we have $N \otimes_S l^m \cong l^{b_0 m}$. We have seen that $\mathrm{Ker}(N \otimes_S \tau) \cong l^{m(b_0 r - b_1)}$, so the sequence (3.8.2) yields the exact sequence

$$0 \to l^{m(b_0 r - b_1)} \to N \otimes_S M \xrightarrow{N \otimes_S \tau} l^{b_0 m} \to 0.$$

Our Ext-vanishing assumption implies that the associated long exact sequence in $\mathrm{Ext}_S(N, -)$ begins as follows:

$$\begin{aligned}0 \to \mathrm{Hom}_S(N, l)^{m(b_0 r - b_1)} &\to \mathrm{Hom}_S(N, N \otimes_S M)\\ &\to \mathrm{Hom}_S(N, l)^{b_0 m} \to \mathrm{Ext}_S^1(N, l)^{m(b_0 r - b_1)} \to 0.\end{aligned}$$

Using the standard isomorphism $\mathrm{Ext}_S^i(N, l) \cong l^{b_i}$, we conclude that this sequence has the following form:

$$0 \to l^{b_0 m(b_0 r - b_1)} \to \mathrm{Hom}_S(N, N \otimes_S M) \to l^{b_0^2 m} \to l^{b_1 m(b_0 r - b_1)} \to 0.$$

Thus, our length assumption explains the second equality in the sequence

$$\begin{aligned}b_1 m(b_0 r - b_1) &= b_0^2 m - \mathrm{length}_S(\mathrm{Hom}_S(N, N \otimes_S M)) + b_0 m(b_0 r - b_1)\\ &= b_0^2 m - \mathrm{length}_S(M) + b_0 m(b_0 r - b_1)\\ &= b_0^2 m - m(r+1) + b_0 m(b_0 r - b_1).\end{aligned}$$

Dividing by m and simplifying, we find that

$$b_1^2 - b_0(r+1)b_1 + (b_0^2 - 1)(r+1) = 0.$$

The desired conclusions now follow from the quadratic formula.

(c) Employ the notation of the exact sequence (3.8.2). We have shown that

$$\mathrm{Ker}(\omega_S \otimes_S \tau) \cong \mathrm{Coker}(\gamma) \cong l^{m(b_0 r - b_1)}$$

so the exact sequence (3.8.2) provides the next exact sequence

$$0 \to l^{m(b_0 r - b_1)} \to \omega_S \otimes_S M \to l^{mb_0} \to 0.$$

Furthermore, the condition $M \in \mathscr{A}(S)$ implies that $\mathrm{Ext}_S^i(\omega_S, \omega_S \otimes_S M) = 0$ for all $i \geq 1$. We conclude from Theorem 3.2(b) that

$$b_i = b_1 \left(\frac{mb_0}{m(b_0 r - b_1)} \right)^{i-1} = b_1 \left(\frac{b_0}{b_0 r - b_1} \right)^{i-1}$$

for all $i \geq 1$. On the other hand, we know that $b_i = r^{i-1} b_1$ for all $i \geq 1$. Since we are assuming that $b_i \neq 0$ for all i, we conclude that $r = b_0/(b_0 r - b_1)$. Solve this equation for b_1 to derive the first desired equality. For the second equality, substitute $b_1 = b_0(r^2 - 1)/r$ into the expression $\frac{b_1 + \sqrt{b_1^2 + 4b_0^2}}{2b_0}$ and simplify. □

Corollary 3.11. *Assume that R is strongly homologically of minimal multiplicity of type (m, n) and with canonical module $\omega_R \ncong R$. If $r = n/m$, then*

$$\beta_1^R(\omega_R) = \beta_0^R(\omega_R)(r^2 - 1)/r \quad \text{and} \quad r = \frac{\beta_1^R(\omega_R) + \sqrt{\beta_1^R(\omega_R)^2 + 4\beta_0^R(\omega_R)^2}}{2\beta_0^R(\omega_R)}.$$

Proof. This follows directly from Theorem 3.10(c) because $\beta_i^R(\omega_R) = \beta_i^S(\omega_S)$. □

The following example is from [10, (3.4)]. It demonstrates how our results can yield exact values for the Betti numbers of canonical modules. It also shows that, if R is strongly homologically of minimal multiplicity with S and M as in Definition 1.2, then M may not be a direct sum of cyclic S-modules. Similar arguments yield the Betti numbers of the canonical modules for the rings constructed in [5].

Example 3.12. Let k be a field, and let $\alpha \in k$ such that $\alpha \neq 0, 1, -1$. Consider the polynomial ring $A = k[X_1, X_2, X_3, X_4]$ and the ideal $I \subseteq A$ generated by the following polynomials:

$$\alpha X_1 X_3 + X_2 X_3, \quad X_1 X_4 + X_2 X_4, \quad X_3^2, \quad X_4^2, \quad X_1^2, \quad X_2^2, \quad X_3 X_4.$$

The ring $R = A/I$ is artinian and local with maximal ideal $\mathfrak{m} = (x_1, x_2, x_3, x_4)R$, and $\mathfrak{m}^3 = 0$. (Here x_i denotes the image of X_i in R.) For each integer n, set

$$d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix}.$$

Consider the following chain complex of R-modules:

$$G = \cdots \xrightarrow{d_{n+1}} R^2 \xrightarrow{d_n} R^2 \xrightarrow{d_{n-1}} \cdots$$

and the R-module $M = \operatorname{Im}(d_0)$. Let ω_R denote a canonical module for R.

Arguing as in [10, (3.1)], one has the following facts. The complexes G and $\operatorname{Hom}_R(G, R)$ are exact. In the language of [6, (4.1.2)], this means that G is a "complete resolution" of M by finite free modules. Using [6, (4.1.3), (4.2.6), (4.4.13)], we conclude that $M \in \mathscr{A}(R)$. Also, one has $\mathfrak{m}^2 M = 0$ and $\beta_0^R(M) = 2$ and $\beta_0^R(\mathfrak{m}M) = 6$, so the ring R is strongly homologically of minimal multiplicity of type $(6, 2)$. In particular, $\operatorname{length}_R(M) = 8$, and the complex

$$G' = \cdots \xrightarrow{d_2} R^2 \xrightarrow{d_1} R^2 \to 0$$

is a minimal free resolution of M. The gist of [10, (3.4)] is that

$$\operatorname{Ker}(d_{n+2}) \ncong \operatorname{Ker}(d_n) \tag{3.12.1}$$

for all $n \geq 1$.

The socle of R is $\mathfrak{m}^2$, which has basis x_1x_2, x_1x_3, x_1x_4. Hence, we have $\beta_0^R(\omega_R) = 3$. Corollary 3.11 implies that $\beta_1^R(\omega_R) = 8$, and Theorem 3.5(a) yields the formula $\beta_n^R(\omega_R) = 8 \cdot 3^{n-1}$ for all $n \geq 1$.[2]

We claim that M is indecomposable. By way of contradiction, suppose that $M \cong M_1 \oplus M_2$, where M_1 and M_2 are both nonzero. The equality $\beta_0^R(M) = 2$ implies that each M_i is cyclic. It follows that $\operatorname{pd}_R(M_i) = \infty$ for $i = 1, 2$. Indeed, if $\operatorname{pd}_R(M_i)$ is finite, then the fact that R is artinian implies that M_i is free. Since M_i is cyclic, we have $M_i \cong R$, and so

$$8 = \operatorname{length}_R(M) = \operatorname{length}_R(M_1) + \operatorname{length}_R(M_2) > \operatorname{length}_R(M_i) = 8,$$

which is impossible.

The resolution G' shows that $\beta_n^R(M) = 2$ for all $n \geq 0$. It follows that $\beta_n^R(M_i) = 1$ for all $n \geq 0$ and for $i = 1, 2$. Let F_i be the minimal free resolution of M_i with nth differential $d_{i,n}$. From [10, (3.8)], it follows that there is an integer $n \geq 1$ such that $\operatorname{Ker}(d_{i,n+2}) \cong \operatorname{Ker}(d_{i,n})$ for $i = 1, 2$. The uniqueness of minimal free resolutions implies that $G' \cong F_1 \oplus F_2$, and hence

$$\operatorname{Ker}(d_{n+2}) \cong \operatorname{Ker}(d_{1,n+2}) \oplus \operatorname{Ker}(d_{2,n+2}) \cong \operatorname{Ker}(d_{1,n}) \oplus \operatorname{Ker}(d_{2,n}) \cong \operatorname{Ker}(d_n).$$

This contradicts (3.12.1). Thus M is indecomposable, as claimed.

[2]Preliminary computations were performed using Macaulay 2 [11].

The next result contains Theorem 1.3(f) from the introduction.

Theorem 3.13. *Assume that R is strongly homologically of minimal multiplicity of type (m, n). If $n = m$, then R is Gorenstein.*

Proof. Using Lemma 2.12(b), we assume that R is complete. Hence R has a canonical module ω_R. The assumption $m = n$ translates as $r = 1$, so Corollary 3.11 implies that $\beta_1^R(\omega_R) = 0$. It follows that R is Gorenstein. □

The following question asks if the conclusion of Theorem 3.13 holds when R is only assumed to be homologically of minimal multiplicity.

Question 3.14. Assume that R is homologically of minimal multiplicity of type (m, n, t). If $n = m$, must R be Gorenstein?

4. ALTERNATE CHARACTERIZATIONS

In this section, we provide alternate characterizations of the rings that are (strongly) homomologically of minimal multiplicity. The first of these results is Theorem 4.5 which says that in the definition of "homologically of minimal multiplicity" one can assume that the ring S is complete with algebraically closed residue field and that the closed fibre $S/\mathfrak{m}S$ is regular. In preparation, we recall some background information on local ring homomorphisms.

Definition 4.1. Let $\varphi: R \to S$ be a local ring homomorphism. A *Cohen factorization* of φ is a diagram of local ring homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ satisfying the following conditions:

(1) One has $\varphi = \varphi'\dot{\varphi}$;
(2) The map $\dot{\varphi}$ is flat with regular closed fibre $R'/\mathfrak{m}R'$;
(3) The local ring R' is complete; and
(4) The map φ' is surjective.

Remark 4.2. Let $\varphi: R \to S$ be a local ring homomorphism. If φ admits a Cohen factorization, then the ring S is a homomorphic image of a complete local ring, so it is complete. Conversely, if S is complete, then φ admits a Cohen factorization by [4, (1.1)].

Definition 4.3. Let $(R, \mathfrak{m}, k)$ be a local ring. The *ith Bass number* of R is the integer $\mu_R^i(R) = \operatorname{rank}_k(\operatorname{Ext}_R^i(k, R))$.

Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism, and assume that R is Cohen–Macaulay. The homomorphism φ has *finite flat dimension* if S has finite flat dimension as an R-module, that is, if S admits a bounded resolution by flat R-modules. The homomorphism φ is *Gorenstein* if it has finite flat dimension and $\mu_S^{i+\operatorname{depth}(S)}(S) = \mu_R^{i+\operatorname{depth}(R)}(R)$ for all i. An ideal $I \subset R$ is *Gorenstein* if the natural surjection $R \to R/I$ is Gorenstein.

Remark 4.4. Let φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism. If φ is flat, then it has finite flat dimension. Also, when φ is flat, it is Gorenstein if and only if the closed fibre $S/\mathfrak{m}S$ is Gorenstein; see [2, (4.2)]. An ideal $I \subset R$ generated by an R-regular sequence is Gorenstein. An ideal $I \subset R$ is Gorenstein if and only if the R-module R/I is perfect and $\beta_g^R(R/I) = 1$ where $g = \mathrm{grade}_R(R/I) = \mathrm{pd}_R(R/I)$; see [2, (4.3)]. If φ has finite flat dimension and ψ: $S \to T$ is another local homomorphism of finite flat dimension, then [2, (4.6)] implies that the composition $\psi\varphi$ is Gorenstein if and only if ψ and φ are both Gorenstein.

Assume that φ admits a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$. The map φ has finite flat dimension if and only if $\mathrm{pd}_{R'}(S)$ is finite; see [4, (3.2)]. The map φ is Gorenstein if and only if $\mathrm{Ker}(\varphi')$ is a Gorenstein ideal of R'; see [4, (3.11)].

Assume that φ is Gorenstein and that S is Cohen–Macaulay. Since $\mathrm{pd}_{R'}(S)$ is finite, it follows that R' is Cohen–Macaulay. Since R' and S are both complete, they each admit a canonical module, and [7, (5.7)] implies that $\omega_S \cong S \otimes_{R'} \omega_{R'}$ and $\mathrm{Tor}_i^{R'}(S, \omega_{R'}) = 0$ for all $i \geq 1$.

Theorem 4.5. *A local ring R is homologically of minimal multiplicity of type (m, n, t) if and only if there exists a local ring homomorphism φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $M \neq 0$ such that*:

(1) *The ring S is complete and Cohen–Macaulay with canonical module ω_S, and l is algebraically closed*;
(2) *The map φ is flat with regular closed fibre $S/\mathfrak{m}S$*;
(3) *One has $\mathrm{Tor}_i^S(\omega_S, M) = 0$ for $i \geq t$; and*
(4) *One has $\mathfrak{n}^2 M = 0$ and $m = \beta_0^S(M)$ and $n = \beta_0^S(\mathfrak{n}M)$.*

Proof. One implication is routine. For the converse, assume that R is homologically of minimal multiplicity of type (m, n, t). We complete the proof in three steps.

Step 1: By definition, there is a local ring homomorphism φ_1: $(R, \mathfrak{m}, k) \to (S_1, \mathfrak{n}_1, l_1)$ and a finitely generated S_1-module $M_1 \neq 0$ such that:

(1′) The ring S_1 has a canonical module ω_{S_1};
(2′) The map φ_1 is flat with Gorenstein closed fibre $S_1/\mathfrak{m}S_1$;
(3′) One has $\mathrm{Tor}_i^{S_1}(\omega_{S_1}, M_1) = 0$ for $i \geq t$; and
(4′) One has $\mathfrak{n}_1^2 M_1 = 0$ and $m = \beta_0^{S_1}(M_1)$ and $n = \beta_0^{S_1}(\mathfrak{n}_1 M_1)$.

Step 2: From Remark 2.5, there is a flat local homomorphism ψ: $(S_1, \mathfrak{n}_1, l_1) \to (S_2, \mathfrak{n}_1 S_2, l)$ such that S_2 is complete and l is the algebraic closure of l_1. Since the map ψ is flat and the maximal ideal of S_2 is $\mathfrak{n}_2 = \mathfrak{n}_1 S_2$, it is straightforward to show that the composition φ_2: $R \xrightarrow{\varphi_1} S_1 \xrightarrow{\psi} S_2$ and the module $M = S_2 \otimes_{S_1} M_1$ satisfy the following conditions:

(1″) The ring S_2 is complete and Cohen–Macaulay with canonical module ω_{S_2} and has an algebraically closed residue field;
(2″) The map φ_2 is flat with Gorenstein closed fibre $S_2/\mathfrak{m}S_2$;

(3″) One has $\operatorname{Tor}_i^{S_2}(\omega_{S_2}, M) = 0$ for $i \geq t$; and
(4″) One has $\mathfrak{n}_2^2 M = 0$ and $m = \beta_0^{S_2}(M)$ and $n = \beta_0^{S_2}(\mathfrak{n}_2 M)$.

Step 3: The ring S_2 is complete, so the local homomorphism φ_2 admits a Cohen factorization $(R, \mathfrak{m}, k) \xrightarrow{\varphi} (S, \mathfrak{n}, l) \xrightarrow{\varphi_2'} (S_2, \mathfrak{n}_2, l)$. Remark 4.4 implies that the ideal $\operatorname{Ker}(\varphi_2') \subset S$ is Gorenstein, the ring S is complete and Cohen–Macaulay, there is an isomorphism $\omega_{S_2} \cong S_2 \otimes_S \omega_S$, and $\operatorname{Tor}_i^S(S_2, \omega_S) = 0$ for all $i \geq 1$.

Let F be a free resolution of ω_S over S. Then $F \otimes_S S_2$ is a free resolution of $S_2 \otimes_S \omega_S \cong \omega_{S_2}$. Hence, for each index i, there are isomorphisms

$$\operatorname{Tor}_i^S(\omega_S, M) \cong \mathrm{H}_i(F \otimes_S M) \cong \mathrm{H}_i((F \otimes_S S_2) \otimes_{S_2} M) \cong \operatorname{Tor}_i^{S_2}(\omega_{S_2}, M). \qquad (4.5.1)$$

It follows that the map φ and the module M satisfy the conditions (1)–(4). □

The next result is a version of Theorem 4.5 for rings that are strongly homologically of minimal multiplicity; it is proved similarly.

Theorem 4.6. *A local ring R is strongly homologically of minimal multiplicity of type (m, n) if and only if there exists a local ring homomorphism φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $M \neq 0$ such that*:

(1) *The ring S is complete and Cohen–Macaulay, and l is algebraically closed*;
(2) *The map φ is flat with regular closed fibre $S/\mathfrak{m}S$*;
(3) *One has $M \in \mathcal{A}(S)$; and*
(4) *One has $\mathfrak{n}^2 M = 0$ and $m = \beta_0^S(M)$ and $n = \beta_0^S(\mathfrak{n}M)$.*

Readers familiar with [3] will recognize that the proof of Theorem 1.3 only requires the homomorphism φ to be *quasi*-Gorenstein. (See Definition 4.7.) One may ask why we require the stronger hypotheses in Definition 1.2. Theorem 4.9 shows that our definition is equivalent to the weaker definition which only requires φ to be quasi-Gorenstein. We have chosen this one since flat maps with Gorenstein closed fibres are more familiar.

Definition 4.7. Let φ: $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism, and assume that R is Cohen–Macaulay. The homomorphism φ has *finite G-dimension* if the $\mathfrak{n}$-adic completion $\widehat{S}$ is in the Auslander class $\mathcal{A}(\widehat{R})$ of the $\mathfrak{m}$-adic completion $\widehat{R}$. The homomorphism φ is *quasi-Gorenstein* if it has finite G-dimension and $\mu_S^{i+\operatorname{depth}(S)}(S) = \mu_R^{i+\operatorname{depth}(R)}(R)$ for all i. An ideal $I \subset R$ is *quasi-Gorenstein* if the natural surjection $R \to R/I$ is quasi-Gorenstein.

Remark 4.8. Let φ: $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism, and assume that R is Cohen–Macaulay. Let $\grave{\varphi}$: $R \to \widehat{S}$ denote the composition of φ with the natural map $S \to \widehat{S}$. Fix a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of $\grave{\varphi}$. Since R is Cohen–Macaulay and $\dot{\varphi}$ is flat with regular closed fibre, the ring R' is Cohen–Macaulay. As R' is complete, it has a canonical module $\omega_{R'}$.

The homomorphism φ has finite G-dimension if and only if $\widehat{S} \in \mathcal{A}(R')$; see [3, (4.1.7) and (4.3)]. In particular, if φ has finite G-dimension, then $\operatorname{Tor}_i^{R'}(\widehat{S}, \omega_{R'}) = 0$

for $i \geq 1$. Moreover, if φ is flat (or more generally, if φ has finite flat dimension), then φ has finite G-dimension. If φ is Gorenstein (e.g., if it is flat with Gorenstein closed fibre), then it is quasi-Gorenstein. The composition of two quasi-Gorenstein homomorphisms is quasi-Gorenstein by [3, (8.9)].

If φ is quasi-Gorenstein, then S is Cohen–Macaulay, and the canonical module of $\widehat{S}$ is $\omega_{\widehat{S}} \cong \widehat{S} \otimes_{R'} \omega_{R'}$. Indeed, from [3, (7.8)] we conclude that the complex $\widehat{S} \otimes^{\mathbf{L}}_{R'} \omega_{R'}$ is a dualizing complex for $\widehat{S}$. (See [3] for an extensive discussion on the topic of dualizing complexes.) The vanishing $\operatorname{Tor}_i^{R'}(\widehat{S}, \omega_{R'}) = 0$ for $i \geq 1$ implies that the complex $\widehat{S} \otimes^{\mathbf{L}}_{R'} \omega_{R'}$ is isomorphic (in the derived category $\mathsf{D}(\widehat{S})$) to the module $\widehat{S} \otimes_{R'} \omega_{R'}$. It follows that this is a canonical module for $\widehat{S}$, and thus S is Cohen–Macaulay.

Theorem 4.9. *A local ring R is homologically of minimal multiplicity of type (m, n, t) if and only if it is Cohen–Macaulay and there exists a local ring homomorphism $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and a finitely generated S-module $M \neq 0$ such that:*

(1) *The ring S has a canonical module ω_S;*
(2) *The map φ is quasi-Gorenstein;*
(3) *One has $\operatorname{Tor}_i^S(\omega_S, M) = 0$ for $i \geq t$; and*
(4) *One has $\mathfrak{n}^2 M = 0$ and $m = \beta_0^S(M)$ and $n = \beta_0^S(\mathfrak{n}M)$.*

Proof. One implication is routine, using the fact that a local homomorphism that is flat with Gorenstein closed fibre is quasi-Gorenstein. For the converse, assume that R is Cohen–Macaulay and there exists a local ring homomorphism $\varphi: R \to S$ and a finitely generated S-module $M \neq 0$ satisfying conditions (1)–(4). By passing to the completion $\widehat{S}$, we assume that S is complete.

Fix a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ of φ. Remark 4.8 implies that R' is Cohen–Macaulay with canonical module $\omega_{R'}$, that $\operatorname{Tor}_i^{R'}(\widehat{S}, \omega_{R'}) = 0$ for $i \geq 1$, and that the canonical module of $\widehat{S}$ is $\omega_{\widehat{S}} \cong \widehat{S} \otimes_{R'} \omega_{R'}$. The argument of Theorem 4.5 now shows that the homomorphism $\dot{\varphi}$ and the R'-module M satisfy the hypotheses of Definition 1.2, so R is homologically of minimal multiplicity of type (m, n, t). $\square$

The next result is a version of Theorem 4.9 for rings that are strongly homologically of minimal multiplicity; it is proved similarly.

Theorem 4.10. *A local ring R is homologically of minimal multiplicity of type (m, n) if and only if it is Cohen–Macaulay and there exists a local ring homomorphism $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and a finitely generated S-module $M \neq 0$ such that:*

(1) *The ring S has a canonical module ω_S;*
(2) *The map φ is quasi-Gorenstein;*
(3) *One has $M \in \mathscr{A}(S)$; and*
(4) *One has $\mathfrak{n}^2 M = 0$ and $m = \beta_0^S(M)$ and $n = \beta_0^S(\mathfrak{n}M)$.*

The final results of this section explain why we do not single out rings that satisfy the conditions that are dual to "(strongly) homologically of minimal multiplicity."

Proposition 4.11. *A local ring R is homologically of minimal multiplicity if and only if there exists a local ring homomorphism φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $N \neq 0$ such that*:

(1) *The ring S has a canonical module ω_S*;
(2) *The map φ is flat with Gorenstein closed fibre $S/\mathfrak{m}S$*;
(3) *One has $\operatorname{Ext}^i_S(\omega_S, N) = 0$ for $i \geq t$; and*
(4) *One has $\mathfrak{n}^2 N = 0$.*

Proof. By Remark 3.3, an S-module N satisfies $\mathfrak{n}^2 N = 0$ if and only if $\mathfrak{n}^2 N^\vee = 0$, and $\operatorname{Ext}^i_S(\omega_S, N) = 0$ if and only if $\operatorname{Tor}^S_i(\omega_S, N^\vee) = 0$. The result now follows. □

Proposition 4.12. *A local ring R is homologically of minimal multiplicity if and only if there exists a local ring homomorphism φ: $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ and a finitely generated S-module $N \neq 0$ such that*:

(1) *The ring S has a canonical module ω_S*;
(2) *The map φ is flat with Gorenstein closed fibre $S/\mathfrak{m}S$*;
(3) *One has $N \in \mathcal{B}(S)$; and*
(4) *One has $\mathfrak{n}^2 N = 0$.*

Proof. The proof is similar to that of Proposition 4.11. □

5. ASCENT AND DESCENT BEHAVIOR

This section culminates in Corollaries 5.15 and 5.16, which describe ascent and descent behavior for our classes of rings along local quasi-Gorenstein ring homomorphisms. We divide the proofs into several pieces.

Lemma 5.1. *Let $I \subset R$ be a quasi-Gorenstein ideal. If the quotient R/I is homologically of minimal multiplicity of type (m, n, t), then R is homologically of minimal multiplicity of type (m, n, t). The converse holds when $I \subseteq \mathfrak{m}^2$.*

Proof. Let τ: $R \to R/I$ denote the canonical surjection.

Assume first that the quotient R/I is homologically of minimal multiplicity of type (m, n, t). Let φ_1: $(R/I, \mathfrak{m}/I) \to (S_1, \mathfrak{n}_1)$ be a ring homomorphism and M_1 an S_1-module as in Theorem 4.5. Since S_1 is complete, Remark 4.2 implies that the composition $\varphi_1\tau$: $R \to S_1$ has a Cohen factorization $R \xrightarrow{\dot{\varphi}_1} R' \xrightarrow{\varphi'_1} S_1$. Since τ and φ_1 are quasi-Gorenstein, Remark 4.8 implies that the composition $\varphi'_1\dot{\varphi}_1 = \varphi_1\tau$ is quasi-Gorenstein. Hence, we have $\omega_{S_1} \cong S_1 \otimes_{R'} \omega_{R'}$ and $\operatorname{Tor}^{R'}_i(S_1, \omega_{R'}) = 0$ for all $i \geq 1$. The isomorphisms (4.5.1) from the proof of Theorem 4.5 shows that

$$\operatorname{Tor}^{R'}_i(\omega_{R'}, M_1) \cong \operatorname{Tor}^{S_1}_i(\omega_{S_1}, M_1) = 0$$

for all $i \geq t$. Let $\mathfrak{m}'$ be the maximal ideal of R'. Since $(\mathfrak{m}')^2 M_1 = \mathfrak{n}_1^2 M_1 = 0$, the homomorphism $\dot{\varphi}_1$: $R \to R'$ and the R'-module M_1 combine to show that R is homologically of minimal multiplicity of type (m, n, t).

For the converse, assume that $I \subseteq \mathfrak{m}^2$ and that R is homologically of minimal multiplicity of type (m, n, t). Let φ_2: $(R, \mathfrak{m}) \to (S_2, \mathfrak{n}_2)$ be a ring homomorphism and M_2 an S_2-module as in Definition 1.2. Since φ_2 is flat, it is straightforward to show that the ideal $IS_2 \subseteq S_2$ is quasi-Gorenstein (see, e.g., [3, (8.6)]) and the induced homomorphism $\overline{\varphi_2}$: $R/I \to S_2/IS_2$ is flat. The closed fibre of the composition $\overline{\varphi_2}\tau = \pi_2\varphi_2$: $R \to S_2/IS_2$ is the ring $(S_2/IS_2) \otimes_R (R/\mathfrak{m}) \cong S_2/\mathfrak{m}S_2$ which is Gorenstein. The assumptions $I \subseteq \mathfrak{m}^2$ and $\mathfrak{n}_2^2 M_2 = 0$ imply that $IS_2M_2 = 0$, so M_2 is naturally an S_2/IS_2-module. As in the previous paragraph, we have $\omega_{S_2/IS_2} \cong S_2/IS_2 \otimes_{S_2} \omega_{S_2}$ and $\operatorname{Tor}_i^{S_2}(S_2/IS_2, \omega_{S_2}) = 0$ for all $i \geq 1$, and

$$\operatorname{Tor}_i^{S_1}(\omega_{S_1}, M_2) \cong \operatorname{Tor}_i^{S_2/IS_2}(\omega_{S_2/IS_2}, M_2) = 0$$

for all $i \geq t$. Since $(\mathfrak{n}_2/IS_2)^2 M_2 = \mathfrak{n}_2^2 M_2 = 0$, the homomorphism $\overline{\varphi_2}$: $R/I \to S_2/IS_2$ and the S_2/IS_2-module M_2 combine to show that R/I is homologically of minimal multiplicity. □

Lemma 5.2. *Let $I \subset R$ be a quasi-Gorenstein ideal. If the quotient R/I is strongly homologically of minimal multiplicity of type (m, n), then R is strongly homologically of minimal multiplicity of type (m, n). The converse holds when $I \subseteq \mathfrak{m}^2$.*

Proof. This is proved like Lemma 5.1. □

The next question asks if the converses in Lemmas 5.1 and 5.2 hold without the assumption $I \subseteq \mathfrak{m}^2$.

Question 5.3. Let $I \subset R$ be a quasi-Gorenstein ideal. If R is (strongly) homologically of minimal multiplicity, must R/I be (strongly) homologically of minimal multiplicity?

Before continuing toward our general results on ascent and descent, we note a few special cases of Lemmas 5.1 and 5.2.

Example 5.4. If $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring of minimal multiplicity and $I \subseteq \mathfrak{m}^2$ is a quasi-Gorenstein ideal, then R/I is strongly homologically of minimal multiplicity; see Proposition 2.6.

If $(S, \mathfrak{n})$ is a Cohen–Macaulay local ring and $\mathbf{x} \in \mathfrak{n}^2$ is an S-regular sequence, then S is (strongly) homologically of minimal multiplicity if and only if $S/(\mathbf{x})S$ is (strongly) homologically of minimal multiplicity. If S has minimal multiplicity and $(\mathbf{x})S \neq 0$, then $S/(\mathbf{x})S$ is strongly homologically of minimal multiplicity, but is not of minimal multiplicity.

Example 5.4 gives a method for constructing rings that are strongly homologically of minimal multiplicity. The next question asks whether this is essentially the only way. In other words, it asks whether there is a structure theorem for rings that are strongly homologically of minimal multiplicity akin to Cohen's structure theorem, where regular rings are replaced by rings of minimal multiplicity.

Question 5.5. If R is strongly homologically of minimal multiplicity, must there be an isomorphism $\widehat{R} \cong Q/I$ where Q is a local Cohen–Macaulay ring of minimal multiplicity and $I \subset Q$ is a quasi-Gorenstein ideal?

Remark 5.6. The ring R from Example 3.12 is strongly homologically of minimal multiplicity, but is not of minimal multiplicity. Furthermore, there does not exist a local ring $(Q, \mathfrak{r})$ with a Q-regular sequence $\mathbf{x} \in \mathfrak{r}^2$ such that $R \cong Q/(\mathbf{x})Q$; see [10, (3.10)]. It follows that there does not exist a local Cohen–Macaulay ring of minimal multiplicity $(Q, \mathfrak{r})$ with a Q-regular sequence $\mathbf{x} \in \mathfrak{r}$ such that $R \cong Q/(\mathbf{x})Q$. However, at this time, we do not know if there exists a local Cohen–Macaulay ring of minimal multiplicity $(Q, \mathfrak{r})$ with a quasi-Gorenstein ideal $I \subset Q$ such that $R \cong Q/I$.

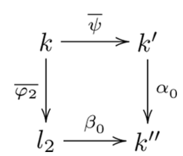
The next two results contain Theorem 1.4 from the introduction.

Theorem 5.7. *Assume that ψ: $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is a flat local ring homomorphism with Gorenstein closed fibre $R'/\mathfrak{m}R'$. If R' is homologically of minimal multiplicity of type (m, n, t), then R is homologically of minimal multiplicity of type (m, n, t). The converse holds when k is perfect and $R'/\mathfrak{m}R'$ is regular.*

Proof. Assume first that R' is homologically of minimal multiplicity of type (m, n, t), and let φ_1: $R' \to S_1$ be a ring homomorphism and M_1 an S_1-module as in Definition 1.2. The composition $\varphi_1\psi$: $R \to S_1$ is flat, and Remark 4.4 implies that it is Gorenstein. It follows readily that this map, with the S_1-module M_1, satisfies the axioms to show that R is homologically of minimal multiplicity of type (m, n, t).

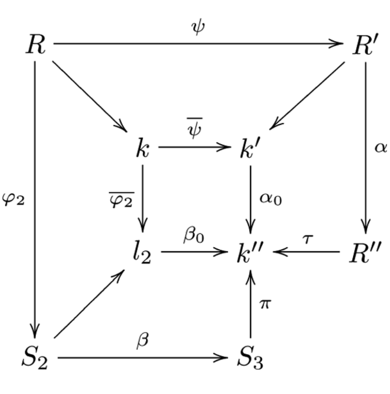
Assume next that R is homologically of minimal multiplicity of type (m, n, t). Assume further that k is perfect and $R'/\mathfrak{m}R'$ is regular. We prove that R' is homologically of minimal multiplicity of type (m, n, t) in two cases.

Case 1. The closed fibre $R'/\mathfrak{m}R'$ is a field. Let φ_2: $(R, \mathfrak{m}, k) \to (S_2, \mathfrak{n}_2, l_2)$ be a ring homomorphism and M_2 an S_2-module as in Theorem 4.5. Since k' and l_2 are extension fields of k, their join k'' fits in a commutative diagram of field extensions



where $\bar{\psi}$ and $\overline{\varphi_2}$ are induced by ψ and φ_2. Remark 2.5 provides flat local ring homomorphisms α: $(R', \mathfrak{m}', k') \to (R'', \mathfrak{m}'R'', k'')$ and β: $(S_2, \mathfrak{n}_2, l_2) \to (S_3, \mathfrak{n}_2S_3, k'')$ such that R'' and S_3 are complete, the map $k' \to k''$ induced by α is precisely α_0, and the map $l_2 \to k''$ induced by β is precisely β_0.

Let τ: $R'' \to k''$ and π: $S_3 \to k''$ denote the natural surjections. It follows that the small quadrilaterals in the following diagram commute:



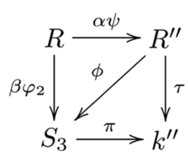
(The unspecified maps are the canonical surjections.) It follows that $\tau\alpha\psi = \pi\beta\varphi_2$.

Note that the composition $\beta\varphi_2$: $R \to S_3$ is flat because β and φ_2 are both flat. Furthermore, the closed fibre $S_3/\mathfrak{m}S_3$ is regular. (Indeed, the map $\overline{\beta}$: $S_2/\mathfrak{m}S_2 \to S_3/\mathfrak{m}S_3$ induced by β is flat because β is flat. The closed fiber of $\overline{\beta}$ is $S_3/\mathfrak{n}_2S_3 = k''$, which is regular. By assumption, the ring $S_2/\mathfrak{m}S_2$ is also regular, and thus $S_3/\mathfrak{m}S_3$ is regular.) It follows that the diagram $R \xrightarrow{\beta\varphi_2} S_3 \xrightarrow{\pi} k''$ is a Cohen factorization of the map $\pi\beta\varphi_2$. Also, since $\mathfrak{n}_2S_3$ is the maximal ideal of S_3, its square annihilates the module $M_3 = S_3 \otimes_{S_2} M_2$, because $\mathfrak{n}_2^2M_2 = 0$. The canonical module of S_3 is $\omega_{S_3} \cong S_3 \otimes_{S_2} \omega_{S_2}$, since β is flat with Gorenstein closed fibre, and it follows that

$$\operatorname{Tor}_i^{S_3}(\omega_{S_3}, M_3) \cong S_3 \otimes_{S_2} \operatorname{Tor}_i^{S_2}(\omega_{S_2}, M_2) = 0$$

for all $i \geq t$. In particular, the map $\beta\varphi_2$: $R \to S_3$ and S_3-module M_3 satisfy the conditions of Definition 1.2.

Similarly, the composition $\alpha\psi$: $R \to R''$ is flat with regular closed fibre, and the diagram $R \xrightarrow{\alpha\psi} R'' \xrightarrow{\tau} k''$ is a Cohen factorization of the map $\tau\alpha\psi$. The diagram $R \xrightarrow{\beta\varphi_2} S_3 \xrightarrow{\pi} k''$ is also a Cohen factorization $\tau\alpha\psi$. Since the field k is perfect, the extension $k \to k''$ is separable, and it follows from [4, (1.7)] that there is a local ring homomorphism ϕ: $R'' \to S_3$ making the following diagram commute:



We claim that ϕ is flat. To show this, we show that $\operatorname{Tor}_i^{R''}(S_3, k'') = 0$ for all $i \geq 1$. Let F be a free resolution of k over R. Since R'' is flat over R, the complex $R'' \otimes_R P$ is a free resolution of $R'' \otimes_R k \cong k''$ over R''. It follows that

$$\operatorname{Tor}_i^{R''}(S_3, k'') \cong \mathrm{H}_i(S_3 \otimes_{R''} (R'' \otimes_R P)) \cong \mathrm{H}_i(S_3 \otimes_R P) \cong \operatorname{Tor}_i^R(S_3, k) = 0$$

for $i \geq 1$; the vanishing comes from the fact that S_3 is flat over R.

Our assumption that $R'/\mathfrak{m}R'$ is a field implies that the maximal ideal of R'' is $\mathfrak{m}'R'' = \mathfrak{m}R''$. Thus, the closed fibre of ϕ is $S_3/\mathfrak{m}'S_3 = S_3/\mathfrak{m}S_3$, which is regular. Hence, the map ϕ: $R'' \to S_3$ with the S_3-module M_3 satisfies the conditions of Definition 1.2, showing that R'' is homologically of minimal multiplicity of type (m, n, t). The local homomorphism α: $R' \to R''$ is flat, so the descent result (established in the first paragraph of this proof) shows that R' is homologically of minimal multiplicity of type (m, n, t). This completes the proof in this case.

Case 2. the general case. Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}'$ be a sequence of elements whose residues modulo $\mathfrak{m}R'$ form a regular system of parameters for the regular ring $R'/\mathfrak{m}R'$. According to [16, Cor. of (22.5)], the sequence $\mathbf{x}$ is R'-regular, and the quotient $R'/\mathbf{x}R'$ is flat as an R-module. Furthermore, the closed fibre of the induced map $R \to R'/\mathbf{x}R'$ is $R'/(\mathbf{x}R' + \mathfrak{m}R') \cong k'$. Since R is homologically of minimal multiplicity of type (m, n, t), Case 1 of our proof shows that $R'/\mathbf{x}R'$ is homologically of minimal multiplicity of type (m, n, t). Since the sequence $\mathbf{x}$ is R'-regular, the descent result in Lemma 5.1 implies that R' is homologically of minimal multiplicity of type (m, n, t). □

Theorem 5.8. *Assume that ψ: $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is a flat local ring homomorphism with $R'/\mathfrak{m}R'$ Gorenstein. If R' is strongly homologically of minimal multiplicity of type (m, n), then R is strongly homologically of minimal multiplicity of type (m, n). The converse holds when k is perfect and $R'/\mathfrak{m}R'$ is regular.*

Proof. The proof is similar to that for Theorem 5.7. □

The next question asks if the converses in Theorems 5.7 and 5.8 hold in general.

Question 5.9. Assume that ψ: $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is a flat local ring homomorphism with Gorenstein closed fibre $R'/\mathfrak{m}R'$. If R is (strongly) homologically of minimal multiplicity, must R' be (strongly) homologically of minimal multiplicity?

The next results contain criteria guaranteeing that a localized tensor product is (strongly) homologically of minimal multiplicity.

Corollary 5.10. *Let $(R, \mathfrak{m}, k)$ and $(R_1, \mathfrak{m}_1, k_1)$ be local k-algebras such that R_1 is Gorenstein and $R \otimes_k R_1$ is noetherian. Set $P = R \otimes_k \mathfrak{m}_1 + \mathfrak{m} \otimes_k R_1$, and $R' = (R \otimes_k R_1)_P$ with maximal ideal $\mathfrak{m}' = PR'$. If R' is homologically of minimal multiplicity of type (m, n, t), then R is homologically of minimal multiplicity of type (m, n, t). The converse holds when k is perfect and R_1 is regular.*

Proof. The natural map $R \to R'$ is flat and local with closed fibre

$$R'/\mathfrak{m}R' \cong R/\mathfrak{m} \otimes_k R_1 \cong k \otimes_k R_1 \cong R_1.$$

The desired conclusions now follow from Theorem 5.7. □

Corollary 5.11. *Let $(R, \mathfrak{m}, k)$ and $(R_1, \mathfrak{m}_1, k_1)$ be local k-algebras such that R_1 is Gorenstein and $R \otimes_k R_1$ is noetherian. Set $P = R \otimes_k \mathfrak{m}_1 + \mathfrak{m} \otimes_k R_1$, and set $R' = (R \otimes_k R_1)_P$ with maximal ideal $\mathfrak{m}' = PR'$. If R' is strongly homologically of minimal multiplicity of type (m, n), then R is strongly homologically of minimal multiplicity of type (m, n). The converse holds when k is perfect and R_1 is regular.*

Proof. This is proved similarly to Corollary 5.10. □

The next questions ask if the converses in Corollaries 5.10 and 5.11 hold when k is not perfect or R_1 is not regular.

Question 5.12. Let $(R, \mathfrak{m}, k)$ and $(R_1, \mathfrak{m}_1, k_1)$ be local k-algebras such that the tensor product $R \otimes_k R_1$ is noetherian. Assume that R_1 is Gorenstein. Set

$P = R \otimes_k \mathfrak{m}_1 + \mathfrak{m} \otimes_k R_1$, and set $R' = (R \otimes_k R_1)_P$ with maximal ideal $\mathfrak{m}' = PR'$. If R is (strongly) homologically of minimal multiplicity, must R' be (strongly) homologically of minimal multiplicity?

Before continuing, we recall the following handy bookkeeping tool.

Definition 5.13. Given a finitely generated R-module M, the *Poincaré series* of M is the formal power series $P_M^R(t) = \sum_{i=0}^{\infty} \beta_i^R(M)t^i$.

The following example shows that the local tensor product of two rings that are strongly homologically of minimal multiplicity need not be homologically of minimal multiplicity. It also shows that, given a flat local homomorphism $R \to R'$, if R and $R'/\mathfrak{m}R'$ are strongly homologically of minimal multiplicity, then R' need not be homologically of minimal multiplicity.

Example 5.14. Assume that k is perfect. Set $R = k[X, Y]/(X, Y)^2$ and $R_1 = k[Z, W]/(Z, W)^2$. These are local artinian rings of minimal multiplicity, type 2 and length 3; see Example 2.4. Hence they are strongly homologically of minimal multiplicity of type $(1, 2)$ by Proposition 2.6. Let ω and ω_1 be canonical modules for R and R_1; their Poincaré series are given by the following formula:

$$P_\omega^R(t) = 2 + 3t\sum_{i=0}^{\infty} 2^i t^i = P_{\omega_1}^{R_1}(t). \tag{5.14.1}$$

The tensor product $R' = R \otimes_k R_1$ is local because it is isomorphic to the local ring $k[X, Y, Z, W]/(X, Y)^2 + (Z, W)^2$. From [14, (2.5.1)], we know that the canonical module of R' is $\omega' = \omega \otimes_k \omega_1$. Thus, the Künneth formula explains the first equality in the next sequence

$$\begin{aligned} P_{\omega'}^{R'}(t) &= P_\omega^R(t)P_{\omega_1}^{R_1}(t) = \left(2 + 3t\sum_{i=0}^{\infty} 2^i t^i\right)^2 \\ P_{\omega'}^{R'}(t) &= 4 + 12t + \sum_{i=2}^{\infty}(9i + 15)2^{i-2}t^i. \end{aligned} \tag{5.14.2}$$

The second equality is from Eq. (5.14.1), and the third one is straightforward.

We show that R' is not homologically of minimal multiplicity. It suffices to show that there are no integers r and t such that $\beta_{t+s}^{R'}(\omega') = r^s \beta_t^{R'}(\omega')$ for all $s \geq 0$; see Theorem 1.3. By way of contradiction, suppose that such integers r and t do exist. Assume without loss of generality that $t \geq 2$. The first two equalities in the next sequence follow directly, and the third one is from Eq. (5.14.2):

$$\begin{aligned} \frac{\beta_{t+1}^{R'}(\omega')}{\beta_t^{R'}(\omega')} &= r = \frac{\beta_{t+2}^{R'}(\omega')}{\beta_{t+1}^{R'}(\omega')} \\ \frac{[9(t+1) + 15]2^{t-1}}{(9t + 15)2^{t-2}} &= \frac{[9(t+2) + 15]2^t}{[9(t+1) + 15]2^{t-1}} \end{aligned}$$

$$[9(t+1)+15]^2 = (9t+15)[9(t+2)+15]$$
$$(9t+24)^2 = (9t+24)^2 - 81.$$

The remaining equalities are straightforward consequences; the final one implies that $0 = -81$, a contradiction.

Next, consider the natural map $R \to R'$, which is flat and local with closed fibre $R'/\mathfrak{m}R \cong R_1$. In particular, the source R and closed fibre $R'/\mathfrak{m}R'$ are strongly homologically of minimal multiplicity, but the target R' is not.

The next results describe our most general ascent and descent properties.

Corollary 5.15. *Assume that ψ: $R \to R'$ is a local, quasi-Gorenstein ring homomorphism. If R' is homologically of minimal multiplicity of type (m, n, t), then R is homologically of minimal multiplicity of type (m, n, t). The converse holds when the residue field k is perfect and when the induced map $\grave{\psi}$: $R \to \widehat{R'}$ admits a Cohen factorization $R \xrightarrow{\dot{\psi}} R'' \xrightarrow{\psi'} \widehat{R'}$ such that* $\mathrm{Ker}(\psi')$ *is contained in the square of the maximal ideal of R''.*

Proof. Assume that R' is homologically of minimal multiplicity of type (m, n, t). Lemma 2.12(a) implies that $\widehat{R'}$ is homologically of minimal multiplicity of type (m, n, t). Let $R \to R'' \to \widehat{R'}$ be a Cohen factorization of the induced map $\grave{\psi}$: $R \to \widehat{R'}$. Lemma 5.1 implies that R'' is homologically of minimal multiplicity of type (m, n, t), and Theorem 5.7 yields the same conclusion for R.

The converse statement is proved similarly. □

Corollary 5.16. *Assume that ψ: $R \to R'$ is a local, quasi-Gorenstein ring homomorphism. If R' is strongly homologically of minimal multiplicity of type (m, n), then R is strongly homologically of minimal multiplicity of type (m, n). The converse holds when the residue field k is perfect and when the induced map $\grave{\psi}$: $R \to \widehat{R'}$ admits a Cohen factorization $R \xrightarrow{\dot{\psi}} R'' \xrightarrow{\psi'} \widehat{R'}$ such that* $\mathrm{Ker}(\psi')$ *is contained in the square of the maximal ideal of R''.*

Proof. This is proved as in Corollary 5.15. □

We conclude with some natural questions.

Question 5.17. Assume that ψ: $R \to R'$ is a local, quasi-Gorenstein ring homomorphism. If R is (strongly) homologically of minimal multiplicity, must R' be (strongly) homologically of minimal multiplicity?

Question 5.18. If R is (strongly) homologically of minimal multiplicity and $\mathfrak{p}$ is a prime ideal of R, must the localization $R_{\mathfrak{p}}$ be (strongly) homologically of minimal multiplicity?

ACKNOWLEDGMENTS

We are grateful to Srikanth Iyengar, Graham Leuschke, Amelia Taylor, and Yuji Yoshino for helpful discussions about this work. We are grateful to the referee for thoughtful comments regarding the manuscript.

REFERENCES

[1] Abhyankar, S. S. (1967). Local rings of high embedding dimension. *Amer. J. Math.* 89:1073–1077.
[2] Avramov, L. L., Foxby, H.-B. (1992). Locally Gorenstein homomorphisms. *Amer. J. Math.* 114:1007–1047.
[3] Avramov, L. L., Foxby, H.-B. (1997). Ring homomorphisms and finite Gorenstein dimension. *Proc. London Math. Soc. (3)* 75:241–270.
[4] Avramov, L. L., Foxby, H.-B., Herzog, B. (1994). Structure of local homomorphisms. *J. Algebra* 164:124–145.
[5] Avramov, L. L., Gasharov, V. N., Peeva, I. V. (1989). A periodic module of infinite virtual projective dimension. *J. Pure Appl. Algebra* 62:1–5.
[6] Christensen, L. W. (2000). *Gorenstein Dimensions*. Lecture Notes in Mathematics, Vol. 1747. Berlin: Springer-Verlag.
[7] Christensen, L. W. (2001). Semi-dualizing complexes and their Auslander categories. *Trans. Amer. Math. Soc.* 353:1839–1883.
[8] Christensen, L. W., Striuli, J., Veliche, O. (2010). Growth in the minimal injective resolution of a local ring. *J. Lond. Math. Soc.* 81:24–44.
[9] Foxby, H.-B. (1972). Gorenstein modules and related modules. *Math. Scand.* 31:267–284.
[10] Gasharov, V. N., Peeva, I. V. (1990). Boundedness versus periodicity over commutative local rings. *Trans. Amer. Math. Soc.* 320:569–580.
[11] Grayson, D. R., Stillman, M. E. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.
[12] Grothendieck, A. (1961). Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.* 11:167.
[13] Hartshorne, R. (1967). *Local Cohomology*. A seminar given by A. Grothendieck, Harvard University, Fall, Vol. 1961. Berlin: Springer-Verlag.
[14] Jorgensen, D. A. (2009). On tensor products of rings and extension conjectures. *J. Comm. Algebra* 1(4):635–646.
[15] Jorgensen, D. A., Leuschke, G. J. (2007). On the growth of the Betti sequence of the canonical module. *Math. Z.* 256:647–659.
[16] Matsumura, H. (1989). *Commutative Ring Theory*. 2nd ed. Studies in Advanced Mathematics, Vol. 8. Cambridge: Cambridge University Press.
[17] Reiten, I. (1972). The converse to a theorem of Sharp on Gorenstein modules. *Proc. Amer. Math. Soc.* 32:417–420.
[18] Sharp, R. Y. (1971). On Gorenstein modules over a complete Cohen–Macaulay local ring. *Quart. J. Math. Oxford Ser. (2)* 22:425–434.