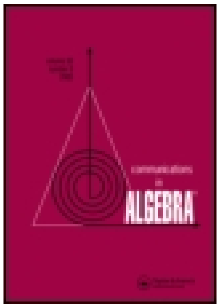

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

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To cite this article: Susan M. Cooper & Sean Sather-Wagstaff (2013) Multiplicities of Semidualizing Modules, Communications in Algebra, 41:12, 4549-4558

To link to this article: http://dx.doi.org/10.1080/00927872.2012.705933

Communications in Algebra®, 41: 4549–4558, 2013

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2012.705933

MULTIPLICITIES OF SEMIDUALIZING MODULES

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A finitely generated module C over a commutative noetherian ring R is semidualizing if $\mathrm{Hom}_R(C, C) \cong R$ ***and*** $\mathrm{Ext}^i_R(C, C) = 0$ ***for all*** $i \geqslant 1$***. For certain local Cohen–Macaulay rings*** $(R, \mathfrak{m})$***, we verify the equality of Hilbert-Samuel multiplicities*** $e_R(J; C) = e_R(J; R)$ ***for all semidualizing R-modules C and all*** $\mathfrak{m}$***-primary ideals J. The classes of rings we investigate include those that are determined by ideals defining fat point schemes in projective space or by monomial ideals.***

Key Words: Betti numbers; Canonical modules; Dualizing modules; Monomial ideals; Fat point schemes; Hilbert-Samuel multiplicities; Semidualizing modules.

2010 Mathematics Subject Classification: 13C14; 13H15.

1. INTRODUCTION

In this section, let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring with a dualizing module D. A finitely generated R-module C is *semidualizing* if $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. Thus, the module D is precisely a semidualizing module of finite injective dimension. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules. (See Section 2 for definitions and background information.) For example, the R-modules R and D are semidualizing. The ring R is Gorenstein if and only if $D \cong R$, equivalently, if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

In this article, we investigate the following question, motivated by the well-known equality $e_R(J; D) = e_R(J; R)$.

Question 1.1. Let C be a semidualzing R-module. For each $\mathfrak{m}$-primary ideal J, must we have an equality of Hilbert-Samuel multiplicities $e_R(J; C) = e_R(J; R)$?

When R is generically Gorenstein (e.g., reduced) an affirmative answer to this question is contained in [10, (2.8(a))]. In Theorems 3.2 and 3.4, we address a few more cases with the following theorem.

Received December 5, 2011. Communicated by R. Wiegand.
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Theorem 1.2. *Assume that R satisfies one of the following conditions*:

(1) $P^2R_P = 0$ *for each* $P \in \mathrm{Ass}(R)$;
(2) $\widehat{R} \cong k[\![X_0, X_1, \dots, X_n]\!]/Ik[\![X_0, X_1, \dots, X_n]\!]$, *where* $I \subseteq k[X_0, X_1, \dots, X_n]$ *is the ideal determining a fat point scheme in* $\mathbb{P}^n_k$; *or*
(3) $\widehat{R} \cong k[\![X_1, \dots, X_n]\!]/I$, *where I is generated by monomials in the* X_i.

For every $\mathfrak{m}$*-primary ideal* $J \subset R$ *and every semidualizing* R*-module* C, *we have* $e_R(J; C) = e_R(J; R)$.

This article is organized as follows. Section 2 consists of background material, and Section 3 contains the proof of Theorem 1.2.

2. BACKGROUND

For the rest of this article, let R and S be commutative noetherian rings of finite Krull dimension.

Definition 2.1. Let C be an R-module. The natural *homothety map*

$$\chi^R_C : R \to \mathrm{Hom}_R(C, C)$$

is the R-module homomorphism given by $\chi^R_C(r)(c) = rc$. The module C is *semidualzing* if it satisfies the following statements:

(1) C is finitely generated;
(2) The homothety map $\chi^R_C : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism; and
(3) $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$.

The module C is *dualizing* if it is semidualizing and has finite injective dimension.[1]

Example 2.2. It is straightforward to show that the free R-module R^1 is semidualizing. It is dualizing if and only if R is Gorenstein.

The following facts will be used in the sequel.

Fact 2.3. Let C be a semidualizing R-module. Then a sequence $x_1, \dots, x_n \in R$ is C-regular if and only if it is R-regular. (See, e.g., [11, (1.4)] for a brief explanation of the local case. The general case has the same proof.)

Fact 2.4. Assume that R is Cohen–Macaulay and that D is a dualizing R-module. Let C be a semidualizing R-module. From [3, (3.1), (3.4)] and [6, (V.2.1)], we have the following:

(a) $\mathrm{Ext}^i_R(C, D) = 0$ for all $i \geqslant 1$;

[1]The assumption $\dim(R) < \infty$ guarantees that a finitely generated R-module C has finite injective dimension over R if and only if $C_\mathfrak{m}$ has finite injective dimension over $R_\mathfrak{m}$. For instance, this removes the need to worry about any distinction between the terms "dualizing" and "locally dualizing," and similarly for "Gorenstein" and "locally Gorenstein." This causes no loss of generality in this article as we are primarily concerned with local and graded situations.

(b) The dual $\mathrm{Hom}_R(C, D)$ is a semidualizing R-module;
(c) The natural biduality map $\delta_C^D : C \to (\mathrm{Hom}_R\mathrm{Hom}_R(C, D)D)$ given by the formula $\delta_C^D(c)(\psi) = \psi(c)$ is an isomorphism;
(d) $\mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, D)) = 0$ for all $i \geqslant 1$; and
(e) The natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, D) \to D$ given by $c \otimes \psi \mapsto \psi(c)$ is an isomorphism.

From (c), we conclude that:

(f) if $\mathrm{Hom}_R(C, D) \cong R$, then $C \cong D$.
Assume that R is local. Because of (d) and (e), the minimal free resolution of D is obtained by tensoring the minimal free resolutions of C and $\mathrm{Hom}_R(C, D)$. In particular, this implies that:
(g) $\beta_i^R(D) = \sum_{j=0}^{i} \beta_j^R(C)\beta_{i-j}^R(\mathrm{Hom}_R(C, D))$ for each $i \geqslant 0$.

Fact 2.5. Let $\varphi : R \to S$ be a homomorphism of commutative noetherian rings. Assume that S has finite flat dimension as an R-module. For example, this is satisfied when S is flat as an R-module, or when φ is surjective with $\mathrm{Ker}(\varphi)$ generated by an R-regular sequence. If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module; the converse holds when φ is faithfully flat; see [3, (4.5)]. Thus, the rule of assignment $[C] \mapsto [S \otimes_R C]$ describes a well-defined function $\mathfrak{S}_0(\varphi) : \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$. If the map φ is local, that is if $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, then the induced map $\mathfrak{S}_0(\varphi)$ is injective; see [3, (4.9)].

Assume that φ is local and satisfies one of the following conditions:

(1) φ is flat with Gorenstein closed fibre $S/\mathfrak{m}S$ (e.g., φ is the natural map from R to its completion $\widehat{R}$); or
(2) φ is surjective with $\mathrm{Ker}(\varphi)$ generated by an R-regular sequence.

Then a semidualizing R-module C is dualizing for R if and only if $S \otimes_R C$ is dualizing for S by [1, (3.1.15),(3.3.14)]. When R is complete and φ satisfies condition (2), the induced map $\mathfrak{S}_0(\varphi) : \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ is bijective; see [4, (4.2)] or [5, (2)].

Fact 2.6. Assume that $(R, \mathfrak{m}, k)$ is local and C is a semidualizing R-module. If C has finite projective dimension, then $C \cong R$; see, e.g., [11, (1.14)]. If R is Gorenstein, then $C \cong R$ by [2, (8.6)]. If $\mathfrak{m}^2 = 0$, then either $C \cong R$ or C is dualizing for R. (Indeed, if $C \ncong R$, then the first syzygy C' of C is a nonzero k-vector space such that $\mathrm{Ext}_R^1(C', C) = 0$, so C is injective.)

The following notions are standard.

Remark/Definition 2.7. Let $(R, \mathfrak{m})$ be a local ring, and let I be an $\mathfrak{m}$-primary ideal of R. Let C be a finitely generated R-module of dimension d. There is a polynomial $H_{I,C}(j) \in \mathbb{Q}[j]$ such that $H_{I,C}(j) = \mathrm{len}_R(I^jC/I^{j+1}C)$ for $j \gg 0$. Moreover, the degree of $H_{I,C}(j)$ is $d-1$, and the leading coefficient is of the form $e_R(I; C)/(d-1)$ for some positive integer $e_R(I; C)$. The integer $e_R(I; C)$ is the *Hilbert–Samuel* multiplicity of C with respect to I.

The next lemma is a version of a result of Herzog [8, (2.3)]. It is proved similarly and is almost certainly well-known.

Lemma 2.8. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that $\mathfrak{m}S = \mathfrak{n}$. Let I be an $\mathfrak{m}$-primary ideal of R, and let C be a finitely generated R-module. Set $\widetilde{C} = S \otimes_R C$ and $\widetilde{I} = IS$. For each j there is an equality* $\mathrm{len}_S(\widetilde{I}^j\widetilde{C}/\widetilde{I}^{j+1}\widetilde{C}) = \mathrm{len}_R(I^jC/I^{j+1}C)$. *In particular, we have $e_S(\widetilde{I}; \widetilde{C}) = e_R(I; C)$.*

We end this section with a discussion of fat point schemes.

Definition 2.9. Let k be a field. Fix distinct points $Q_1, \ldots, Q_r \in \mathbb{P}^n_k$ and integers $m_1, \ldots, m_r \geqslant 1$. Set $S_0 = k[X_0, X_1, \ldots, X_n]$ with irrelevant maximal ideal $\mathfrak{n}_0 = (X_0, X_1, \ldots, X_n)S_0$. For each index j, let $I(Q_j) \subset S_0$ be the (reduced) vanishing ideal of Q_j. The subscheme of $\mathbb{P}^n_k$ defined by the ideal $I = \cap_{j=1}^r I(Q_j)^{m_j} \subseteq S_0$ is the *fat point scheme* determined by the points $Q_1, \ldots, Q_r$ with multiplicities $m_1, \ldots, m_r$.

Remark 2.10. Continue with the notation of Definition 2.9.

Set $S = k[\![X_0, X_1, \ldots, X_n]\!]$ with maximal ideal $\mathfrak{n} = (X_0, X_1, \ldots, X_n)S$. The local rings $(S_0)_{\mathfrak{n}_0}/I_{\mathfrak{n}_0}$ and $R = S/IS$ are Cohen–Macaulay of dimension 1.

Note that the quotient $S_0/I(Q_j)$ is isomorphic (as a graded k-algebra) to a polynomial ring $k[Y]$. In particular, the completion of the local ring $(S_0)_{\mathfrak{n}_0}/I(Q_j)(S_0)_{\mathfrak{n}_0}$ (isomorphic to $S/I(Q_j)S$) is isomorphic to the formal power series ring $k[[Y]]$. In particular, the ideal $I(Q_j)S$ is prime. It follows that the associated primes of $R = S/I$ are of the form $P_j = I(Q_j)S/I$. Localizing at one of these primes yields

$$R_{P_j} \cong S_{I(Q_j)}/IS_{I(Q_j)} \cong S_{I(Q_j)}/I(Q_j)^{m_j}S_{I(Q_j)} \cong S_{I(Q_j)}/(I(Q_j)S_{I(Q_j)})^{m_j}.$$

In other words, we have $R_{P_j} \cong S_j/\mathfrak{n}_j^{m_j}$ for some regular local ring $(S_j, \mathfrak{n}_j)$.

3. MULTIPLICITIES OF SEMIDUALIZING MODULES

In this section, we consider Question 1.1 for certain classes of rings.

Lemma 3.1. *Let $(S, \mathfrak{n})$ be a regular local ring containing a field. Let e be a positive integer, and set $R = S/\mathfrak{n}^e$. Let C be a semidualizing R-module. Then either $C \cong R$ or C is dualizing for R. In particular, we have* $\mathrm{len}_R(C) = \mathrm{len}_R(R)$.

Proof. Fact 2.6 deals with the case $e = 1$, so assume that $e \geqslant 2$. The ring R is artinian and local. Hence, it is complete and has a dualizing module D. There are isomorphisms

$$R \cong \widehat{R} \cong \widehat{S}/\mathfrak{n}^e\widehat{S} \cong k[\![X_1, \ldots, X_n]\!]/(X_1, \ldots, X_n)^e$$

where k is a field and $n = \mathrm{edim}(S)$. We now conclude from [10, (4.11)] that $C \cong R$ or $C \cong D$. The conclusion $\mathrm{len}_R(C) = \mathrm{len}_R(R)$ follows from the well-known equality $\mathrm{len}_R(D) = \mathrm{len}_R(R)$. $\square$

The next result contains cases (1) and (2) of Theorem 1.2 from the introduction.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring, and let C be a semidualizing R-module. Assume that R satisfies one of the following conditions*:

(1) $P^2R_P = 0$ *for each* $P \in \mathrm{Ass}(R)$;
(2) $\widehat{R} \cong k[\![X_0, X_1, \ldots, X_n]\!]/Ik[\![X_0, X_1, \ldots, X_n]\!]$ *where* $I \subseteq k[X_0, X_1, \ldots, X_n]$ *is the ideal determining a fat point scheme in* $\mathbb{P}^n_k$.

Then for every $\mathfrak{m}$*-primary ideal* $J \subset R$, *we have* $e_R(J; C) = e_R(J; R)$.

Proof. (1) Assume that $P^2R_P = 0$ for each $P \in \mathrm{Ass}(R)$. Fact 2.6 implies that $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ for each $P \in \mathrm{Ass}(R)$, hence the second equality in the following sequence wherein each sum is taken over all $P \in \mathrm{Ass}(R)$:

$$e(J; C) = \sum_P \mathrm{len}_{R_P}(C_P)e(J; R/P) = \sum_P \mathrm{len}_{R_P}(R_P)e(J; R/P) = e(J; R).$$

The remaining equalities follow from the additivity formula [1, (4.7.t)].

(2) Using Fact 2.5 and Lemma 2.8 we may pass to the completion to assume that $R \cong \widehat{R}$. For each $P \in \mathrm{Ass}(R)$, Remark 2.10 implies that $R_P \cong S/\mathfrak{n}^m$ for some regular local ring $(S, \mathfrak{n})$. Lemma 3.1 implies that $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ for each $P \in \mathrm{Ass}(R)$, hence the desired conclusion follows as in case (1). $\square$

The next result contains part of case (3) of Theorem 1.2 from the introduction. The general case is in Theorem 3.4.

Lemma 3.3. *Let $(A, \mathfrak{r})$ be a complete reduced local ring, and set $S = A[\![x_1, \ldots, x_n]\!]$, the formal power series ring, with maximal ideal $\mathfrak{n} = (\mathfrak{r}, x_1, \ldots, x_n)S$. Let $I \subset S$ be an ideal generated by monomials in the x_i, and set $R = S/I$ with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is Cohen–Macaulay, and let C be a semidualizing R-module. Then for each $P \in \mathrm{Spec}(R)$ and for each PR_P-primary ideal $J \subset R_P$, we have $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$.*

Proof. Here is an outline of the proof. We show that the theory of polarization for monomial ideals yields a complete reduced Cohen–Macaulay local ring R^* and a surjection $\tau : R^* \to R$ such that $\mathrm{Ker}(\tau)$ is generated by an R^*-regular sequence $\mathbf{y}$. Facts 2.3 and 2.5 yield a semidualizing R^*-module such that the sequence $\mathbf{y}$ is C^*-regular and $C^*/(\mathbf{y})C^* \cong C$. Because R^* is complete and reduced, the desired conclusion follows from [10, (2.8.b)].

Set $S_0 = A[x_1, \ldots, x_n] \subset S$. The ideal $I_0 = I \cap S_0$ is generated by monomials in the x_i, in fact, by the same list of monomial generators used to generate I.

The theory of polarization for monomial ideals yields the following:

(1) A polynomial ring $S_0^* = A[x_{1,1}, \ldots, x_{1,t_1}, x_{2,1}, \ldots, x_{2,t_2}, \ldots, x_{n,1}, \ldots, x_{n,t_n}]$ with irrelevant maximal ideal $\mathfrak{n}_0^* = (\mathfrak{r}, \{x_{i,j}\})S_0^*$;
(2) An ideal $I_0^* \subseteq S_0^*$ generated by *square-free* monomials in the $x_{i,j}$;
(3) A sequence $\mathbf{y} = y_1, \ldots, y_r \in \mathfrak{n}_0^*$ that is both S_0^*-regular and (S_0^*/I_0^*)-regular and such that $S_0^*/(\mathbf{y})S_0^* \cong S_0$ and $S_0^*/(I_0^* + (\mathbf{y})S_0^*) \cong S_0/I_0$.

Localizing at $\mathfrak{n}_0^*$ and passing to the completion yields the following:

(1′) A power series ring $S^* = A[\![x_{1,1}, \ldots, x_{1,t_1}, x_{2,1}, \ldots, x_{2,t_2}, \ldots, x_{n,1}, \ldots, x_{n,t_n}]\!]$ over k with maximal ideal denoted $\mathfrak{n}^* = (\mathfrak{r}, \{x_{i,j}\})S^*$;
(2′) An ideal $I^* = I_0^* S^* \subseteq S^*$ generated by square-free monomials in the $x_{i,j}$;
(3′) A sequence $\mathbf{y} = y_1, \ldots, y_r \in \mathfrak{n}^*$ that is both S^*-regular and (S^*/I^*)-regular and such that $S^*/(\mathbf{y})S^* \cong S$ and $S^*/(I^* + (\mathbf{y})S^*) \cong S/I = R$.

Setting $R^* = S^*/I^*$, we have the following:

(1″) Since I^* is generated by square-free monomials, the ring R^* is reduced;
(2″) The sequence $\mathbf{y}$ is R^*-regular such that $R^*/(\mathbf{y})R^* \cong R$. In particular, since R is Cohen-Macaulay, so is R^*;
(3″) Since S^* is complete, so is R^*; Thus, Fact 2.5 provides a semidualizing R^*-module C^* such that $C \cong C^* \otimes_{R^*} R$;
(4″) Since the sequence $\mathbf{y}$ is R^*-regular, it is also C^*-regular by Fact 2.3.

Let $\tau : R^* \to R$ be the canonical surjection, and set $P^* = \tau^{-1}(P)$. We then have the following:

(1‴) Since R^* is reduced, so is the localization $R^*_{P^*}$;
(2‴) Since R^* is Cohen-Macaulay, so is the localization $R^*_{P^*}$. In particular, the ring $R^*_{P^*}$ is equidimensional. Also, we have $R^*_{P^*}/(\mathbf{y})R^*_{P^*} \cong R_P$;
(3‴) Since R^* is complete, it is excellent, and it follows that the localization $R^*_{P^*}$ is also excellent. In particular, for every $\mathfrak{p} \in \operatorname{Min}(R^*_{P^*})$ the ring $(R^*_{P^*})_{\mathfrak{p}}/\mathfrak{p}(R^*_{P^*})_{\mathfrak{p}} \otimes_{R^*_{P^*}} \widehat{R^*_{P^*}}$ is Gorenstein;
(4‴) The $R^*_{P^*}$-module $C^*_{P^*}$ is semidualizing and satisfies $C^*_{P^*}/(\mathbf{y})C^*_{P^*} \cong C_P$.

Using the conditions (1‴)—(4‴), the conclusion $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$ now follows from [10, (2.8.b)]. □

The next result contains case (3) of Theorem 1.2 from the introduction. In preparation, recall that a prime ideal P in a local ring R is *analytically unramified* if the completion $\widehat{R/P}$ is reduced. For example, if R is excellent, then every prime ideal of R is analytically unramified.

Theorem 3.4. *Let $(A, \mathfrak{r})$ be a complete reduced local ring, and $S = A[\![x_1, \ldots, x_n]\!]$ the formal power series ring, with maximal ideal $\mathfrak{n} = (\mathfrak{r}, x_1, \ldots, x_n)S$. Let $I \subset S$ be an ideal generated by monomials in the x_i. Let R be a local Cohen–Macaulay ring such that $\widehat{R} \cong S/I$, and let C be a semidualizing R-module.*

(a) *Let $P \in \operatorname{Spec}(R)$ be analytically unramified. Then for every PR_P-primary ideal $J \subset R_P$, we have $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$.*
(b) *For every $\mathfrak{m}$-primary ideal $J \subset R$, we have $e_R(J; C) = e_R(J; R)$.*

Proof. (a) Since the natural map $R \to \widehat{R}$ is flat and local, there is a prime $\widetilde{P} \in \operatorname{Spec}(\widehat{R})$ such that $P = \widetilde{P} \cap R$ and that the induced map $R_P \to \widehat{R}_{\widetilde{P}}$ is flat and local.

The R_P-module C_P is semidualizing. Furthermore, by flat base-change, the $\widehat{R}_{\widetilde{P}}$-module $\widehat{R}_{\widetilde{P}} \otimes_{R_P} C_P$ is semidualizing. The fact that P is analytically unramified implies that the maximal ideal of R_P extends to the maximal ideal of $\widehat{R}_{\widetilde{P}}$. Hence, Lemma 2.8 yields the first and third equalities in the following sequence:

$$e_{R_P}(J; C_P) = e_{\widehat{R}_{\widetilde{P}}}(J\widehat{R}_{\widetilde{P}}; \widehat{R}_{\widetilde{P}} \otimes_{R_P} C_P) = e_{\widehat{R}_{\widetilde{P}}}(J\widehat{R}_{\widetilde{P}}; \widehat{R}_{\widetilde{P}}) = e_{R_P}(J; R_P).$$

The second equality is from Lemma 3.3.

(b) Since $R/\mathfrak{m}$ is a field, it is complete. Hence, the prime ideal $\mathfrak{m}$ is analytically unramified, so the desired conclusion follows from part (a). □

Corollary 3.5. *Let $(S, \mathfrak{n})$ be a regular local ring containing a field, and let $\mathbf{x} = x_1, \ldots, x_n$ be a regular system of parameters for S. Let $I \subset S$ be an ideal generated by monomials in the x_i, and set $R = S/I$ with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is Cohen-Macaulay, and let C be a semidualizing R-module. For every $P \in \operatorname{Ass}(R)$, we have $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$.*

Proof. Since R is Cohen–Macaulay, we have $P \in \operatorname{Min}(R)$. This explains the first and third equalities in the next sequence:

$$\operatorname{len}_{R_P}(C_P) = e_{R_P}(PR_P; C_P) = e_{R_P}(PR_P; R_P) = \operatorname{len}_{R_P}(R_P).$$

For the second equality, it suffices to show that P is analytically unramified; then the equality follows from Theorem 3.4(a).

Since I is generated by monomials in the x_i, the associated prime P has the form $P = (x_{i_1}, \ldots, x_{i_j})R$. This is, of course, standard when S is a polynomial ring. Since S is not a polynomial ring, we justify this statement. First note that each ideal $(x_{i_1}, \ldots, x_{i_j})R$ is prime because the sequence $\mathbf{x}$ is a regular system of parameters. Since $R = S/I$ is Cohen–Macaulay, the prime P is minimal in $\operatorname{Spec}(R)$. Let $\pi : S \to R$ be the canonical surjection, and set $Q = \pi^{-1}(P)$. The prime Q is a minimal prime for any primary decomposition of I, and it follows that Q is a minimal prime for any primary decomposition of the radical $\sqrt{I}$.

Because the sequence $\mathbf{x}$ is regular and contained in the Jacobson radical of S, a result of Heinzer, Mirbagheri, Ratliff, and Shah [7, (4.10)] implies that there are non-negative integers $u, e_{1,1}, \ldots, e_{1,n}, e_{2,1}, \ldots, e_{2,n}, \ldots, e_{u,1}, \ldots, e_{u,n}$ such that

$$I = \cap_{s=1}^{u}(x_1^{e_{s,1}}, \ldots, x_n^{e_{s,n}})S.$$

Since each ideal $(x_{i_1}, \ldots, x_{i_j})S$ is prime, it is straightforward to show that one has $\sqrt{(x_1^{e_{s,1}}, \ldots, x_n^{e_{s,n}})S} = (x_{i_1}, \ldots, x_{i_j})S$ and hence

$$\sqrt{I} = \cap_{s=1}^{u}(x_1^{\epsilon_{s,1}}, \ldots, x_n^{\epsilon_{s,n}})S \tag{3.5.1}$$

where

$$\epsilon_{s,i} = \begin{cases} 0 & \text{if } e_{s,i} = 0 \\ 1 & \text{if } e_{s,i} \neq 0. \end{cases}$$

Since each $(x_1^{\epsilon_{s,1}}, \ldots, x_n^{\epsilon_{s,n}})S$ is prime, the intersection (3.5.1) is a primary decomposition. It follows that $P = (x_1^{\epsilon_{s,1}}, \ldots, x_n^{\epsilon_{s,n}})S$ for some index s, so P has the desired form.

It follows that $R/P \cong S/(x_{i_1}, \ldots, x_{i_j})S$ is a regular local ring. Thus, the completion $\widehat{R/P}$ is also a regular local ring. In particular, the ring $\widehat{R/P}$ is an integral domain, so it is reduced, and P is analytically unramified by definition. □

We conclude with three results relating lengths and multiplicities to Betti numbers of semidualizing modules, starting with a general result for modules of infinite projective dimension.

Lemma 3.6. *Let R be a local ring such that $\operatorname{Ass}(R) = \operatorname{Min}(R)$. Let C be a finitely generated R-module of infinite projective dimension, and consider an exact sequence*

$$R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

Assume that for each $P \in \operatorname{Ass}(R)$ one has $\operatorname{len}_{R_P}(C_P) \leqslant \operatorname{len}_{R_P}(R_P)$. Then $a_1 \geqslant a_0$.

Proof. Suppose that $a_1 < a_0$, that is, that $a_1 - a_0 + 1 \leqslant 0$. Set $K = \operatorname{Ker}(\partial)$, and consider the exact sequence

$$0 \to K \to R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

Localize this sequence at an arbitrary $P \in \operatorname{Ass}(R)$, and count lengths to find that

$$0 \leqslant \operatorname{len}_{R_P}(K_P) \leqslant (a_1 - a_0 + 1)\operatorname{len}_{R_P}(R_P) \leqslant 0.$$

It follows that $K_P = 0$ for all $P \in \operatorname{Ass}(R)$.

Set $L = \operatorname{Im}(\partial)$ and localize the exact sequence

$$0 \to K \to R^{a_1} \xrightarrow{\tau} L \to 0$$

to conclude that $L_P \cong R_P^{a_1}$ for each $P \in \operatorname{Ass}(R)$. That is, the R-module L has rank a_1. Hence, we have the third step in the next sequence:

$$a_1 \geqslant \mu_R(L) \geqslant \operatorname{rank}_R(L) = a_1.$$

The first step is from the surjection τ. It follows that $\mu_R(L) = \operatorname{rank}_R(L)$, hence we conclude that L is free; see, e.g., [12, (1.12)]. The exact sequence

$$0 \to L \to R^{a_0} \to C \to 0$$

implies that $\operatorname{pd}_R(C)$ is finite, a contradiction. So, we have $a_1 \geqslant a_0$, as desired. □

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a local ring such that $\operatorname{Ass}(R) = \operatorname{Min}(R)$. Let C be a semidualizing R-module such that $C \not\cong R$, and consider an exact sequence*

$$R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

For each $P \in \mathrm{Ass}(R)$, assume that one of the following conditions holds:

(1) R_P *is Gorenstein*;
(2) $P^2 R_P = 0$;
(3) $R_P \cong S/\mathfrak{n}^e$ *for some regular local ring* $(S, \mathfrak{n})$ *containing a field and some integer* $e \geqslant 1$; *or*
(4) R_P *is isomorphic to a localization of a Cohen–Macaulay ring of the form S/I where S is a regular local ring containing a field with $x_1, \ldots, x_n \in S$ a regular system of parameters for S such that I is generated by monomials in the x_i.*

Then $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ *for each* $P \in \mathrm{Ass}(R)$. *It follows that $a_1 \geqslant a_0$ and that $e(J; C) = e(J; R)$ for each $\mathfrak{m}$-primary ideal J.*

Proof. We first show that $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ for each $P \in \mathrm{Ass}(R)$. If P satisfies condition (1) or (2), this is a consequence of Fact 2.6. Under conditions (3) and (4), we apply Lemma 3.1 and Corollary 3.5, respectively.

Now, the conclusion $a_1 \geqslant a_0$ follows from Lemma 3.6, since Fact 2.6 implies that $\mathrm{pd}_R(C) = \infty$. The equality $e(J; C) = e(J; R)$ for each $\mathfrak{m}$-primary ideal J follows from the additivity formula as in the proof of Theorem 3.2. □

The next result shows how the existence of a nontrivial semidualizing module yields an affirmative answer to [9, (2.6)].

Corollary 3.8. *Let R be a Cohen–Macaulay local ring with a dualizing module D. Let C be a semidualizing R-module such that $D \ncong C \ncong R$. If for each $P \in \mathrm{Ass}(R)$ one of the conditions* (1)–(4) *from Theorem* 3.7 *holds, then* $\beta_1^R(D) \geqslant 2\beta_0^R(D)$.

Proof. Set $C^\dagger = \mathrm{Hom}_R(C, D)$. The condition $D \ncong C$ implies that $C^\dagger \ncong R$ by Fact 2.4(f). Hence, Theorem 3.7 implies that $\beta_1^R(C^\dagger) \geqslant \beta_0^R(C^\dagger)$ and $\beta_1^R(C) \geqslant \beta_0^R(C)$. This explains the second step in the next sequence:

$$\beta_1^R(D) = \beta_1^R(C)\beta_0^R(C^\dagger) + \beta_0^R(C)\beta_1^R(C^\dagger) \geqslant 2\beta_0^R(C)\beta_0^R(C^\dagger) = 2\beta_0^R(D).$$

The first and third steps follow from Fact 2.4(g). □

ACKNOWLEDGMENTS

We are grateful to Cătălin Ciupercă for helpful discussions about this material. Sean Sather-Wagstaff was supported in part by a grant from the NSA.

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