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Descent via Koszul extensions

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Abstract

Let R be a commutative noetherian local ring with completion $\hat{R}$. We apply differential graded (DG) algebra techniques to study descent of modules and complexes from $\hat{R}$ to R' where R' is either the henselization of R or a pointed étale neighborhood of R: We extend a given $\hat{R}$-complex to a DG module over a Koszul complex; we describe this DG module equationally and apply Artin approximation to descend it to R'.

This descent result for Koszul extensions has several applications. When R is excellent, we use it to descend the dualizing complex from $\hat{R}$ to a pointed étale neighborhood of R; this yields a new version of P. Roberts' theorem on uniform annihilation of homology modules of perfect complexes. As another application we prove that the Auslander Condition on uniform vanishing of cohomology ascends to $\hat{R}$ when R is excellent, henselian, and Cohen–Macaulay.

Keywords: Artin approximation; Descent; Koszul extensions; Liftings; Semidualizing; Semi-dualizing complexes

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Introduction

Let $(R, \mathfrak{m})$ be a commutative noetherian local ring with $\mathfrak{m}$-adic completion $\hat{R}$. We investigate a pervasive question in local algebra: When is a given $\hat{R}$-module N extended from R, i.e., when is there an R-module M such that $N \cong \hat{R} \otimes_R M$?

If R is Cohen–Macaulay, a classical approach to this question is a two-step analysis that treats the ring and the module separately. Let $\boldsymbol{x}$ be a maximal R-regular sequence and consider the commutative diagram of local ring homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ R/(\boldsymbol{x}) \arrow[r, "\cong"] & \hat{R}/(\boldsymbol{x}). \end{tikzcd} \tag{$*$}$$

The bottom map is an isomorphism because $\boldsymbol{x}$ is a system of parameters. For every finitely generated $\hat{R}$-module N, the module $N/\boldsymbol{x}N$ is finitely generated over $R/(\boldsymbol{x})$ and, hence, also over R. The first step is to identify conditions on R guaranteeing that $N/\boldsymbol{x}N$ has the form $M/\boldsymbol{x}M$ for some finitely generated R-module M. The next step is to identify conditions on N such that the isomorphism $N/\boldsymbol{x}N \cong M/\boldsymbol{x}M$ forces an isomorphism $N \cong \hat{R} \otimes_R M$. Often, this second step hinges on the good homological properties of the vertical maps.

If R is not Cohen–Macaulay, then this construction is problematic. A maximal R-regular sequence is not a system of parameters, so the map $R/(\boldsymbol{x}) \to \hat{R}/(\boldsymbol{x})$ will not be an isomorphism in general. This could be remedied by replacing $\boldsymbol{x}$ with a system of parameters, but then the good homological properties of the vertical maps would be lost. To circumvent these problems, we leave the realm of rings.

We replace the rings $R/(\boldsymbol{x})$ and $\hat{R}/(\boldsymbol{x})$ in $(*)$ with Koszul complexes $K^R(\boldsymbol{a})$ and $K^{\hat{R}}(\boldsymbol{a})$ where $\boldsymbol{a}$ is a list of elements in $\mathfrak{m}$ such that $R/(\boldsymbol{a})$ is complete. For example, $\boldsymbol{a}$ can be a system of parameters or a generating sequence for $\mathfrak{m}$. The natural morphism $K^R(\boldsymbol{a}) \to K^{\hat{R}}(\boldsymbol{a})$ is a homology isomorphism and induces an equivalence between the derived categories $\mathsf{D}(K^R(\boldsymbol{a}))$ and $\mathsf{D}(K^{\hat{R}}(\boldsymbol{a}))$ of differential graded (DG) modules. Hence the resulting commutative diagram of differential graded algebra homomorphisms supports a two-step analysis parallel to the one described above.

In step one, contained in (3.3), we identify conditions on R under which a DG module over $K^{\hat{R}}(\boldsymbol{a})$ that is extended from $\hat{R}$ is also extended from R.

Theorem A. *Let $(R, \mathfrak{m})$ be an excellent henselian local ring and $\boldsymbol{a} \in \mathfrak{m}$ a list of elements such that $R/(\boldsymbol{a})$ is complete. For every $\hat{R}$-complex N whose homology is finitely generated over $\hat{R}$, there is a complex M of finitely generated R-modules such that $K^{\hat{R}}(\boldsymbol{a}) \otimes_R M$ is isomorphic to $K^{\hat{R}}(\boldsymbol{a}) \otimes_{\hat{R}} N$ in the derived category $\mathsf{D}(K^{\hat{R}}(\boldsymbol{a}))$.*

In this theorem, if N is a module then, under additional conditions on $\boldsymbol{a}$ or N, also M is a module. As applications we obtain the next two theorems. The first is contained in (4.2); it extends (the commutative case of) lifting results of Auslander, Ding, and Solberg [4]; for definitions see (4.1).

Theorem B. *Let $(R, \mathfrak{m})$ be an excellent henselian local ring and $\boldsymbol{x} \in \mathfrak{m}$ an R-regular sequence such that $S = R/(\boldsymbol{x})$ is complete. Let N be a finitely generated S-module. If $\operatorname{Ext}_S^2(N, N) = 0$, then N has a lifting to R. If $\operatorname{Ext}_S^1(N, N) = 0$, then any two liftings of N to R are isomorphic.*

The second application, contained in (4.4), is an ascent result for Auslander's conditions on vanishing of cohomology for finitely generated modules; see (4.3).

Theorem C. *Let R be an excellent henselian Cohen–Macaulay local ring. If R satisfies the (Uniform) Auslander Condition, then so does $\hat{R}$.*

In step two of the analysis, we give a condition on N sufficient to ensure that an isomorphism of DG modules $K^{\hat{R}}(\boldsymbol{a}) \otimes_R M \simeq K^{\hat{R}}(\boldsymbol{a}) \otimes_{\hat{R}} N$ forces an isomorphism of complexes $\hat{R} \otimes_R M \cong N$. Semidualizing $\hat{R}$-complexes (5.3) satisfy this condition, and we obtain Theorem D, which is part of (5.4). It subsumes Hinich's result [19] that an excellent henselian ring admits a dualizing complex; see also Rotthaus [29].

Theorem D. *Let R be an excellent henselian local ring. There is a bijective correspondence, induced by the functor $\hat{R} \otimes_R -$, between the sets of (shift-)isomorphism classes of semidualizing complexes in the derived categories $\mathsf{D}(R)$ and $\mathsf{D}(\hat{R})$.*

Semidualizing complexes also furnish an example of how the conclusion of Theorem A may fail for rings that are not excellent and henselian; see (5.5).

Much of this work is done in a setting broader than suggested by the above results. In (5.4) we show that, if R is excellent, then every semidualizing $\hat{R}$-complex descends to the henselization R^h and, moreover, that any finite collection of semidualizing $\hat{R}$-complexes descends to a pointed étale neighborhood of R. This allows us to prove, in (6.1), a new version of Roberts' theorem [27] on uniform annihilation of homology modules of perfect complexes. This, in turn, applies to recover a recent result of Zhou [36] on uniform annihilation of local cohomology modules.

As to the organization of the paper, background material is collected in Section 1, and Theorem A is proved in Sections 2–3. Applications, including Theorems B and C, are given in Sections 4 and 6. Theorem D is proved in Section 5.

1. Algebra and module structures

In this paper, $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring with $\mathfrak{m}$-adic completion $(\hat{R}, \hat{\mathfrak{m}}, k)$. For a list of elements $\boldsymbol{a} = a_1, \ldots, a_e$ in $\mathfrak{m}$, we denote the Koszul complex on $\boldsymbol{a}$ by $K^R(\boldsymbol{a})$. If $R/(\boldsymbol{a})$ is complete, then we call $\boldsymbol{a}$ a *co-complete sequence*.

For the rest of this section, fix a list of elements $\boldsymbol{a} \in \mathfrak{m}$ and set $K^R = K^R(\boldsymbol{a})$.

1.1. Complexes. We employ homological grading for complexes of R-modules

$$M = \cdots \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} \cdots$$

and call them *R-complexes* for short. Let M be an R-complex and m an integer. The *m-fold shift* of M is denoted $\Sigma^m M$; it is the complex with $(\Sigma^m M)_n = M_{n-m}$ and $\partial_n^{\Sigma^m M} = (-1)^m \partial_{n-m}^M$. The *hard right truncation of M at m*, denoted $M_{\geqslant m}$, is given by

$$(M_{\geqslant m})_n = \begin{cases} M_n & \text{if } n \geqslant m, \\ 0 & \text{if } n < m \end{cases} \quad \text{and} \quad \partial_n^{M_{\geqslant m}} = \begin{cases} \partial_n^M & \text{if } n > m, \\ 0 & \text{if } n \leqslant m. \end{cases}$$

The *hard left truncation of M at m* is denoted $M_{\leqslant m}$ and defined similarly.

A complex M is *bounded* if $M_n = 0$ when $|n| \gg 0$. The quantities $\sup M$ and $\inf M$ are the supremum and infimum, respectively, of the set $\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$. We say that M is *homologically bounded* if $\mathrm{H}(M)$ is bounded, and M is *homologically degreewise finite* if each module $\mathrm{H}_n(M)$ is finitely generated. A complex is *homologically finite* if it is homologically both bounded and degreewise finite.

Isomorphisms in the category of R-complexes are identified by the symbol $\cong$. Isomorphisms in $\mathsf{D}(R)$, the derived category of the category of R-modules, are identified by the symbol $\simeq$. A morphism α between R-complexes corresponds to an isomorphism in $\mathsf{D}(R)$ if and only if the induced morphism $\mathrm{H}(\alpha)$ in homology is an isomorphism or, equivalently, the mapping cone $\mathrm{Cone}\,\alpha$ is exact; when these conditions are satisfied, α is called a *quasiisomorphism*.

Every R-complex M has a semifree resolution $P \xrightarrow{\simeq} M$; see [12, Prop. 6.6]. Such resolutions allow definition of derived tensor product and Hom functors $- \otimes_R^{\mathbf{L}} -$ and $\mathbf{R}\mathrm{Hom}_R(-,-)$ because the functors $P \otimes_R -$ and $\mathrm{Hom}_R(P,-)$ preserve quasiisomorphisms of R-complexes.

The Koszul complex K^R is a bounded complex of finite rank free R-modules, in particular, it is semifree and so the functors $K^R \otimes_R -$ and $K^R \otimes_R^{\mathbf{L}} -$ are naturally isomorphic. This fact will be used without further mention. If M is homologically degreewise finite, then [13, 1.3] provides the (in)equalities

$$\inf\big(K^R \otimes_R M\big) = \inf M \quad \text{and} \quad \sup M \leqslant \sup\big(K^R \otimes_R M\big) \leqslant e + \sup M. \tag{1.1.1}$$

Hence, the complexes M and $K^R \otimes_R M$ are simultaneously homologically bounded.

It is straightforward to verify the following special case of tensor-evaluation. For R-complexes M and N there is an isomorphism in $\mathsf{D}(R)$

$$K^R \otimes_R \mathbf{R}\mathrm{Hom}_R(M,N) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R\big(M, K^R \otimes_R N\big). \tag{1.1.2}$$

1.2. DG modules over Koszul complexes. The Koszul complex K^R can be realized as an exterior algebra, and the wedge product endows it with a differential graded (DG)[3] algebra structure that is commutative; see e.g. [8, Prop. 1.6.2]. That is, the product is unitary and associative, and it satisfies

$$uv = (-1)^{|u||v|} vu \quad \text{and} \quad u^2 = 0 \quad \text{when } |u| \text{ is odd}$$
$$\partial^{K^R}(uv) = \partial^{K^R}(u)v + (-1)^{|u|} u \partial^{K^R}(v)$$

for all u, v in K^R, where $|u|$ denotes the degree of u.

[3] Once available, [7] will be an authoritative reference for DG algebra.

A *DG K^R-module* is an R-complex M equipped with a K^R-multiplication: a morphism of R-complexes $K^R \otimes_R M \to M$, written $u \otimes m \mapsto um$, that is unitary and associative and satisfies the Leibniz rule

$$\partial^M(um) = \partial^{K^R}(u)m + (-1)^{|u|}u\partial^M(m)$$

for all $u \in K^R$ and $m \in M$. A DG K^R-module M is *homologically finite* if the homology module $\mathrm{H}(M)$ is finitely generated over $\mathrm{H}_0(K^R) \cong R/(\boldsymbol{a})$, equivalently if M is homologically finite as an R-complex.

If M is an R-complex, then $K^R \otimes_R M$ has a DG K^R-module structure given by $u(v \otimes x) = (uv) \otimes x$. Moreover, if M is a homologically finite R-complex, then $K^R \otimes_R M$ is a homologically finite DG K^R-module.

A morphism of DG K^R-modules is a morphism of R-complexes that is also K^R-linear. Isomorphisms in the category of DG K^R-modules are identified by the symbol $\cong$. The derived category of the category of DG K^R-modules is denoted $\mathsf{D}(K^R)$; isomorphisms in this category are identified by the symbol $\simeq$. A morphism of DG K^R-modules corresponds to an isomorphism in $\mathsf{D}(K^R)$ if and only if it does so in $\mathsf{D}(R)$ and is then called a *quasiisomorphism*.

Every DG K^R-module M has a semifree resolution $P \xrightarrow{\simeq} M$; see [12, Prop. 6.6]. Such resolutions allow definition of derived tensor product and Hom functors, $- \otimes^{\mathbf{L}}_{K^R} -$ and $\mathbf{R}\mathrm{Hom}_{K^R}(-,-)$ because the functors $P \otimes_{K^R} -$ and $\mathrm{Hom}_{K^R}(P,-)$ preserve quasiisomorphisms of DG K^R-modules.

1.3. Local homomorphisms and Koszul complexes. Let $\vartheta : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local ring homomorphism, that is, $\vartheta(\mathfrak{m}) \subseteq \mathfrak{n}$. Set $K^S = K^S(\vartheta(\boldsymbol{a}))$; there is then an isomorphism of S-complexes and of DG K^R-modules

$$S \otimes_R K^R \cong K^S. \tag{1.3.1}$$

Assume ϑ is flat, i.e. it gives S the structure of a flat R-module, and assume $\hat{R} \cong \hat{S}$. If $\boldsymbol{a}$ is co-complete, then ϑ induces an isomorphism of rings $R/(\boldsymbol{a}) \cong S/(\vartheta(\boldsymbol{a}))$ and, further, a quasi-isomorphism of R-complexes

$$K^R \xrightarrow{\simeq} K^S \tag{1.3.2}$$

which also respects the DG algebra structures. In particular, every (homologically finite) DG K^S-module is a (homologically finite) DG K^R-module. Moreover, the functor $K^S \otimes_{K^R} -$ is an equivalence between $\mathsf{D}(K^R)$ and $\mathsf{D}(K^S)$; it conspires with (1.3.2) to yield an isomorphism in $\mathsf{D}(K^S)$

$$K^S \xrightarrow{\simeq} K^S \otimes_{K^R} K^S. \tag{1.3.3}$$

The homology inverse is the multiplication morphism.

2. Equational descriptions of Koszul extensions

The next lemma is a crucial step towards Theorem A from the introduction.

2.1. Lemma. *Let $(R,\mathfrak{m})$ be a local ring, m a positive integer, and P a complex of finite rank free $\hat{R}$-modules such that $P_n = 0$ when $n < 0$ or $n > m$. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^R = K^R(\boldsymbol{a})$ and $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$. There exists a finite system $\mathcal{S}$ of polynomial equations with coefficients in R such that:*

(a) *The system $\mathcal{S}$ has a solution in $\hat{R}$.*
(b) *A solution to $\mathcal{S}$ in R yields a complex A of finite rank free R-modules such that $A_n = 0$ when $n < 0$ or $n > m$, and $K^{\hat{R}} \otimes_R A \simeq K^{\hat{R}} \otimes_{\hat{R}} P$ in $\mathsf{D}(K^{\hat{R}})$.*

The rest of the section is devoted to the proof of this result; the argument proceeds in ten steps, the first of which sets up notation.

2.2. Differentials on K^R and $K^{\hat{R}}$. Fix a basis $\varepsilon_1, \ldots, \varepsilon_{2^e}$ for K^R over R. For each $i = 1, \ldots, 2^e$ set $\hat{\varepsilon}_i = 1 \otimes \varepsilon_i \in K^{\hat{R}}$, cf. (1.3.1). The elements $\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_{2^e}$ form a basis for $K^{\hat{R}}$ over $\hat{R}$. The differential $\partial_n^{K^R}$ is given by a matrix of size $\binom{e}{n-1} \times \binom{e}{n}$ with entries in R

$$K_n^R \xrightarrow{[d_{nij}]} K_{n-1}^R.$$

Note that $\partial_n^{K^R} = [d_{nij}] = 0$ when $n < 1$ or $n > e$. By (1.3.1) the matrix $[d_{nij}]$ also describes the nth differential on $K^{\hat{R}}$.

Multiplication on the degree n component of K^R by a basis vector ε_h is given by a matrix of size $\binom{e}{n+|\varepsilon_h|} \times \binom{e}{n}$ with entries in R

$$K_n^R \xrightarrow{[t_{nij}^h]} K_{n+|\varepsilon_h|}^R.$$

By (1.3.1) the matrices $[t_{nij}^h]$ also describe multiplication by $\hat{\varepsilon}_h$ on $K^{\hat{R}}$.

2.3. A resolution of $K^{\hat{R}} \otimes_{\hat{R}} P$ over K^R. Without loss of generality, we can assume the complex P is minimal; that is, $\partial^P(P) \subseteq \mathfrak{m}P$. The DG $K^{\hat{R}}$-module $K^{\hat{R}} \otimes_{\hat{R}} P$ is a homologically finite DG K^R-module through (1.3.2). By [1, Prop. 2] there exists a K^R-semifree resolution $F \xrightarrow{\simeq} K^{\hat{R}} \otimes_{\hat{R}} P$, such that the differential of $k \otimes_{K^R} F$ is 0 and $F^\natural = \coprod_{i \geqslant 0} \Sigma^i ((K^R)^\natural)^{\beta_i}$ with $\beta_i \in \mathbb{N}_0$. (Here $F^\natural$ denotes the graded R-module underlying the DG K^R-module F.) Applying the functor $K^{\hat{R}} \otimes_{K^R} -$ yields a $K^{\hat{R}}$-semifree resolution

$$K^{\hat{R}} \otimes_{K^R} F \xrightarrow{\simeq} K^{\hat{R}} \otimes_{K^R} K^{\hat{R}} \otimes_{\hat{R}} P,$$

and (1.3.3) induces a quasiisomorphism of DG $K^{\hat{R}}$-modules

$$K^{\hat{R}} \otimes_{\hat{R}} P \xrightarrow{\simeq} K^{\hat{R}} \otimes_{K^R} K^{\hat{R}} \otimes_{\hat{R}} P.$$

As $K^{\hat{R}} \otimes_{K^R} F$ is $K^{\hat{R}}$-semifree, [12, Prop. 6.4] provides a $K^{\hat{R}}$-morphism

$$\varphi : K^{\hat{R}} \otimes_{K^R} F \xrightarrow{\simeq} K^{\hat{R}} \otimes_{\hat{R}} P. \tag{2.3.1}$$

Since φ is a quasiisomorphism between semifree DG $K^{\hat{R}}$-modules, the induced map

$$k \otimes_{K^{\hat{R}}} \varphi : k \otimes_{K^{\hat{R}}} \left(K^{\hat{R}} \otimes_{K^R} F\right) \longrightarrow k \otimes_{K^{\hat{R}}} \left(K^{\hat{R}} \otimes_{\hat{R}} P\right)$$

is a quasiisomorphism; see [12, Prop. 6.7]. Each of these complexes has zero differential (P is minimal), so $k \otimes_{K^{\hat{R}}} \varphi$ is an isomorphism, hence the underlying graded k-vector spaces have the same rank. The semifreeness of $K^{\hat{R}} \otimes_{K^R} F$ and $K^{\hat{R}} \otimes_{\hat{R}} P$ over $K^{\hat{R}}$ implies

$$\operatorname{rank}_R F_n = \operatorname{rank}_{\hat{R}}\left(K^{\hat{R}} \otimes_{K^R} F\right)_n = \operatorname{rank}_{\hat{R}}\left(K^{\hat{R}} \otimes_{\hat{R}} P\right)_n \tag{2.3.2}$$

for all n. In particular, $F_n = 0$ when $n < 0$ or $n > m + e$.

2.4. DG structures on F and $K^{\hat{R}} \otimes_{K^R} F$. For each integer n, set $r_n = \operatorname{rank}_R F_n$ and fix an R-basis for F_n. The differential ∂_n^F is given by an $r_{n-1} \times r_n$ matrix

$$F_n \xrightarrow{[u_{nij}]} F_{n-1}$$

with entries in R. Note that $[u_{nij}] = 0$ when $n < 1$ or $n > m + e$. There is an isomorphism of DG $K^{\hat{R}}$-modules

$$K^{\hat{R}} \otimes_{K^R} F \cong \hat{R} \otimes_R F \tag{2.4.1}$$

cf. (1.3.1). In the $\hat{R}$-basis induced by the R-basis for F, the nth differential on $K^{\hat{R}} \otimes_{K^R} F$ is also given by the matrix $[u_{nij}]$.

For each basis vector $\varepsilon_h \in K^R$, cf. (2.2), multiplication by ε_h on F is given by matrices with entries in R

$$F_n \xrightarrow{[v^h_{nij}]} F_{n+|\varepsilon_h|}.$$

By (2.4.1) the matrices $[v^h_{nij}]$ also describe multiplication by $\hat{\varepsilon}_h$ on $K^{\hat{R}} \otimes_{K^R} F$.

2.5. DG structure on $K^{\hat{R}} \otimes_{\hat{R}} P$. For each integer n, fix a basis for the free $\hat{R}$-module P_n and set $s_n = \operatorname{rank}_{\hat{R}} P_n$. The nth differential of P is then given by an $s_{n-1} \times s_n$ matrix with entries in $\hat{R}$

$$P_n \xrightarrow{[x_{nij}]} P_{n-1}$$

which is zero when $n > m$ or $n < 1$. In the basis on $K^{\hat{R}} \otimes_{\hat{R}} P$, coming from the bases chosen for $K^{\hat{R}}$ and the modules $P_0, \dots, P_m$, the differential

$$\partial_n^{K^{\hat{R}} \otimes_{\hat{R}} P} : \bigoplus_{p=0}^{m} K^{\hat{R}}_{n-p} \otimes_{\hat{R}} P_p \longrightarrow \bigoplus_{q=0}^{m} K^{\hat{R}}_{n-1-q} \otimes_{\hat{R}} P_q$$

is given by a block matrix

$$\partial_n^{K^{\hat{R}} \otimes_{\hat{R}} P} = \begin{pmatrix} [b_{nij}^{00}] & \cdots & [b_{nij}^{0m}] \\ \vdots & & \vdots \\ [b_{nij}^{m0}] & \cdots & [b_{nij}^{mm}] \end{pmatrix} = \left[b_{nij}^{qp}\right] \tag{2.5.1}$$

where

$$b_{nij}^{qp} = \begin{cases} d_{(n-p)ij} & \text{if } p = q, \\ (-1)^{n-p} x_{pij} & \text{if } p = q+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Note that $\partial_n^{K^{\hat{R}} \otimes_{\hat{R}} P} = 0$ when $n < 1$ or $n > m + e$.

For each basis vector $\hat{\varepsilon}_h \in K^{\hat{R}}$, cf. (2.2), multiplication by $\hat{\varepsilon}_h$ on $K^{\hat{R}} \otimes_{\hat{R}} P$ is given by the formula $\hat{\varepsilon}_h(f \otimes g) = (\hat{\varepsilon} f) \otimes g$; see (1.2). In the chosen basis for $K^{\hat{R}} \otimes_{\hat{R}} P$, multiplication by $\hat{\varepsilon}_h$ on the degree n component is given by

$$\bigoplus_{p=0}^{m} \left[t_{(n-p)ij}^{h}\right] \otimes_{\hat{R}} P_p : \bigoplus_{p=0}^{m} K_{n-p}^{\hat{R}} \otimes_{\hat{R}} P_p \longrightarrow \bigoplus_{p=0}^{m} K_{n-p+|\hat{\varepsilon}_h|}^{\hat{R}} \otimes_{\hat{R}} P_p.$$

Hence, multiplication by $\hat{\varepsilon}_h$ on $K^{\hat{R}} \otimes_{\hat{R}} P$ is given by matrices with entries in R

$$\left(K^{\hat{R}} \otimes_{\hat{R}} P\right)_n \xrightarrow{[w_{nij}^h]} \left(K^{\hat{R}} \otimes_{\hat{R}} P\right)_{n+|\varepsilon_h|}.$$

2.6. First set of variables. We introduce a finite set of variables

$$\{X_{nij} \mid n = 1, \dots, m;\ i = 1, \dots, s_{n-1};\ j = 1, \dots, s_n\}.$$

The equality $\partial_n^P \partial_{n+1}^P = 0$ says that the elements $x_{nij} \in \hat{R}$ satisfy a system $\mathcal{S}_1$ of quadratic equations in the variables X_{nij} with coefficients 1 and 0, namely the system coming from the matrix equations

$$[X_{nij}][X_{(n+1)ij}] = [0] \quad \text{for } n = 1, \dots, m-1. \tag{2.6.1}$$

For later reference, define $[B_{nij}^{qp}]$ for $p, q = 0, \dots, m$ and $n = 1, \dots, m$ to be the block matrix described as in (2.5.1) and (2.5.2) by

$$B_{nij}^{qp} = \begin{cases} d_{(n-p)ij} & \text{if } p = q, \\ (-1)^{n-p} X_{pij} & \text{if } p = q+1, \\ 0 & \text{otherwise.} \end{cases}$$

2.7. The map φ. Since φ is a morphism, it satisfies the equation

$$\varphi_{n-1} \partial_n^{K^{\hat{R}} \otimes_{K^R} F} - \partial_n^{K^{\hat{R}} \otimes_{\hat{R}} P} \varphi_n = 0 \tag{2.7.1}$$

for each n. The $K^{\hat{R}}$-linearity of φ means that there are equalities

$$\varphi(\hat{\varepsilon}_h f) = \hat{\varepsilon}_h \varphi(f)$$

for all $f \in K^{\hat{R}} \otimes_{K^R} F$ and $h = 1, \ldots, 2^e$. Each map φ_n is between free $\hat{R}$-modules of rank r_n, cf. (2.3.2), so it is given by an $r_n \times r_n$ matrix with entries in $\hat{R}$

$$\big(K^{\hat{R}} \otimes_{K^R} F\big)_n \xrightarrow{[y_{nij}]} \big(K^{\hat{R}} \otimes_{\hat{R}} P\big)_n$$

which is 0 when $n < 0$ or $n > m + e$. The $K^{\hat{R}}$-linearity of φ can, therefore, be expressed by commutativity of diagrams

$$\begin{tikzcd}
(K^{\hat{R}} \otimes_{K^R} F)_n \arrow[r, "{[y_{nij}]}"] \arrow[d, "{[v^h_{nij}]}"] & (K^{\hat{R}} \otimes_{\hat{R}} P)_n \arrow[d, "{[w^h_{nij}]}"'] \\
(K^{\hat{R}} \otimes_{K^R} F)_{n+|\varepsilon_h|} \arrow[r, "{[y_{(n+|\varepsilon_h|)ij}]}"] & (K^{\hat{R}} \otimes_{\hat{R}} P)_{n+|\varepsilon_h|}
\end{tikzcd} \tag{2.7.2}$$

for $n = 0, \ldots, m + e$ and $h = 1, \ldots, 2^e$.

2.8. Second set of variables. We introduce another finite set of variables

$$\{Y_{nij} \mid n = 0, \ldots, m + e;\ i = 1, \ldots, r_n;\ j = 1, \ldots, r_n\}.$$

By (2.7.1) the elements $x_{nij}, y_{nij} \in \hat{R}$ satisfy a system $\mathcal{S}_2$ of equations in the variables X_{nij}, Y_{nij} with coefficients in R, namely the system coming from the matrix equations

$$[Y_{(n-1)ij}][u_{nij}] - \big[B^{qp}_{nij}\big][Y_{nij}] = [0] \quad \text{for } n = 1, \ldots, m + e. \tag{2.8.1}$$

By (2.7.2) the elements y_{nij} satisfy a second system $\mathcal{S}_3$ of equations in Y_{nij} with coefficients in R, namely those coming from the matrix equations

$$[Y_{(n+|\varepsilon_h|)ij}]\big[v^h_{nij}\big] - \big[w^h_{nij}\big][Y_{nij}] = [0] \tag{2.8.2}$$

for $h = 1, \ldots, 2^e$ and $n = 0, \ldots, m + e - |e_h|$.

2.9. The mapping cone of φ. The complex $\operatorname{Cone}\varphi$ consists of finite rank free $\hat{R}$-modules. In the basis for $\operatorname{Cone}\varphi$ coming from the bases chosen for $K^{\hat{R}} \otimes_{K^R} F$ and $K^{\hat{R}} \otimes_{\hat{R}} P$, the nth differential is given by the block matrix

$$\partial_n^{\operatorname{Cone}\varphi} = \begin{pmatrix} [b^{qp}_{nij}] & [y_{(n-1)ij}] \\ [0] & -[u_{(n-1)ij}] \end{pmatrix}$$

which is zero when $n < 1$ or $n > m + e + 1$. Since φ is a quasiisomorphism, the mapping cone is an exact complex of free $\hat{R}$-modules and bounded (below). Hence, there exists a homotopy between 0 and the identity on $\operatorname{Cone}\varphi$, i.e. a degree 1 homomorphism σ on $\operatorname{Cone}\varphi$ such that

$$\sigma_{n-1}\partial_n^{\operatorname{Cone}\varphi} + \partial_{n+1}^{\operatorname{Cone}\varphi}\sigma_n = 1_n^{\operatorname{Cone}\varphi} \tag{2.9.1}$$

for every n. Each map σ_n is given by a matrix of size $(r_{n+1} + r_n) \times (r_n + r_{n-1})$ with entries in $\hat{R}$

$$(\operatorname{Cone}\varphi)_n \xrightarrow{[z_{nij}]} (\operatorname{Cone}\varphi)_{n+1}$$

which is 0 when $n < 0$ or $n > m + e$.

2.10. Third set of variables. We introduce a third finite set of variables

$$\{Z_{nij} \mid n = 0, \dots, m+e;\ i = 1, \dots, r_{n+1} + r_n;\ j = 1, \dots, r_n + r_{n-1}\}.$$

Eq. (2.9.1) means that the elements x_{nij}, y_{nij}, and z_{nij} satisfy a system of equations in X_{nij}, Y_{nij}, and Z_{nij} with coefficients in R, namely the system $\mathcal{S}_4$ coming from the matrix equations

$$[Z_{(n-1)ij}]\begin{pmatrix} [B^{qp}_{nij}] & [Y_{(n-1)ij}] \\ [0] & -[u_{(n-1)ij}] \end{pmatrix} + \begin{pmatrix} [B^{qp}_{(n+1)ij}] & [Y_{nij}] \\ [0] & -[u_{nij}] \end{pmatrix}[Z_{nij}] = [\delta_{ij}] \tag{2.10.1}$$

for $n = 0, \dots, m + e + 1$, where δ_{ij} is the Kronecker delta.

2.11. Solutions to $\mathcal{S}$. By construction, the system $\mathcal{S} = \bigsqcup_{i=1}^4 \mathcal{S}_i$ has a solution in $\hat{R}$, namely x_{nij}, y_{nij}, z_{nij}; see (2.6.1), (2.8.1), (2.8.2), and (2.10.1). This proves (a).

For part (b), assume that $\mathcal{S}$ has a solution $\tilde{x}_{nij}$, $\tilde{y}_{nij}$, $\tilde{z}_{nij}$ in R. In view of the isomorphism φ, see (2.3.1), it suffices to show that this yields a complex A of finite rank free R-modules such that $K^R \otimes_R A \simeq F$ in $\mathsf{D}(K^R)$ and $A_n = 0$ when $n < 0$ or $n > m$. For each n, let A_n be a free R-module of rank $s_n = \operatorname{rank}_{\hat{R}} P_n$ and fix an R-basis for A_n; note that $A_n = 0$ when $n < 0$ or $n > m$. For each n let

$$\begin{aligned}
&\partial_n^A : A_n \longrightarrow A_{n-1} && \text{be given by } \partial_n^A = [\tilde{x}_{nij}],\\
&\tilde{\varphi}_n : F_n \longrightarrow \big(K^R \otimes_R A\big)_n && \text{be given by } \tilde{\varphi}_n = [\tilde{y}_{nij}],\\
&\tilde{\sigma}_n : (\operatorname{Cone}\tilde{\varphi})_n \longrightarrow (\operatorname{Cone}\tilde{\varphi})_{n+1} && \text{be given by } \tilde{\sigma}_n = [\tilde{z}_{nij}].
\end{aligned}$$

Since the elements $\tilde{x}_{nij}$ satisfy $\mathcal{S}_1$, one has $\partial_n^A\partial_{n+1}^A = 0$; so A is a complex. The elements $\tilde{x}_{nij}$, $\tilde{y}_{nij}$ satisfy $\mathcal{S}_2$ and $\mathcal{S}_3$, so the map $\tilde{\varphi}$ is a K^R-linear morphism of R-complexes. Moreover, the elements $\tilde{x}_{nij}$, $\tilde{y}_{nij}$, $\tilde{z}_{nij}$ satisfy $\mathcal{S}_4$, so the map $\tilde{\sigma}$ is a homotopy between 0 and the identity on $\operatorname{Cone}\tilde{\varphi}$. In particular, the cone is exact and, therefore, $\tilde{\varphi}$ is the desired quasiisomorphism. $\square$

3. Descent of Koszul extensions

In this section we accomplish step one of the analysis described in the introduction. In particular, Theorem A is a special case of (3.3)(a).

3.1. The approximation property. The ring R is said to have the *approximation property* if it satisfies the following: Given any finite system $\mathcal{S}$ of polynomial equations with coefficients in R and any positive integer t, if $\mathcal{S}$ has a solution in $\hat{R}$ then it also has a solution in R, which can be chosen congruent to the original solution modulo $\hat{\mathfrak{m}}^t$.

By work of D. Popescu [25, Thm. (1.3)], Rotthaus [28, Thm. 1], and Spivakovsky [32, Thm. 11.3], a local ring has the approximation property if and only if it is excellent and henselian. For example, every local analytic algebra over a perfect field has the approximation property [31]; see also [34, (1.19)].

3.2. Henselization. A pointed étale neighborhood of R is a flat local homomorphism $R \to R' = R[X]_{\mathfrak{n}}/(f)$, where f is a monic polynomial whose derivative is a unit in $R[X]_{\mathfrak{n}}$, and $\mathfrak{n}$ is a prime ideal lying over $\mathfrak{m}$. The set of pointed étale neighborhoods of R forms a filtered direct system $\{R_\lambda \mid \lambda \in \Lambda\}$, and the henselization of R is the limit $R^h = \varinjlim_\lambda R_\lambda$. The natural map $R \to R^h$ is a flat local ring homomorphism, and there is an isomorphism $\widehat{R^h} \cong \hat{R}$. See [16, §18] and [26].

Assume R is excellent; by [16, Cor. (18.7.6)] and (3.1) the henselization R^h then has the approximation property. Let $\mathcal{S}$ be a finite set of polynomial equations with coefficients in R. If $\mathcal{S}$ has a solution in $\hat{R}$, then $\mathcal{S}$ has a solution in R^h, and it follows that there is a pointed étale neighborhood $R \to R'$, such that $\mathcal{S}$ has a solution in R'.

3.3. Theorem. *Let $(R, \mathfrak{m})$ be an excellent local ring. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$.*

(a) *For every homologically finite $\hat{R}$-complex N, there exists a homologically finite R^h-complex M such that $K^{\hat{R}} \otimes_{R^h} M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ in $\mathsf{D}(K^{\hat{R}})$.*
(b) *For every list of homologically finite $\hat{R}$-complexes $N^{(1)}, \dots, N^{(t)}$ there exists a pointed étale neighborhood $R \to R'$ and homologically finite R'-complexes $M^{(1)}, \dots, M^{(t)}$ such that $K^{\hat{R}} \otimes_{R'} M^{(i)} \simeq K^{\hat{R}} \otimes_{\hat{R}} N^{(i)}$ in $\mathsf{D}(K^{\hat{R}})$.*

Proof. Let N be any homologically finite $\hat{R}$ complex and identify it with its minimal semifree resolution; see [1, Prop. 2]. After a shift, we may assume that $N_n = 0$ for $n < 0$. Set $s = \sup N$ and $m = s + 2e + 1$. Consider the complex $P = N_{\leqslant m}$ and the system $\mathcal{S}$ of equations, whose existence and solvability in $\hat{R}$ is given by Lemma (2.1). It is sufficient to prove the following:

3.3.1. Claim. *If the system $\mathcal{S}$ has a solution in R, then there exists a homologically finite R-complex M such that $K^{\hat{R}} \otimes_R M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ in $\mathsf{D}(K^{\hat{R}})$.*

Indeed, R^h has the approximation property, see (3.1), so part (a) follows by applying (3.3.1) to $R = R^h$. A list $N^{(1)}, \dots, N^{(t)}$ of homologically finite $\hat{R}$-complexes also results in a finite system $\mathcal{S}^{(1)} \sqcup \cdots \sqcup \mathcal{S}^{(t)}$ of polynomial equations. As noted in (3.2), there is a pointed étale neighborhood $R \to R'$, such that the compound system and, in particular, each subsystem $\mathcal{S}^{(i)}$ has a solution in R'. Part (b) now follows by applying (3.3.1) to $R = R'$.

Proof of (3.3.1). By Lemma (2.1) there exists a complex A of finite rank free R-modules such that $A_n = 0$ when $n < 0$ or $n > m$, and

$$K^{\hat{R}} \otimes_R A \simeq K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m} \tag{1}$$

in $\mathsf{D}(K^{\hat{R}})$. Augment A by an R-free resolution of $\operatorname{Ker} \partial_m^A$; this yields a complex M of finite rank free R-modules with $\sup M < m$ and $M_{\leqslant m} \cong A$. In particular, the isomorphism (1) can be rewritten as

$$K^{\hat{R}} \otimes_R M_{\leqslant m} \simeq K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m}. \tag{2}$$

Next we show that $\sup(K^{\hat{R}} \otimes_R M) < m$. Apply $K^{\hat{R}} \otimes_{\hat{R}} -$ to the triangle

$$N_{\leqslant m} \xrightarrow{\xi_m^N} N \longrightarrow N_{\geqslant m+1} \longrightarrow \Sigma^1 N_{\leqslant m}$$

and inspect the long exact homology sequence

$$\cdots \longrightarrow \mathrm{H}_{i+1}\big(K^{\hat{R}} \otimes_{\hat{R}} N_{\geqslant m+1}\big) \longrightarrow \mathrm{H}_i\big(K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m}\big) \longrightarrow \mathrm{H}_i\big(K^{\hat{R}} \otimes_{\hat{R}} N\big) \longrightarrow \cdots.$$

The module $\mathrm{H}_{i+1}(K^{\hat{R}} \otimes_{\hat{R}} N_{\geqslant m+1})$ vanishes for $i < m$ while $\mathrm{H}_i(K^{\hat{R}} \otimes_{\hat{R}} N)$ vanishes for $i > s+e$ by (1.1.1). Hence

$$\mathrm{H}_i\big(K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m}\big) = 0 \quad \text{when } s+e < i < m. \tag{3}$$

The isomorphisms $(K^{\hat{R}} \otimes_R M)_i \cong (K^{\hat{R}} \otimes_R M_{\leqslant m})_i$ for $i \leqslant m$ yield the first isomorphism in the next chain; the second is by (2), and the vanishing is by (3).

$$\mathrm{H}_i\big(K^{\hat{R}} \otimes_R M\big) \cong \mathrm{H}_i\big(K^{\hat{R}} \otimes_R M_{\leqslant m}\big) \cong \mathrm{H}_i\big(K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m}\big) = 0 \quad \text{when } s+e < i < m.$$

Since the modules $\mathrm{H}_i(M)$ are finitely generated, Nakayama's lemma implies that

$$\mathrm{H}_i(M) = 0 \quad \text{when } s+e < i < m.$$

As $\sup M < m$ it follows that $\sup M \leqslant s+e$. Hence $\sup(K^{\hat{R}} \otimes_R M) \leqslant s+2e < m$.

Next we construct a commutative diagram in the category of DG $K^{\hat{R}}$-modules

$$\begin{tikzcd}
K^{\hat{R}} \otimes_R M_{\leqslant m} \arrow[r, "\alpha", "\simeq"'] \arrow[d, "K^{\hat{R}} \otimes \xi_m^M"'] & K^{\hat{R}} \otimes_{\hat{R}} N_{\leqslant m} \arrow[d, "K^{\hat{R}} \otimes \xi_m^N"] \\
K^{\hat{R}} \otimes_R M \arrow[r, dotted, "\rho"] & K^{\hat{R}} \otimes_{\hat{R}} N.
\end{tikzcd} \tag{4}$$

The top horizontal map exists by (2) and [12, Prop. 6.4] as $K^{\hat{R}} \otimes_R M_{\leqslant m}$ is $K^{\hat{R}}$-semifree. To find a morphism ρ making the diagram commute, consider the triangle

$$K^{\hat{R}} \otimes_R M_{\leqslant m} \longrightarrow K^{\hat{R}} \otimes_R M \longrightarrow K^{\hat{R}} \otimes_R M_{\geqslant m+1} \longrightarrow \Sigma^1 K^{\hat{R}} \otimes_R M_{\leqslant m}$$

and apply the functor $\mathrm{Mor}_{\mathsf{D}(K^{\hat{R}})}(-, K^{\hat{R}} \otimes_{\hat{R}} N) \cong \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_{K^{\hat{R}}}(-, K^{\hat{R}} \otimes_{\hat{R}} N))$ to obtain the exact sequence of abelian groups

$$\begin{aligned} \mathrm{Mor}_{\mathsf{D}(K^{\hat{R}})}\big(K^{\hat{R}} \otimes_R M, K^{\hat{R}} \otimes_{\hat{R}} N\big) &\xrightarrow{\zeta} \mathrm{Mor}_{\mathsf{D}(K^{\hat{R}})}\big(K^{\hat{R}} \otimes_R M_{\leqslant m}, K^{\hat{R}} \otimes_{\hat{R}} N\big) \\ &\longrightarrow \mathrm{Mor}_{\mathsf{D}(K^{\hat{R}})}\big(\Sigma^{-1} K^{\hat{R}} \otimes_R M_{\geqslant m+1}, K^{\hat{R}} \otimes_{\hat{R}} N\big). \end{aligned}$$

Note that $\mathrm{Mor}_{\mathsf{D}(K^{\hat{R}})}(\Sigma^{-1} K^{\hat{R}} \otimes_R M_{\geqslant m+1}, K^{\hat{R}} \otimes_{\hat{R}} N) = 0$ because

$$\inf\big(\Sigma^{-1} K^{\hat{R}} \otimes_R M_{\geqslant m+1}\big) \geqslant m \quad \text{and} \quad \sup\big(K^{\hat{R}} \otimes_{\hat{R}} N\big) \leqslant s + e < m.$$

Now ρ can be chosen as any preimage of $(K^{\hat{R}} \otimes_R \xi_m^N) \circ \alpha$ under ζ. The map ρ is a quasiisomorphism, as the vertical maps in (4) induce isomorphisms on homology in degrees less than m and $\mathrm{H}_i(K^{\hat{R}} \otimes_R M) = 0 = \mathrm{H}_i(K^{\hat{R}} \otimes_{\hat{R}} N)$ for $i \geqslant m$. $\square$

3.4. Remark. By the existence of minimal semifree resolutions [1, Prop. 2] and the equality of infima (1.1.1), the complex M in (3.3.1) can be chosen as a minimal complex of finite rank free R-modules with $M_n = 0$ for $n < \inf N$.

Recall that a finitely generated R-module M is *maximal Cohen–Macaulay* if the depth of M equals the Krull dimension of R.

3.5. Proposition. *Let $(R, \mathfrak{m})$ be an excellent local ring. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$.*

(a) *For every maximal Cohen–Macaulay $\hat{R}$-module N, there exists a maximal Cohen–Macaulay R^h-module M such that $K^{\hat{R}} \otimes_{R^h} M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ in $\mathsf{D}(K^{\hat{R}})$.*
(b) *For every list of maximal Cohen–Macaulay $\hat{R}$-modules $N^{(1)}, \dots, N^{(t)}$ there exists a pointed étale neighborhood $R \to R'$ and maximal Cohen–Macaulay R'-modules $M^{(1)}, \dots, M^{(t)}$ such that $K^{\hat{R}} \otimes_{R'} M^{(i)} \simeq K^{\hat{R}} \otimes_{\hat{R}} N^{(i)}$ in $\mathsf{D}(K^{\hat{R}})$.*

Proof. (a) By Theorem (3.3)(a) there is a homologically finite R^h-complex M such that there is an isomorphism $K^{\hat{R}} \otimes_{R^h} M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ in $\mathsf{D}(K^{\hat{R}})$. It suffices to show that M is (isomorphic in $\mathsf{D}(R^h)$ to) a maximal Cohen–Macaulay R^h-module.

Assume for the moment that the sequence $\boldsymbol{a} \in \mathfrak{m}$ is a system of parameters for R. Then $\boldsymbol{a}$ is an N-regular sequence, and so the depth-sensitivity of $K^{\hat{R}}$ implies that $\mathrm{H}_i(K^{\hat{R}} \otimes_{\hat{R}} N) = 0$ for $i > 0$. Combining this with (1.1.1) it readily follows that M is (isomorphic in $\mathsf{D}(R^h)$ to) a module:

$$0 = \inf\bigl(K^{\hat{R}} \otimes_{\hat{R}} N\bigr) = \inf\bigl(K^{\hat{R}} \otimes_{R^h} M\bigr) = \inf M$$
$$\leqslant \sup M \leqslant \sup\bigl(K^{\hat{R}} \otimes_{R^h} M\bigr) = \sup\bigl(K^{\hat{R}} \otimes_{\hat{R}} N\bigr) = 0.$$

By depth-sensitivity of the Koszul complex K^{R^h}, the equality

$$0 = \sup\bigl(K^{\hat{R}} \otimes_{R^h} M\bigr) = \sup\bigl(K^{R^h} \otimes_{R^h} M\bigr)$$

implies that M is a maximal Cohen–Macaulay R^h-module.

Now consider the general situation, wherein we assume only that $R/(\boldsymbol{a})$ is complete. Let $\boldsymbol{x}$ be a system of parameters for R. Applying $K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} -$ to the isomorphism $K^{\hat{R}} \otimes_{R^h} M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ yields the first isomorphism below, while the second one uses associativity and commutativity of tensor products.

$$K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} \bigl(K^{\hat{R}} \otimes_{R^h} M\bigr) \simeq K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} \bigl(K^{\hat{R}} \otimes_{\hat{R}} N\bigr),$$
$$K^{\hat{R}} \otimes_{\hat{R}} \bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M\bigr) \simeq K^{\hat{R}} \otimes_{\hat{R}} \bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} N\bigr).$$

The computations from the previous paragraph show that $K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} N$ is isomorphic in $\mathsf{D}(\hat{R})$ to the finite-length module $N/(\boldsymbol{x})N$. The second isomorphism above and [22, Thm. 2.3] now yield the next sequence of equalities

$$\sup\bigl(K^{\hat{R}} \otimes_{\hat{R}} \bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M\bigr)\bigr) = \sup\bigl(K^{\hat{R}} \otimes_{\hat{R}} \bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} N\bigr)\bigr) = e.$$

The complex $K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M$ has total homology of finite length, and so another application of [22, Thm. 2.3] yields

$$\sup\bigl(K^{\hat{R}} \otimes_{\hat{R}} \bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M\bigr)\bigr) = e + \sup\bigl(K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M\bigr).$$

It follows that $\sup(K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M) = 0$, and as in the previous argument we conclude that M is (isomorphic in $\mathsf{D}(R^h)$ to) a maximal Cohen–Macaulay R^h-module.

Part (b) is proved similarly using Theorem (3.3)(b). $\square$

4. Applications I: Vanishing of cohomology

4.1. Liftings. Let $\boldsymbol{x} \in \mathfrak{m}$ be an R-regular sequence and set $S = R/(\boldsymbol{x})$. A *lifting* of a homologically finite S-complex N to R is a homologically finite R-complex M such that $N \simeq S \otimes_R^{\mathbf{L}} M$ in $\mathsf{D}(S)$. Note that, if N is a module, then a lifting of N to R is (isomorphic in $\mathsf{D}(R)$ to) a module M with $N \cong S \otimes_R M$ and $\operatorname{Tor}^R_{\geqslant 1}(S, M) = 0$.

In [4] Auslander, Ding, and Solberg show that vanishing of the cohomology modules $\operatorname{Ext}^i_S(N, N)$ for $i = 1, 2$ guarantees existence and uniqueness of a lifting of N to R when R is complete. Yoshino extended these results to complexes in [35].

The next result uses Theorem (3.3) to relax the conditions on the ring in [4,35]; it contains Theorem B from the introduction. Note that the assumption that $\boldsymbol{x}$ is co-complete yields isomorphisms $\hat{R}/(\boldsymbol{x}) \cong R^h/(\boldsymbol{x}) \cong R/(\boldsymbol{x})$.

4.2. Theorem. *Let $(R, \mathfrak{m})$ be an excellent local ring. Fix a co-complete R-regular sequence $\boldsymbol{x} \in \mathfrak{m}$ and set $S = R/(\boldsymbol{x})$.*

(a) *Every homologically finite S-complex N with $\operatorname{Ext}^2_S(N, N) = 0$ lifts to R^h. If $\operatorname{Ext}^1_S(N, N) = 0$, then any two liftings of N to R^h are isomorphic in $\mathsf{D}(R^h)$.*
(b) *Let $N^{(1)}, \dots, N^{(t)}$ be homologically finite S-complexes. If $\operatorname{Ext}^2_S(N^{(i)}, N^{(i)}) = 0$ for each $i = 1, \dots, t$, then there is a pointed étale neighborhood $R \to R'$ such that each $N^{(i)}$ lifts to R'. If $\operatorname{Ext}^1_S(N^{(i)}, N^{(i)}) = 0$, then any two liftings of $N^{(i)}$ to R' are isomorphic in $\mathsf{D}(R')$.*

Proof. (a) First, assume $\operatorname{Ext}^2_S(N, N) = 0$. By [35, Lem. (3.2)] there is a homologically finite $\hat{R}$-complex L, such that $N \simeq S \otimes^{\mathbf{L}}_{\hat{R}} L$ in $\mathsf{D}(S)$. As R is excellent, Theorem (3.3)(a) provides a homologically finite R^h-complex M such that $K^{\hat{R}}(\boldsymbol{x}) \otimes_{R^h} M \simeq K^{\hat{R}}(\boldsymbol{x}) \otimes_{\hat{R}} L$ in $\mathsf{D}(K^{\hat{R}}(\boldsymbol{x}))$. Now the augmentation morphism $K^{\hat{R}}(\boldsymbol{x}) \xrightarrow{\simeq} S$ yields the second of the following isomorphisms in $\mathsf{D}(S)$

$$N \simeq S \otimes^{\mathbf{L}}_{\hat{R}} L \simeq S \otimes^{\mathbf{L}}_{R^h} M.$$

By the isomorphism $S \cong R^h/(\boldsymbol{x})$ this shows that M is a lifting of N to R^h.

Next, assume $\operatorname{Ext}^1_S(N, N) = 0$ and let M and M' be liftings of N to R^h. It follows that $\hat{R} \otimes^{\mathbf{L}}_{R^h} M$ and $\hat{R} \otimes^{\mathbf{L}}_{R^h} M'$ are both liftings of N to $\hat{R}$, and so [35, Lem. (3.3)] implies $\hat{R} \otimes^{\mathbf{L}}_{R^h} M \simeq \hat{R} \otimes^{\mathbf{L}}_{R^h} M'$ in $\mathsf{D}(\hat{R})$. Now [14, Lem. 1.10] yields the desired isomorphism $M \simeq M'$ in $\mathsf{D}(R^h)$.

(b) The proof is similar, using Theorem (3.3)(b). □

4.3. Auslander conditions. The ring R is said to satisfy the *Auslander Condition* if for every finitely generated R-module M there exists an integer b_M such that $\operatorname{Ext}^{\gg 0}_R(M, X) = 0$ implies $\operatorname{Ext}^{>b_M}_R(M, X) = 0$ for every finitely generated R-module X. Moreover, R satisfies the *Uniform Auslander Condition* if there is an integer $b \geqslant 0$ such that $\operatorname{Ext}^{\gg 0}_R(M, X) = 0$ implies $\operatorname{Ext}^{>b}_R(M, X) = 0$ for all finitely generated R-modules M and X. Examples of rings that do not satisfy the Auslander Condition were first given by Jorgensen and Şega [23]. For a list of rings that are known to satisfy the Auslander Condition, see [10, App. A].

The next result contains Theorem C from the introduction.

4.4. Theorem. *Let R be an excellent henselian Cohen–Macaulay local ring. The completion $\hat{R}$ satisfies the* (*Uniform*) *Auslander Condition if and only if R satisfies the* (*Uniform*) *Auslander Condition.*

Proof. It is straightforward to verify the "only if" part, cf. [10, Prop. (5.5)].

For the "if" part, let N and Y be finitely generated $\hat{R}$-modules and assume that $\operatorname{Ext}^{\gg 0}_{\hat{R}}(N, Y) = 0$. By a standard argument we can without loss of generality assume that N and Y are maximal Cohen–Macaulay $\hat{R}$-modules; see [11]. This reduction involves replacing N by a high syzygy and Y by a maximal Cohen–Macaulay $\hat{R}$-module that approximates it in the sense of [3, Thm. A].

Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$. By Proposition (3.5) there exist finitely generated R-modules M and X such that $K^{\hat{R}} \otimes_R M \simeq K^{\hat{R}} \otimes_{\hat{R}} N$ and $K^{\hat{R}} \otimes_R X \simeq$

$K^{\hat{R}} \otimes_{\hat{R}} Y$ in $\mathsf{D}(K^{\hat{R}})$. This accounts for the last equality in the computation below; the first three follow by (1.1.1), (1.1.2), and adjointness.

$$\begin{aligned}
\inf \mathbf{R}\mathrm{Hom}_{\hat{R}}(N, Y) &= \inf\bigl(K^{\hat{R}} \otimes_{\hat{R}} \mathbf{R}\mathrm{Hom}_{\hat{R}}(N, Y)\bigr) \\
&= \inf \mathbf{R}\mathrm{Hom}_{\hat{R}}\bigl(N, K^{\hat{R}} \otimes_{\hat{R}} Y\bigr) \\
&= \inf \mathbf{R}\mathrm{Hom}_{K^{\hat{R}}}\bigl(K^{\hat{R}} \otimes_{\hat{R}} N, K^{\hat{R}} \otimes_{\hat{R}} Y\bigr) \\
&= \inf \mathbf{R}\mathrm{Hom}_{K^{\hat{R}}}\bigl(K^{\hat{R}} \otimes_R M, K^{\hat{R}} \otimes_R X\bigr).
\end{aligned}$$

Combined with a parallel computation starting from $\mathbf{R}\mathrm{Hom}_R(M, X)$, this yields the second equality in the next sequence

$$\begin{aligned}
\sup\bigl\{m \in \mathbb{Z} \bigm| \mathrm{Ext}^m_{\hat{R}}(N, Y) = 0\bigr\} &= -\inf \mathbf{R}\mathrm{Hom}_{\hat{R}}(N, Y) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(M, X) \qquad (1)\\
&= \sup\bigl\{m \in \mathbb{Z} \bigm| \mathrm{Ext}^m_R(M, X) = 0\bigr\}.
\end{aligned}$$

In particular, we have $\mathrm{Ext}^{\gg 0}_R(M, X) = 0$. If R satisfies the Auslander Condition, then there is an integer $b = b_M$ such that $\mathrm{Ext}^{>b}_R(M, X) = 0$, and (2) shows that $\mathrm{Ext}^{>b}_{\hat{R}}(N, Y) = 0$. It follows that $\hat{R}$ satisfies the Auslander Condition. Ascent of the Uniform Auslander Condition follows from the same argument. $\square$

5. Descent of semidualizing complexes

We now focus on step two of the analysis described in the introduction, namely, transfer of information from Koszul extensions to $\hat{R}$-complexes.

5.1. Definition. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^R = K^R(\boldsymbol{a})$. A class C of homologically finite R-complexes is *K^R-distinguishable* if it satisfies the following property: Given homologically finite R-complexes C and X, if C is in C and $K^R \otimes_R X \simeq K^R \otimes_R C$ in $\mathsf{D}(K^R)$, then $X \simeq C$ in $\mathsf{D}(R)$.

5.2. Lemma. *Let $(R, \mathfrak{m})$ be a local ring. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$. Given a $K^{\hat{R}}$-distinguishable class C of homologically finite $\hat{R}$-complexes, if C is in C and there is a homologically finite R-complex B such that $K^{\hat{R}} \otimes_R B \simeq K^{\hat{R}} \otimes_{\hat{R}} C$ in $\mathsf{D}(K^{\hat{R}})$, then $\hat{R} \otimes_R B \simeq C$ in $\mathsf{D}(\hat{R})$.*

Proof. By the assumption on C, the claim is immediate from the isomorphisms

$$K^{\hat{R}} \otimes_{\hat{R}} (\hat{R} \otimes_R B) \simeq K^{\hat{R}} \otimes_R B \simeq K^{\hat{R}} \otimes_{\hat{R}} C. \qquad \square$$

The main result of this section concerns semidualizing complexes; the definition is recalled below. This notion is wide enough to encompass dualizing complexes in the sense of

Grothendieck [17] and the relative dualizing complexes of Avramov and Foxby [6]; yet it is narrow enough to admit a rich theory [9]. This notion has been introduced independently by several authors; for example in Wakamatsu's work on generalized tilting modules [33].

5.3. Semidualizing complexes. A homologically finite R-complex C is *semidualizing* if the homothety morphism $\chi^R_C : R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism in $\mathsf{D}(R)$. Further, C is *dualizing* in the sense of [17, V.§2] if it is semidualizing and isomorphic in $\mathsf{D}(R)$ to a bounded complex of injective modules.

Theorem D from the introduction is a special case of part (a) in the next result.

5.4. Theorem. *Let R be an excellent local ring.*

(a) *For every semidualizing $\hat{R}$-complex C there exists a semidualizing R^h-complex B such that $C \simeq \hat{R} \otimes_{R^h} B$. In particular, R^h has a dualizing complex.*
(b) *For every list of semidualizing $\hat{R}$-complexes $C^{(1)}, \dots, C^{(t)}$ there is a pointed étale neighborhood $R \to R'$ and semidualizing R'-complexes $B^{(1)}, \dots, B^{(t)}$ such that $C^{(i)} \simeq \hat{R} \otimes_{R'} B^{(i)}$ for $i = 1, \dots, t$. In particular, there exists a pointed étale neighborhood $R \to R'$ such that R' admits a dualizing complex.*

Proof. (a) Let C be a semidualizing $\hat{R}$-complex. Fix a co-complete sequence $\boldsymbol{a} \in \mathfrak{m}$ and set $K^{\hat{R}} = K^{\hat{R}}(\boldsymbol{a})$. By Theorem (3.3)(a) there exists a homologically finite R^h-complex B such that $K^{\hat{R}} \otimes_{R^h} B \simeq K^{\hat{R}} \otimes_{\hat{R}} C$ in $\mathsf{D}(K^{\hat{R}})$. The class C of semidualizing $\hat{R}$-complexes is $K^{\hat{R}}$-distinguishable by Lemma (A.3), so Lemma (5.2) yields an isomorphism $\hat{R} \otimes_{R^h} B \simeq C$ in $\mathsf{D}(\hat{R})$, and it follows by (A.1) that the complex B is semidualizing for R^h. Because $\hat{R}$ admits a dualizing complex, this shows that R^h also admits a dualizing complex; see [5, Thm. (5.1)].

(b) The proof is similar, using Theorem (3.3)(b). $\square$

The next example demonstrates how badly the conclusion of Theorem (5.4) (and hence Theorem (3.3)) can fail when R is not excellent.

5.5. Example. Let S_0 be a field of characteristic zero. For $n > 0$ let $[V_{nij}]$ be a 2×3 matrix of indeterminants and consider the complete normal Cohen–Macaulay local domain

$$S_n = S_{n-1}[\![V_{nij}]\!]/I_2(V_{nij}).$$

By [30, Cor. 4.9(c)] there are exactly 2^n distinct shift-isomorphism classes of semidualizing complexes in $\mathsf{D}(S_n)$; by [9, Cor. (3.7)] each class contains a module. For each $n > 0$ there exists, by [18, Thm. 8], a Cohen–Macaulay local unique factorization domain R_n such that $S_n \cong \hat{R}_n$. By [30, Prop. 3.4] each ring R_n has only the trivial semidualizing complex R_n up to shift-isomorphism. Hence, the only semidualizing $\hat{R}_n$-complex that descends to R_n is the trivial one $\hat{R}_n$.

Also Theorem (4.2) has an application to semidualizing complexes. The next result extends part of [15, Prop. 4.2].

5.6. Proposition. *Let* $(R,\mathfrak{m})$ *be an excellent henselian local ring and* $\boldsymbol{x} \in \mathfrak{m}$ *a co-complete* R*-regular sequence. There is a bijective correspondence, induced by the functor* $R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R -$*, between the sets of* (*shift-*)*isomorphism classes of semidualizing complexes in* $\mathsf{D}(R)$ *and* $\mathsf{D}(R/(\boldsymbol{x}))$.

Proof. This follows directly from (A.1) and Theorem (4.2)(a). □

6. Applications II: Annihilation of (co)homology

For every local ring R, the completion $\hat{R}$ has a dualizing complex. If R is excellent then, by Theorem (5.4), a dualizing complex is available much closer to R. This is the key to the next result; if R itself has a dualizing complex, then the conclusion holds by a result of Roberts [27, Thm. 1].

6.1. Theorem. *Let* R *be an excellent local ring of Krull dimension* d. *There exists a chain of ideals* $\mathfrak{b}_d \subseteq \cdots \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_0$ *satisfying the following conditions*:

(a) *For each* $i = 0, \dots, d$ *there is an inequality* $\dim R/\mathfrak{b}_i \leqslant i$.
(b) *If* $F = 0 \to F_r \to \cdots \to F_0 \to 0$ *is a complex of finite rank free* R*-modules with* $\operatorname{length}_R \mathrm{H}(F) < \infty$, *then* $\mathfrak{b}_i$ *annihilates* $\mathrm{H}_j(F)$ *for each* $j \geqslant r - i$.

Proof. By Theorem (5.4)(b) there exists a pointed étale neighborhood $R \to R'$ such that R' admits a dualizing complex. Note that $\dim R' = d$. By [27, Thm. 1 and preceding Prop. and Def.] there exists a chain of ideals $\mathfrak{b}'_d \subseteq \cdots \subseteq \mathfrak{b}'_1 \subseteq \mathfrak{b}'_0$ in R' such that: (a$'$) $\dim R'/\mathfrak{b}'_i \leqslant i$, and (b$'$) if $F' = 0 \to F'_r \to \cdots \to F'_0 \to 0$ is a complex of finite rank free R'-modules with $\operatorname{length}_{R'} \mathrm{H}(F') < \infty$, then $\mathfrak{b}'_i \mathrm{H}_j(F') = 0$ for $j \geqslant r - i$.

For each $i = 0, \dots, d$ set $\mathfrak{b}_i = \mathfrak{b}'_i \cap R$. It is not difficult to verify that $\dim R/\mathfrak{b}_i = \dim R'/\mathfrak{b}'_i \leqslant i$. Let F be a complex satisfying the hypothesis of (b). The complex $F' = R' \otimes_R F$ satisfies the hypothesis of (b$'$). Indeed, the isomorphisms $R'/\mathfrak{m}R' \cong k$ and $\mathrm{H}_j(F') \cong \mathrm{H}_j(F) \otimes_R R'$ guarantee $\operatorname{length}_{R'} \mathrm{H}(F') < \infty$. This provides R-isomorphisms $\mathrm{H}_j(F') \cong \mathrm{H}_j(F)$. The ideal $\mathfrak{b}'_i$ annihilates $\mathrm{H}_j(F')$ for $j \geqslant r - i$ and contains $\mathfrak{b}_i$; hence $\mathfrak{b}_i$ annihilates $\mathrm{H}_j(F)$ for $j \geqslant r - i$. □

6.2. Remark. By unpublished examples of Nishimura, an excellent local ring need not possess a dualizing complex [24, Exa. 6.1], and a ring with a dualizing complex need not be excellent [24, Exa. 4.2]. In view of this, the hypothesis in (6.1) is neither stronger nor weaker than the hypothesis in [27, Thm. 1].

A classical application of Roberts' theorem [27, Thm. 1] is to find uniform annihilators of local cohomology modules $\mathrm{H}^j_{\mathfrak{m}}(R)$. Hochster and Huneke [20,21], for instance, do this when R is an equidimensional local ring that admits a dualizing complex or is unmixed and excellent. Theorem (6.1) allows us to drop the unmixedness condition, thus recovering a recent result of Zhou [36, Cor. 3.3(ii)].

6.3. Remark. Let R be an equidimensional excellent local ring of Krull dimension $d > 1$. For each j the local cohomology module $\mathrm{H}^j_{\mathfrak{m}}(R)$ is a direct limit of homology modules in degree $d - j$ of Koszul complexes on powers of a system of parameters. Thus, for each $i = 0, \dots, d$ the ideal $\mathfrak{b}_i$ annihilates $\mathrm{H}^j_{\mathfrak{m}}(R)$ for $j \leqslant i$. In particular, $\mathfrak{b}_{d-1}$ annihilates $\mathrm{H}^j_{\mathfrak{m}}(R)$ for $j < d$ and $\dim R/\mathfrak{b}_{d-1} \leqslant d - 1$. Since R is equidimensional, $\mathfrak{b}_{d-1}$ is not contained in any minimal prime of R. In the terminology of [36], this means that $\mathfrak{b}_{d-1}$ contains a uniform cohomological annihilator for R.

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Appendix A. Semidualizing complexes are K^R-distinguishable

We start by recalling a few facts about semidualizing complexes; see (5.3).

A.1. Ascent and descent. Let $R \to S$ be a local ring homomorphism such that S has finite flat dimension as an R-module, and let C, C' be degreewise homologically finite R-complexes. The complex $S \otimes_R^{\mathbf{L}} C$ is S-semidualizing if and only if C is R-semidualizing. Furthermore, if C and C' are semidualizing R-complexes such that $S \otimes_R^{\mathbf{L}} C \simeq S \otimes_R^{\mathbf{L}} C'$ in $\mathsf{D}(S)$, then $C \simeq C'$ in $\mathsf{D}(R)$; see [14, Thm. 4.5 and 4.9].

The following definition is introduced as a convenience for Lemma (A.3).

A.2. Definition. Fix a list of elements $\boldsymbol{a} \in \mathfrak{m}$ and set $K^R = K^R(\boldsymbol{a})$. A *semidualizing DG K^R-module* is a homologically finite DG K^R-module M such that the homothety morphism $\chi_M^{K^R} : K^R \longrightarrow \mathbf{R}\mathrm{Hom}_{K^R}(M, M)$ is an isomorphism in $\mathsf{D}(K^R)$.

The next lemma shows that the class of semidualizing R-complexes is K^R-distinguishable, as defined in (5.1).

A.3. Lemma. *Let $(R, \mathfrak{m})$ be a local ring and let C and C' be degreewise homologically finite R-complexes. Fix a list of elements $\boldsymbol{a} \in \mathfrak{m}$ and set $K^R = K^R(\boldsymbol{a})$.*

(a) *The DG K^R-module $K^R \otimes_R C$ is K^R-semidualizing if and only if C is R-semidualizing.*
(b) *If C and C' are semidualizing and $K^R \otimes_R C \simeq K^R \otimes_R C'$ in $\mathsf{D}(K^R)$, then $C \simeq C'$ in $\mathsf{D}(R)$.*

Proof. For brevity set $K = K^R$. Recall from [9] that a homologically finite R-complex X is *C-reflexive* if $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded and the biduality morphism $\delta_X^C : X \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ is an isomorphism in $\mathsf{D}(R)$.

(a) Recall from (1.2) that $K \otimes_R C$ is homologically finite over K if and only if C is homologically finite over R. In the commutative diagram

$$\begin{tikzcd} K \otimes_R R \arrow[r, "\cong"'] \arrow[d, "K\otimes\chi_C^R"'] & K \arrow[r, "\chi_{K\otimes C}^K"] & \mathbf{R}\mathrm{Hom}_K(K \otimes_R C, K \otimes_R C) \arrow[d, "\simeq"'] \\ K \otimes_R \mathbf{R}\mathrm{Hom}_R(C, C) \arrow[rr, "\simeq"] & & \mathbf{R}\mathrm{Hom}_R(C, K \otimes_R C) \end{tikzcd}$$

the right-hand vertical isomorphism is by adjointness, and the lower horizontal one is tensor-evaluation (1.1.2). The diagram shows that $\chi_{K\otimes_R C}^K$ is an isomorphism in $\mathsf{D}(K)$ if and only if

$K \otimes_R \chi_C^R$ is so. The latter is tantamount to χ_C^R being an isomorphism in $\mathsf{D}(R)$; to see this apply (1.1.1) to $\operatorname{Cone}(K \otimes_R \chi_C^R) \cong K \otimes_R \operatorname{Cone}\chi_C^R$.

(b) Assume $K \otimes_R C \simeq K \otimes_R C'$ in $\mathsf{D}(K)$. The first isomorphism in the following diagram is the homothety morphism

$$K \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_K(K \otimes_R C', K \otimes_R C) \xleftarrow[\beta]{\simeq} K \otimes_R \mathbf{R}\mathrm{Hom}_R(C', C)$$

while the second is the composition of adjunction and tensor-evaluation (1.1.2). It follows that $K \otimes_R \mathbf{R}\mathrm{Hom}_R(C', C)$ is homologically bounded, and hence so is $\mathbf{R}\mathrm{Hom}_R(C', C)$ by (1.1.1). In the commutative diagram

$$\begin{tikzcd}
K \otimes_R C' \arrow[r, "\delta^{K\otimes C}_{K\otimes C'}", "\simeq"'] \arrow[d, "K\otimes\delta^C_{C'}"'] & \mathbf{R}\mathrm{Hom}_K(\mathbf{R}\mathrm{Hom}_K(K \otimes_R C', K \otimes C), K \otimes C) \arrow[d, "\mathbf{R}\mathrm{Hom}(\beta, K\otimes_R C)", "\simeq"'] \\
K \otimes_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C', C), C) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_K(K \otimes_R \mathbf{R}\mathrm{Hom}_R(C', C), K \otimes_R C)
\end{tikzcd}$$

the lower horizontal arrow is the composition of adjointness and tensor-evaluation. The diagram shows that $K \otimes_R \delta^C_{C'}$ is an isomorphism in $\mathsf{D}(K)$. An application of (1.1.1) to $\operatorname{Cone}(K \otimes_R \delta^C_{C'}) \cong K \otimes_R \operatorname{Cone}\delta^C_{C'}$ implies that $\delta^C_{C'}$ is an isomorphism in $\mathsf{D}(R)$. This proves that C' is C-reflexive. Symmetrically, C is C'-reflexive, and by [2, Thm. 5.3] it then follows that C and C' are isomorphic up to shift in $\mathsf{D}(R)$. From (1.1.1) we have $C \simeq C'$. $\square$

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