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On Zero Divisor Graphs

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Abstract. We survey the research conducted on zero divisor graphs, with a focus on zero divisor graphs determined by equivalence classes of zero divisors of a commutative ring R. In particular, we consider the problem of classifying star graphs with any finite number of vertices. We study the pathology of a zero divisor graph in terms of cliques, we investigate when the clique and chromatic numbers are equal, and we show that the girth of a Noetherian ring, if finite, is 3. We also introduce a graph for modules that is useful for studying zero divisor graphs of trivial extensions.

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1 Introduction

In this paper, the term "ring" (unless explicitly stated otherwise) means "commutative ring with identity," and ring homomorphisms are assumed to respect identities.

This paper continues with the overarching goal of research in the area of zero divisor graphs, namely the investigation of the interplay between the ring-theoretic properties of a ring R and the graph theoretic properties of certain graphs obtained from R. Our particular focus is on $\Gamma_E(R)$, the zero divisor graph determined by equivalence classes, introduced in [29], and further studied in [8, 35] (see Definition 2.13). We sometimes discuss $G(R)$, the graph defined by I. Beck, and $\Gamma(R)$, the graph defined by D. F. Anderson and P. S. Livingston; see Definitions 2.1 and 2.4. A survey of the research concerning these graphs is given in Section 2.

The graph $\Gamma_E(R)$ is a condensed version of $G(R)$ and $\Gamma(R)$, constructed in such a way as to reduce the "noise" produced by individual zero divisors. (In [8], this is called the "compressed" zero divisor graph.) Accordingly, $\Gamma_E(R)$ is smaller and simpler than $G(R)$ and $\Gamma(R)$. One might expect that these graphs are finite or at least have a finite clique number[1] if some finiteness condition is imposed on the ring, for example, if the ring is Noetherian or Artinian. However, in [35], S. Spiroff and C. Wickham show that the Noetherian condition is not enough to ensure a finite graph by exhibiting a

[1] See Appendix B for a brief dictionary of terms from graph theory.

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Noetherian ring R such that $\Gamma_E(R)$ is an infinite star. Moreover, in the current paper, we show how to construct an Artinian ring R such that $\Gamma_E(R)$ is an infinite star; see Examples 3.28–3.30. Proposition 5.3 shows that these examples are minimal with respect to length.

Recall that Anderson and Livingston [10] completely characterized the star graphs of the form $\Gamma(R)$ where R is finite, proving that the star graphs G that occur as $\Gamma(R)$ are precisely those such that $|G|$ is a prime power. This leads to the question of whether or not a star graph of any size could be realized as $\Gamma_E(R)$. In Section 3, we investigate this question. In particular, Example 3.25 shows how to find rings R such that $\Gamma_E(R)$ is a star with c vertices where c is any positive number of the following form

$$\begin{array}{llll}
2^n - 4, & 2^n - 3, & 2^n - 2, & 2^n - 1, \\
2^n, & 2^n + 1, & 2^n + 2, & 2^n + 3, \\
2^n \cdot 3 - 2, & 2^n \cdot 3 - 1, & 2^n \cdot 3, & 2^n \cdot 3 + 1, \\
2^n \cdot 3 + 2, & 2^n \cdot 3 + 3, & 2^n \cdot 7 - 4, & 2^n \cdot 7 - 3, \\
2^n \cdot 7 - 2, & 2^n \cdot 7 - 1, & 2^n \cdot 7, & 2^n \cdot 7 + 1, \\
2^n \cdot 7 + 2, & 2^n \cdot 7 + 3, & 2^n \cdot 15 - 12, & 2^n \cdot 15 - 11, \\
2^n \cdot 15 - 6, & 2^n \cdot 15 - 5, & 2^n \cdot 15 - 4, & 2^n \cdot 15 - 3 \\
2^n \cdot 15, & 2^n \cdot 15 + 1, & 2^n \cdot 15 + 2, & 2^n \cdot 15 + 3
\end{array}$$

with n a non-negative integer. At this time, the smallest star graph we do not know how to obtain is the star with 36 vertices; see Examples 3.14, 3.15, and 3.24.

In addition, we show that the Artinian condition not is enough to guarantee finite clique number. In particular, in Section 5, we construct an Artinian local ring with length 6 whose graph contains an infinite clique; see Example 5.2. Our method uses the trivial extension of the ring by its dualizing module. This is facilitated by our use of a graph associated to an R-module M, called the torsion graph of M. In an effort to show that our example with infinite clique number is minimal, we show that for rings R of length at most 4, the graph $\Gamma_E(R)$ has a finite clique number; see Propositions 5.3 and 5.8. The case where R is local of length 5 is still open.

In terms of the classification of these zero divisor graphs, we investigate cut vertices, girth, and edge domination in Section 6 when R is Noetherian. In particular, we show that a cut vertex of $\Gamma_E(R)$ corresponds to an associated prime, and the girth of the graph, if finite, is 3; see Proposition 6.9 and Theorem 6.6. (If R is non-Noetherian and $\operatorname{girth}(\Gamma_E(R)) < \infty$, then $\operatorname{girth}(\Gamma_E(R)) \leq 4$; see Proposition 6.1 (iii).)

In keeping with the previous research on zero divisor graphs, we also consider graph homomorphisms and colorings in Sections 4 and 7, respectively. For colorings, we are able to address a version of Beck's conjecture regarding chromatic numbers for the graph $\Gamma_E(R)$ by constructing a ring R such that $\omega(\Gamma_E(R)) = 3$, but $\chi(\Gamma_E(R)) = 4$; see Example 7.7.

2 Survey of Past Research on Zero Divisor Graphs

Because so much literature has been written on the topic of various zero divisor graphs, often from very different points of view, we collect here an overview of the material. The terms in **bold** are defined within the text, while the *italicized* terms appear in Appendix B. Throughout, and unless otherwise stated, R will be a commutative ring with unity.

2.1 Beck's Zero Divisor Graph

The idea of a zero divisor *graph* originated with I. Beck [12].

Definition 2.1 ([12]). Given a ring R, let $G(R)$ denote the graph whose vertex set is R, such that distinct vertices r and s are *adjacent* provided that $rs = 0$.

By definition, $G(R)$ is a *simple graph*, so there are no loops; thus the existence of self-annihilating elements of R is not encoded in the graph. Moreover, because the zero vertex is adjacent to every ring element, the graph $G(R)$ is *connected* with *diameter* at most 2.

Beck's main interest was the *chromatic number* $\chi(G(R))$ of the graph $G(R)$. He conjectured that $\chi(G(R))$ equals $\omega(G(R))$, the *clique number* of $G(R)$. The clique number is a lower bound for the chromatic number since all the vertices in a *clique* are adjacent to one another and require distinct colors. Moreover, we have the following:

Theorem 2.2 ([12, Theorems 3.9, 6.13, 7.3, Propositions 7.1, 7.2]). *Let R be a ring.*

(i) *The following conditions are equivalent:*

   (a) $\chi(G(R))$ *is finite;*

   (b) $\omega(G(R))$ *is finite;*

   (c) *the nilradical of R is finite and is a finite intersection of prime ideals; and*

   (d) $G(R)$ *does not contain an infinite clique.*

(ii) *Let R be such that $\chi(G(R))$ is finite. If R is a finite product of reduced rings and principal ideal rings, then $\omega(G(R)) = \chi(G(R))$.*

(iii) *If $\chi(G(R)) < \infty$, then $\chi(G(R)) = n$ if and only if $\omega(G(R)) = n$, for $n \leq 4$.*

(iv) *If $\chi(G(R)) = 5$, then $\omega(G(R)) = 5$.*

In addition, Beck lists all the finite rings R with $\chi(G(R)) \leq 3$.

Although this result provides evidence for Beck's conjecture, D. D. Anderson and M. Naseer [4] provided a example where the chromatic number is strictly greater than the clique number.

Example 2.3 ([4, Theorem 2.1]). If R is the factor ring of $\mathbb{Z}_4[X, Y, Z]$ determined by the ideal $(X^2 - 2, Y^2, Z^2, 2X, 2Y, 2Z, YX, XZ, YZ - 2)$, then $\chi(G(R)) = 6$ and $\omega(G(R)) = 5$.

Moreover, they extended Beck's classification of finite rings with small chromatic number to $\chi(G(R)) = 4$.

2.2 Anderson and Livingston's Zero Divisor Graph

The first simplification of Beck's zero divisor graph was introduced by D. F. Anderson and P. S. Livingston [10]. Their motivation was to give a better illustration of the zero divisor structure of the ring. In this new zero divisor graph, which is still a simple graph with edges defined the same way as above, only the **zero divisors** of the ring are included; i.e., non-zero elements r of R such that $\mathrm{Ann}_R(r) \neq (0)$.

Definition 2.4 ([10]). Given a ring R, let $Z^*(R)$ denote the set of zero divisors of R. Let $\Gamma(R)$ denote the graph whose vertex set is $Z^*(R)$, such that distinct vertices r and s are adjacent provided that $rs = 0$.

In general, we have the following, despite the absence of the zero vertex:

Theorem 2.5 ([10, Theorem 2.3]). *Given a ring R, the graph $\Gamma(R)$ is connected with diameter at most* 3.

Anderson and Livingston often focus on the case when R is finite, as these rings yield finite graphs. They determine for which rings the graph is *complete* or a *star*. For the stars, we have the following:

Theorem 2.6 ([10, Theorem 2.13]). *Given a finite ring R, if the graph $\Gamma(R)$ is a star with at least four vertices, then $|\Gamma(R)| = p^n$, for some prime p and integer $n \geq 0$. Moreover, each star graph of order p^n can be realized as $\Gamma(R)$ for some R.*

Anderson and Livingston, and others, e.g., [1, 2, 7, 29], investigate the interplay between the graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of R. For example, D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston [7] study the clique number of $\Gamma(R)$ and the relationship between graph isomorphisms and ring isomorphisms. A particularly important and surprising result is the following:

Theorem 2.7 ([7, Theorem 4.1]). *Given finite reduced rings R and S that are not fields, the graphs $\Gamma(R)$ and $\Gamma(S)$ are graph isomorphic if and only if R and S are ring isomorphic.*

The authors [7] also determine all n for which $\Gamma(\mathbb{Z}_n)$ is *planar*, and pose the question of which finite rings in general determine a planar zero divisor graph. This was partially answered by S. Akbari, H. R. Maimani, and S. Yassemi [1], who were able to refine the question to local rings of cardinality at most thirty-two:

Theorem 2.8 ([1, Theorems 1.2 and 1.4]). *If R is a finite local ring that is not a field and contains at least thirty-three elements, then $\Gamma(R)$ is not planar.*

However, at the same time, N. O. Smith [33], independently provided a complete answer, as well as a classification of which rings are planar, listing forty-four isomorphism classes in all.

Theorem 2.9 ([33, Theorems 3.7 and Corollary 3.8]). *If R is a finite local ring that is not a field and contains at least twenty-eight elements or ten zero divisors, then $\Gamma(R)$ is not planar.*

Some of these results were recovered by R. Belshoff and J. Chapman [13], who have also worked on questions concerning planarity, also known as genus zero. Additionally, Smith [34] studied planarity of infinite rings, as well as zero divisor graphs with genus one, also known as toroidal zero divisor graphs. In particular, H.-J. Chiang-Hsieh, N. O. Smith, and H.-J. Wang [16] consider rings with toroidal zero divisor graphs. C. Wickham [36] is another researcher who has studied zero divisor graphs of genus one. Moreover, along with N. Bloomfield, C. Wickham [14] considers graphs of genus two.

A key component to proofs concerning planarity is Kuratowski's Theorem, which says that a graph is planar if and only if it contains no subgraph homeomorphic to the complete graph K_5 or the *complete bipartite* graph $K_{3,3}$. Akbari, Maimani, and Yassemi [1] list the rings that determine a *complete r-partite* graph. In particular, they show the following:

Theorem 2.10 ([1, Theorems 2.4 and 3.2]). *Let R be a finite ring such that $\Gamma(R)$ is r-partite.*

(i) *Then r is a prime power.*

(ii) *If $r \geq 3$, then at most one partitioning subset of $\Gamma(R)$ can have more than one vertex.*

(iii) *If R is reduced, then $\Gamma(R)$ is bipartite (i.e., $r = 2$) if and only if there exist two distinct primes in R with trivial intersection.*

(iv) *If R is reduced and $\Gamma(R)$ is bipartite, then $\Gamma(R)$ is complete bipartite.*

These results are similar to those of Theorem 2.6 which describe the rings R such that $\Gamma(R)$ is a star, i.e., a complete bipartite graph of the form $K_{1,n}$.

Another graph invariant that is studied for zero divisor graphs is the *girth*. Anderson and Livingston showed that if R is Artinian and $\Gamma(R)$ contains a cycle, then the girth is no more than four, and they conjectured that this upper bound would hold in general. This conjecture was subsequently, and independently, established as fact by F. DeMeyer and K. Schneider [19] and S. B. Mulay [29].

Theorem 2.11 ([19, Theorem 1.6], [29, (1.4)]). *Given a ring R, if $\Gamma(R)$ is not acyclic, then the girth of $\Gamma(R)$ is at most* 4.

This bound on the girth is sharp, given the following:

Example 2.12 ([10, Example 2.1 (b)]). The graph $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a 4-cycle.

Moreover, this example shows that zero divisor graphs are not *chordal*. This jibes with the fact chordal graphs are *perfect*. Additional results concerning the girth of $\Gamma(R)$ can also be found in [1, 7].

2.3 Mulay's Zero Divisor Graph

S. B. Mulay [29] introduces the next zero divisor graph associated to a ring.

Definition 2.13 ([29]). Given a ring R, two zero divisors $r, s \in Z^*(R)$ are **equivalent** if $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(s)$. The equivalence class of r is denoted $[r]$. The graph $\Gamma_E(R)$ has vertex set equal to the set of equivalence classes $\{[r] \mid r \in Z^*(R)\}$, and distinct classes $[r]$ and $[s]$ are adjacent in $\Gamma_E(R)$ provided that $rs = 0$ in R.

It is shown in [29] that this is well-defined, that is, that adjacency in $\Gamma_E(R)$ is independent of representatives of $[r]$ and $[s]$. By definition, the graph $\Gamma_E(R)$ is simple. Furthermore, we have the following:

Theorem 2.14 ([29]; see also [18, Theorem 1.2] and [35, Proposition 1.4]). *Given a ring R, the graph $\Gamma_E(R)$ is connected with diameter at most* 3.

In [35], S. Spiroff and C. Wickham compare and contrast $\Gamma_E(R)$ with $\Gamma(R)$. One important difference between this new graph and its two predecessors is that $\Gamma_E(R)$ can be finite even when R is infinite, thus giving a more succinct visual description of the zero divisor structure of the ring. Another difference is found in the set of graphs that can be realized as $\Gamma_E(R)$. For instance, we have the following, in contrast with the results of [1, 10]:

Theorem 2.15 ([35]). *Let R be a Noetherian ring.*

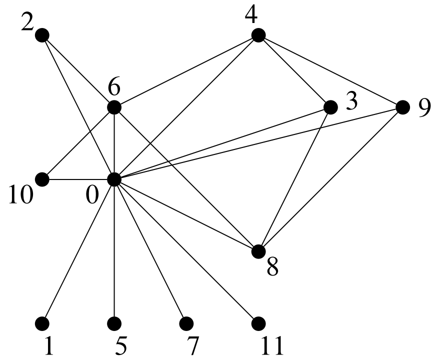
(i) *If $\Gamma_E(R)$ is complete K_n, then $n = 2$.*
(ii) *If $\Gamma_E(R)$ is complete bipartite $K_{n,m}$, then $n = 1$, i.e., $\Gamma_E(R)$ is a star.*
(iii) *If $\Gamma_E(R)$ has at least three vertices, then it is not a cycle, more generally, it is not regular.*

One important aspect of this graph is that, since the vertices in the graph correspond to annihilator ideals in the ring, the associated primes of R are represented in $\Gamma_E(R)$.

In order to illustrate the difference between the three zero divisor graphs discussed so far, we provide an example of each for the same ring.

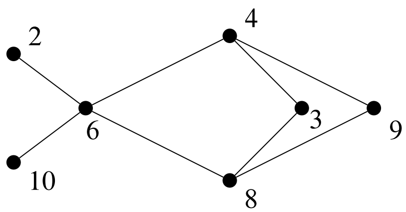
Example 2.16. Let $R = \mathbb{Z}/12\mathbb{Z}$.

$G(\mathbb{Z}/12\mathbb{Z})$



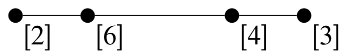
In Beck's graph above, every element of $\mathbb{Z}/12\mathbb{Z}$ is represented by a distinct vertex.

$\Gamma(\mathbb{Z}/12\mathbb{Z})$



Anderson and Livingston's graph includes only the zero divisors, but each such element determines a distinct vertex.

$\Gamma_E(\mathbb{Z}/12\mathbb{Z})$



In Mulay's graph, the four distinct classes are determined by $\mathrm{Ann}_R(2) = (6)$, $\mathrm{Ann}_R(3) = (4)$, $\mathrm{Ann}_R(4) = (3)$, and $\mathrm{Ann}_R(6) = (2)$.

2.4 Other Zero Divisor Graphs

S. P. Redmond [31] introduces a zero divisor graph with respect to an ideal.

Definition 2.17 ([31]). Given a ring R and an ideal I, the graph $\Gamma_I(R)$ has vertices x from $R \backslash I$ such that $(I :_R x) \neq I$. Distinct vertices x and y are adjacent if $xy \in I$.

Of course, if $I = (0)$, then $\Gamma_I(R)$ is just $\Gamma(R)$. If I is prime, then $\Gamma_I(R) = \emptyset$. Redmond discusses the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$.

Theorem 2.18 ([31, Corollary 2.7 and Remark 2.8]). *Given a ring R and an ideal I, the graph $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$. Moreover, if $\Gamma(R/I)$ is a graph on n vertices, then $\Gamma_I(R)$ has $n \cdot |I|$ vertices.*

In contrast with Theorem 2.7, we have the following:

Example 2.19 ([31, Remark 2.3]). Set $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $S = \mathbb{Z}_{24}$, with ideals $I = (0) \times \mathbb{Z}_3$ and $J = (8)$, respectively. Then the graphs $\Gamma(R/I)$ and $\Gamma(S/J)$ are isomorphic, but $\Gamma_I(R)$ and $\Gamma_J(S)$ are not.

H. R. Maimani, M R. Pournaki, and S. Yassemi [27] continue the study of this new graph and take up the question of when $\Gamma_I(R) \cong \Gamma_J(S)$ might imply $\Gamma(R/I) \cong \Gamma(S/J)$. They show the following:

Theorem 2.20 ([27, Theorem 2.2]). *If I and J are finite radical ideals of the rings R and S, respectively, then $\Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$ iff $\Gamma_I(R) \cong \Gamma_J(S)$.*

Further incarnations of zero divisor graphs involve objects other than commutative rings. For example, given a commutative semigroup S, expressed multiplicatively, which contains 0, F. DeMeyer, T. McKenzie, and K. Schneider [18] defined a graph in the spirit of Anderson and Livingston.

Definition 2.21 ([18]). Let S be a commutative multiplicative semigroup with 0. Denote by $\Gamma(S)$ the graph whose vertex set is the (non-zero) zero divisors of S, with an edge drawn between distinct zero divisors x and y if and only if $xy = 0$.

F. DeMeyer and L. DeMeyer [17] further this construction and give some necessary conditions for a graph G to be of the form $\Gamma(S)$. For example:

Theorem 2.22 ([17, Theorem 1]). *If G is the graph of a semigroup, then for each pair x, y of non-adjacent vertices of G, there is a vertex z with $\mathcal{N}(x) \cup \mathcal{N}(y) \subseteq \overline{\mathcal{N}(z)}$, where $\overline{\mathcal{N}(z)} = \mathcal{N}(z) \cup \{z\}$ is the* closure *of the* neighborhood *$\mathcal{N}(z)$ of z.*

In addition, they provide some classes of graphs that can be realized from semigroups, e.g., G is complete, complete bipartite, or has at least one *end* and diameter 2. See [17, Theorems 1 and 3]. They also consider a zero divisor graph more in line with Beck's original one by including 0 in the vertex set. Denote this graph by $G(S)$. In [30], S. K. Nimbhokar, M. P. Wasadikar, and L. DeMeyer study these graphs under the additional assumption that every element of S is idempotent, in which case S is called a meet-semilattice, and show that a version of Beck's conjecture regarding the chromatic number holds in this setting:

Theorem 2.23 ([30, Theorem 2 and Corollary 1]). *Let S be a commutative multiplicative semigroup with 0 such that every element of S is idempotent. If $\omega(G(S)) < \infty$, then $\chi(G(S)) = \omega(G(S))$ and $\chi(\Gamma(S)) = \omega(\Gamma(S))$.*

Zero divisor graphs associated to semigroups are also studied by L. DeMeyer, L. Greve, A. Sabbaghi, and J. Wang [21] and L. DeMeyer, M. D'Sa, I. Epstein, A. Geiser, and K. Smith [20], among others.

Remark 2.24. It is important to note that when R is a commutative ring, $\Gamma_E(R)$ is the zero divisor graph of a semigroup, namely the semigroup determined by the equivalence classes of zero divisors. Therefore, some of the results on semigroups may be applied to $\Gamma_E(R)$; e.g., connected and diameter less than or equal to three [18, Theorem 1.2]. However, not every semigroup graph can be obtained as $\Gamma_E(R)$ for some commutative ring R; e.g., if G is complete or complete bipartite but not a star graph, then it can be realized as $\Gamma(S)$ [17, Theorems 3], but not as $\Gamma_E(R)$ [35, Propositions 1.5 and 1.7].

Yet another interpretation of a zero divisor graph focuses on posets P containing 0, a concept which was introduced by R. Halaš and M. Jukl [24]. Their graph is in the spirit of Beck's original definition.

Definition 2.25 ([24]). Given a poset P containing 0, let $G(P)$ be the graph whose vertex set is P, such that distinct vertices x and y are adjacent provided that 0 is the only element lying below x and y.

Theorem 2.26 ([24, Theorem 2.9]). *Given a poset P containing* 0*, if $\omega(G(P))$ is finite, then $\chi(G(P)) = \omega(G(P))$.*

Subsequently, D. Lu and T. Wu [26] define a zero divisor graph for posets à la Anderson and Livingston:

Definition 2.27 ([26]). Let P be a poset containing 0. A non-zero element $x \in P$ is a **zero divisor** if there is a non-zero element $y \in P$ such that 0 is the only element lying below x and y. Let $\Gamma(P)$ be the graph whose vertex set is $Z^*(P)$, such that distinct vertices x and y are adjacent provided that 0 is the only element lying below x and y.

Of particular importance in the work in [26] is the notion of a *compact graph*. For instance, we have the following:

Theorem 2.28 ([26, Theorems 3.1 and 3.2]). *A simple graph G is the zero divisor graph of a poset if and only if G is compact. Moreover, if G is compact with $\omega(G) < \infty$, then $\omega(G) = \chi(G)$.*

In general, zero divisor graphs of rings are not compact because of the possible existence of nilpotent elements in the ring and the absence of loops in the graph. Accordingly, it can be shown that reduced rings yield compact zero divisor graphs (using any of the definitions), and hence satisfy $\omega(G) = \chi(G)$ whenever either is finite, or $\omega(G)$ is infinite. Similarly, we have the following:

Theorem 2.29 ([26, Propositions 2.2 (1) and 4.1]). *Given a commutative reduced multiplicative semigroup S with* 0*, the graph $\Gamma(S)$ is compact and if $\Gamma(S)$ has finite girth, then it has girth at most* 4.

Lastly, zero divisor graphs have also been defined for non-commutative rings.

Definition 2.30 ([32]). In a non-commutative ring D, a non-zero element x is a **zero divisor** if either $xy = 0$ or $yx = 0$ for some non-zero element y. Let $\Gamma(D)$ be the *directed graph* whose vertices are the zero divisors of D, with an edge $x \to y$ drawn between distinct vertices provided that $xy = 0$.

From the viewpoint that an undirected edge is a pair of directed edges, this definition reverts to that of $\Gamma(R)$ in the case of a commutative ring R. S. P. Redmond showed that connectivity in the non-commutative case depends upon whether or not the set of left and right zero-divisors coincide. S. Akbari and A. Mohammadian [3] continue the study of this directed graph, giving an example of the smallest zero divisor graph associated to a non-commutative ring, namely $\Gamma(D)$, for $D = \left\{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] : a, b \in \mathbb{Z}_2\right\}$, which has the form $E_{11} \leftarrow E_{12} \to (E_{11} + E_{12})$.

Redmond also defined an undirected graph for a non-commutative ring.

Definition 2.31 ([32]). Given a non-commutative ring D, let $\Gamma'(D)$ be the graph whose vertices are the zero divisors of D, with an edge drawn between distinct vertices x and y provided that $xy = 0$ or $yx = 0$.

For this graph, the properties of connectivity, diameter less than or equal to 3, and girth less than or equal to 4 when finite, all hold, as with earlier zero divisor graphs. Moreover, Akbari and Mohammadian show the following:

Theorem 2.32 ([3, Corollary 10]). *A finite star graph can be realized as $\Gamma'(D)$ if and only if the vertices number p^n or $2p^n - 1$, for some prime p and some integer $n \geq 0$.*

It should be noted that over 100 papers, by many authors, have been written on the topic of zero divisor graphs, and hence we only highlight a handful of results from a few papers. Our aim in the above survey is to give a flavor of the available research on zero divisor graphs, especially as it pertains to the current project, which focuses on $\Gamma_E(R)$, the zero divisor graph determined by equivalence classes. For another survey article on the topic of zero divisor graphs, see D. F. Anderson, M. C. Axtell, and J. A. Stickles [6].

3 Star Graphs

In this section, we describe some rings R such that $\Gamma_E(R)$ is a star. The constructions are not only different, but more complicated than for $\Gamma(R)$, which have a nice characterization, namely having exactly p^n vertices for p a prime and $n \geq 0$. By our methods, stars with $c < 100$ vertices can be constructed as $\Gamma_E(R)$ for the following c values: 1–35, 42–67, 90–99. At the time of this publication, we do not have a complete characterization of which stars are possible.

To motivate our constructions, note that small stars, as well as an infinite star, are constructed in [35]; see also Examples 3.28–3.30. Thus, we may assume that $\Gamma_E(R)$

has at least four vertices. When R is Noetherian, then $\mathrm{Ass}(R) = \{\mathfrak{p}\}$, with $\mathfrak{p}^3 = 0$, and the characteristic of R is either 2, 4, or 8 by [35, Proposition 2.4]. Furthermore, since localization at $\mathfrak{p}$ does not change the graph (see Corollary 4.3), we may take R to be Artinian and local with maximal ideal $\mathfrak{m} = \mathfrak{p}$.

For simplicity, we focus on the case where R is finite of characteristic 2. Also, we take all zero divisors of R to be square-zero, since at most one class of zero divisor can fail to have this property. To aid in computations, we focus on the case where R is a standard graded algebra over $\mathbb{F}_2$ with irrelevant maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^3 = 0$. This implies that the socle of R contains the graded ideal $R_2 = \mathfrak{m}^2$. Thus, the only non-trivial zero divisors are in R_1, that is, they are linear forms in the generators of $\mathfrak{m}$. Furthermore, we have $R \cong \mathbb{F}_2 \oplus R_1 \oplus R_2$.

We begin with a construction $R^\flat$ that yields stars with even numbers of vertices. The distinct annihilator ideals of $R^\flat$ are described in Proposition 3.11. The structure of $\Gamma_E(R^\flat)$ is given in Theorems 3.12 and 3.13, and some specific star graphs are described in Examples 3.14–3.15. In addition, at the end of the section we detail some examples (3.28–3.30) that are relevant to our study in Section 5 of clique numbers of “small”rings.

Construction 3.1. Let R be a $\mathbb{Z}$-graded ring $R = \mathbb{F}_2 \oplus R_1 \oplus R_2$ generated over $\mathbb{F}_2$ by R_1 such that $r^2 = 0$ for all $r \in R_1$. Set $d = \dim_{\mathbb{F}_2}(R_1)$, and choose a basis $X_1, \ldots, X_d$ for R_1 over $\mathbb{F}_2$. Assume that $d \geq 1$, and fix integers e, t such that $e \geq 1$ and $1 \leq t \leq \min(d, e)$. Let $\mathbf{Y} = Y_1, \ldots, Y_e$ be a sequence of indeterminates, and set

$$\begin{aligned} R' &= R[\mathbf{Y}]/(R_2 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) + (\mathbf{Y})^2) \\ R^\flat &= R[\mathbf{Y}]/(R_2 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) + (\mathbf{Y})^2 + (X_i X_j + X_i Y_j + X_j Y_i \mid 1 \leq i < j \leq t)) \\ &\cong R'/(X_i X_j + X_i Y_j + X_j Y_i \mid 1 \leq i < j \leq t). \end{aligned}$$

Remark 3.2. Under the assumptions and notation of Construction 3.1, the ring R is local with maximal ideal $\mathfrak{m} = R_+ = 0 \oplus R_1 \oplus R_2$ because the ideal $\mathfrak{m}$ is maximal and $\mathfrak{m}^3 = 0$. The ring R' is the special case of $R^\flat$ where $t = 1$. The polynomial ring $R[\mathbf{Y}]$ is $\mathbb{Z}^2$-graded where $R[\mathbf{Y}]_{(i,j)}$ consists of all the homogeneous forms in $R[\mathbf{Y}]$ of degree j with coefficients in R_i. For instance, this provides

$$R_i \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) = R[\mathbf{Y}]_{(i,1)} \cong \bigoplus_{k=1}^{e} R_i Y_k.$$

The next example shows how to build rings R that satisfy the assumptions of Construction 3.1.

Example 3.3. Fix an integer $d \geq 1$, and let $\mathbf{X} = X_1, \ldots, X_d$ be indeterminates.

(i) First, we consider the ring $R = \mathbb{F}_2[\mathbf{X}]/(\mathbf{X})^2$. Since the ideal $(\mathbf{X})^2$ is homogeneous, the quotient ring R is graded as follows:

$$R = \mathbb{F}_2 \oplus \operatorname{Span}_{\mathbb{F}_2}(\mathbf{X}).$$

It follows that $R_2 = 0$, so $r^2 = 0$ for all $r \in R_+ = 0 \oplus R_1$. We conclude that $\operatorname{Soc}(R) = R_1$, and $\Gamma_E(R)$ is a single vertex $[X_1]$.

(ii) Next, we consider the ring

$$R = \mathbb{F}_2[\mathbf{X}]/((X_1^2, \ldots, X_d^2) + (\mathbf{X})^3).$$

Since the ideal $(X_1^2, \ldots, X_d^2) + (\mathbf{X})^3$ is homogeneous, the quotient ring R is graded as follows:

$$R = \mathbb{F}_2 \oplus \operatorname{Span}_{\mathbb{F}_2}(\mathbf{X}) \oplus \operatorname{Span}_{\mathbb{F}_2}(\{X_i X_j \mid i \neq j\}).$$

Since $X_i^2 = 0$ in R for all i, it follows that $r^2 = 0$ for all $r \in R_+ = 0 \oplus R_1 \oplus R_2$.

If $d = 1$, then R is the same as the ring constructed in part (i), so $\Gamma_E(R)$ is a single vertex $[X_1]$. Assume then that $d \geq 2$. In this event, $\Gamma_E(R)$ is a star with 2^d vertices. Specifically, the zero divisors of R are the non-zero elements of R_+, and for each $l \in R_1$ and $f \in R_2$ with $l, f \neq 0$, one has

$$\begin{aligned} \operatorname{Ann}_R(f) &= R_+ = R_1 \oplus R_2 \\ \operatorname{Ann}_R(l) &= \operatorname{Ann}_R(l + f) = 0 \oplus \operatorname{Span}_{\mathbb{F}_2}(l) \oplus R_2. \end{aligned}$$

(Argue as in the proofs of Propositions 3.8 and 3.11.) Thus, $\Gamma_E(R)$ is a star with central vertex $[f]$ and with edges $[f]-[l_1], \ldots, [f]-[l_{2^d-1}]$ where $l_1, \ldots, l_{2^d-1}$ are the distinct non-zero elements of R_1.

Proposition 3.4. *Continue with the assumptions and notation of Construction* 3.1.

(i) *The ring R' is $\mathbb{Z}^2$-graded with*

$$\begin{aligned} R' &= R'_{(0,0)} \oplus [R'_{(1,0)} \oplus R'_{(0,1)}] \oplus [R'_{(2,0)} \oplus R'_{(1,1)}] \\ &\cong \mathbb{F}_2 \oplus [R_1 \oplus \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})] \oplus [R_2 \oplus R_1 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})]. \end{aligned}$$

(ii) *The ring R' is local with maximal ideal*

$$\mathfrak{m}' = R'_+ = 0 \oplus [R'_{(1,0)} \oplus R'_{(0,1)}] \oplus [R'_{(2,0)} \oplus R'_{(1,1)}].$$

(iii) *For each non-unit $f \in R'$, we have $f^2 = 0$.*

Proof. (i) Following Remark 3.2, we have $R_2 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) = R[\mathbf{Y}]_{(2,1)}$, and the ideal $(\mathbf{Y})^2$ is generated by $R[\mathbf{Y}]_{(0,2)}$. It follows that the ideal $I = (R_2 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) + (\mathbf{Y})^2) \subseteq R[\mathbf{Y}]$ is $\mathbb{Z}^2$-graded, generated by $R[\mathbf{Y}]_{(2,1)} + R[\mathbf{Y}]_{(0,2)}$. In other words, I is the direct sum of $R[\mathbf{Y}]_{(i,j)}$ taken over the set of all ordered pairs (i, j) such that either $(j \geq 2)$ or $(i \geq 2$ and $j \geq 1)$. Since $R_i = 0$ for all $i \geq 3$, the only bi-graded pieces of $R[\mathbf{Y}]$ that survive in the quotient R' are the following

$$\begin{aligned} R'_{(0,0)} &= R[\mathbf{Y}]_{(0,0)} = \mathbb{F}_2, \quad & R'_{(0,1)} &= R[\mathbf{Y}]_{(0,1)} = \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}), \\ R'_{(1,0)} &= R[\mathbf{Y}]_{(1,0)} = R_1, \quad & R'_{(1,1)} &= R[\mathbf{Y}]_{(1,1)} = R_1 \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}), \\ R'_{(2,0)} &= R[\mathbf{Y}]_{(2,0)} = R_2. \end{aligned}$$

(ii) Since $R'_{(1,0)} = R_1$ and $R'_{(0,1)} = \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})$ consist of square-zero elements, the ideal $\mathfrak{m}$ they generate is nilpotent. Since $\mathfrak{m}$ is maximal, it is therefore the unique maximal ideal.

(iii) Every non-unit $f \in R'$ is of the form $f_1 + \sum_j g_j Y_j$ where f_1 is a non-unit of R and $g_j \in R$. Since we are working over $\mathbb{F}_2$, the assumption $f_1^2 = 0$ implies that $f^2 = f_1^2 + \sum_j g_j^2 Y_j^2 = 0 + \sum_j g_j^2 0 = 0$. □

We will often make use of the $\mathbb{Z}^2$-graded structure of R' from Proposition 3.4 (i). Sometimes, though, we only need to know that R' is $\mathbb{Z}$-graded, where we use the standard induced $\mathbb{Z}$-grading:

$$\begin{aligned}
R'_i &= \bigoplus_{p+q=i} R_{(p,q)},\\
R'_0 &\cong \mathbb{F}_2 \oplus 0 \oplus 0,\\
R'_1 &\cong 0 \oplus [R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})] \oplus 0,\\
R'_2 &\cong 0 \oplus 0 \oplus [R_2 \oplus R_1 \,\mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})],\\
R'_i &= 0 \quad \text{for all } i \geq 3.
\end{aligned}$$

Proposition 3.5. *Continue with the assumptions and notation of Construction* 3.1.

(i) *The ring $R^\flat$ is $\mathbb{Z}$-graded with*

$$\begin{aligned}
R^\flat &= R^\flat_0 \oplus R^\flat_1 \oplus R^\flat_2\\
&\cong \mathbb{F}_2 \oplus [R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})]\\
&\quad \oplus \left[\frac{R_2 \oplus R_1 \,\mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})}{\mathrm{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \leq i < j \leq t)}\right].
\end{aligned}$$

(ii) *The ring $R^\flat$ is local with maximal ideal*

$$\mathfrak{m}^\flat = R^\flat_+ = 0 \oplus R^\flat_1 \oplus R^\flat_2.$$

(iii) *For each non-unit $f \in R^\flat$, we have $f^2 = 0$.*

(iv) *For $1 \leq i \leq j \leq t$, we have $(X_i + Y_i)(X_j + Y_j) = 0$ in $R^\flat$.*

(v) $\dim_{\mathbb{F}_2}(R^\flat_1) = d + e$.

Proof. (i) The elements $X_iX_j + X_iY_j + X_jY_i \in R'$ are $\mathbb{Z}$-homogeneous of degree 2, so the quotient

$$R^\flat \cong R'/(X_iX_j + X_iY_j + X_jY_i \mid 1 \leq i < j \leq t)$$

is $\mathbb{Z}$ graded with $\deg(X_i) = 1 = \deg(Y_j)$. The ring R' only has non-zero summands in degrees 0, 1, and 2. Since the elements $X_iX_j + X_iY_j + X_jY_i$ have degree 2, the summands R'_p and $R^\flat_p$ are isomorphic (for $p = 0, 1$) via the natural surjection $R' \to R^\flat$. Since R'_2 is in $\operatorname{Soc}(R')$, the ideal generated by the $X_iX_j + X_iY_j + X_jY_i$ is just $\operatorname{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \le i < j \le t)$. Hence, the desired descriptions of $R^\flat$ follow from Proposition 3.4 (i).

(ii) Since R' is local and $\operatorname{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \le i < j \le t) \subseteq \mathfrak{m}'$, it follows that $R^\flat$ is local with maximal ideal

$$\mathfrak{m}^\flat = \mathfrak{m}' / \operatorname{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \le i < j \le t).$$

(iii) Since every non-unit in R' is square-zero, the same is true of the homomorphic image $R^\flat$.

(iv) When $i = j$, this follows from part (iii). When $i < j$, we have

$$(X_i + Y_i)(X_j + Y_j) = X_iX_j + X_iY_j + X_jY_i + Y_iY_j = X_iX_j + X_iY_j + X_jY_i = 0$$

in $R^\flat$ because $Y_iY_j = 0 = X_iX_j + X_iY_j + X_jY_i$ by construction.

(v) This follows from the description of $R^\flat_1$ in part (i). □

Lemma 3.6. *Continue with the assumptions and notation of Construction* 3.1. *In* $R[\mathbf{Y}]$, *let* $l \in R_1$ *and* $m \in \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$ *such that* $lm \in \operatorname{Span}_{\mathbb{F}_2}(X_iY_j + X_jY_i \mid 1 \le i < j \le t)$. *Then* $l = 0$ *or* $m = 0$.

Proof. Consider the polynomial ring $S = \mathbb{F}_2[\mathbf{X}]$ with the natural $\mathbb{Z}$-graded surjection $\tau\colon S \to R$. Note that τ_i is an isomorphism for $i = 0, 1$. It follows that the induced $\mathbb{Z}^2$-graded surjection $\tau[\mathbf{Y}]\colon S[\mathbf{Y}] \to R[\mathbf{Y}]$ is an isomorphism for multi-degrees (i, j) with $i \le 1$. (Here $\deg(X_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$.) Thus, the condition $lm \in \operatorname{Span}_{\mathbb{F}_2}(X_iY_j + X_jY_i \mid 1 \le i < j \le t)$ in $R[\mathbf{Y}]_{(1,1)} \cong S[\mathbf{Y}]_{(1,1)}$ implies that there are elements $\gamma_{i,j} \in \mathbb{F}_2$ such that $lm = \sum_{1 \le i < j \le t} \gamma_{i,j}(X_iY_j + X_jY_i)$ in $S[\mathbf{Y}]$. It follows that

$$lm \in I = I_2\begin{pmatrix} X_1 & \cdots & X_t \\ Y_1 & \cdots & Y_t \end{pmatrix} \subseteq S[\mathbf{Y}]$$

where I is the ideal of $S[\mathbf{Y}]$ generated by the size-2 minors of the matrix of variables. Since I is generated by elements of degree $(1, 1)$, we have $I \cap S[\mathbf{Y}]_{(1,0)} = 0$ and $I \cap S[\mathbf{Y}]_{(0,1)} = 0$.

From [15, Theorem 2.10] we know that the ideal I' in $\mathbb{F}_2[X_1, \ldots, X_t, Y_1, \ldots, Y_t]$ generated by the size-2 minors of the matrix of variables is prime. It follows that the ideal $I = I'S[\mathbf{Y}]$ is prime in $S[\mathbf{Y}]$. Hence, either $l \in I \cap S[\mathbf{Y}]_{(1,0)} = 0$ or $m \in I \cap S[\mathbf{Y}]_{(0,1)} = 0$, as desired. □

Remark 3.7. Set $H = \operatorname{Span}_{\mathbb{F}_2}(X_i + Y_i \mid 1 \le i \le t) \subseteq R^\flat_1 = R'_1 = R[\mathbf{Y}]_1$. Given an element $f = \sum_{i=1}^d a_iX_i + \sum_{j=1}^e b_jY_j \in R^\flat_1$, it is straightforward to show that $f \in H$ if and only if the following conditions are satisfied:

(i) $a_i = 0 = b_j$ for all $i, j > t$; and

(ii) $a_i = b_i$ for all $i \leq t$.

Furthermore, in the case $t = 1$, we have $H = 0$.

Proposition 3.8. *Continue with the assumptions and notation of Construction* 3.1 *and Remark* 3.7.

(i) *For all $l \in R_1 \setminus \{0\}$ and $m \in \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \setminus \{0\}$, we have $lm \neq 0$ in $R^\flat$.*

(ii) $\mathrm{Soc}(R^\flat) = R^\flat_2$.

(iii) $\left|[R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})] \setminus [R_1 \cup \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \cup H]\right| = 2^{d+e} - 2^d - 2^e - 2^t + 2$.

Proof. (i) Suppose by way of contradiction that $lm = 0$ in $R^\flat$. In the polynomial ring $R[\mathbf{Y}]$ we have $lm \in R[\mathbf{Y}]_{(1,1)}$. From the relations used to create $R^\flat$, it follows that

$$lm \in \mathrm{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \leq i < j \leq t) \subseteq R[\mathbf{Y}].$$

It follows that there are elements $\gamma_{i,j} \in \mathbb{F}_2$ such that

$$\begin{aligned} lm &= \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iX_j + X_iY_j + X_jY_i) \\ &= \sum_{1 \leq i < j \leq t} \gamma_{i,j}X_iX_j + \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iY_j + X_jY_i) \end{aligned}$$

in $R[\mathbf{Y}]$. The elements lm and $\sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iY_j + X_jY_i)$ are in $R[\mathbf{Y}]_{(1,1)}$, and the element $\sum_{1 \leq i < j \leq t} \gamma_{i,j}X_iX_j$ is in $R[\mathbf{Y}]_{(2,0)}$. It follows that $\sum_{1 \leq i < j \leq t} \gamma_{i,j}X_iX_j = 0$, and hence

$$lm = \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iY_j + X_jY_i) \in \mathrm{Span}_{\mathbb{F}_2}(X_iY_j + X_jY_i \mid 1 \leq i < j \leq t) \subseteq R[\mathbf{Y}]$$

so Lemma 3.6 implies that $l = 0$ or $m = 0$, a contradiction.

(ii) Since $R^\flat_i = 0$ for all $i \geq 3$, the containment $\mathrm{Soc}(R^\flat) \supseteq R^\flat_2$ is routine. For the reverse containment $\mathrm{Soc}(R^\flat) \subseteq R^\flat_2$, we use the fact that the socle of $R^\flat$ is a $\mathbb{Z}$-graded ideal; this follows from the fact that $R^\flat$ is $\mathbb{Z}$-graded. Since $R^\flat_0$ consists of units, it suffices to show that the only elements of degree 1 in $\mathrm{Soc}(R^\flat)$ are 0.

Let $f \in \mathrm{Soc}(R^\flat)_1 \subseteq R^\flat_1 = R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})$, and fix $l \in R_1$ and $m \in \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})$ such that $f = l + m$. The assumption $f \in \mathrm{Soc}(R^\flat)$ implies that

$$0 = fY_1 = lY_1 + mY_1 = lY_1$$

since $(\mathbf{Y})^2 = 0$ in $R^\flat$. Thus, part (i) implies that $l = 0$. Thus, we have $f = m$, and it follows that

$$0 = X_1 f = X_1 m$$

so part (i) implies that $f = m = 0$, as desired.

(iii) Since R_1, $\mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})$, and H have pair-wise trivial intersection, the number of elements of $[R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})] \setminus [R_1 \cup \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \cup H]$ is given by the formula $2^{d+e} - 2^d - (2^e - 1) - (2^t - 1)$. □

Notation 3.9. Set $v = 2^{d+e} - 2^d - 2^e - 2^t + 2$. Write

$$[R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})] \setminus [R_1 \cup \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \cup H] = \{f_1, \ldots, f_v\}$$

and let $0 \neq z \in \mathrm{Soc}(R)$.

Lemma 3.10. *Continue with the assumptions and notation of Construction* 3.1*, Remark* 3.7*, and Notation* 3.9*. Let $l, p \in R_1$ and $m, q \in \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})$ such that $l, m \neq 0$. Set $f = l + m$ and $g = p + q$, and assume that $fg = 0$ in $R^\flat$.*

(i) *If $f \in H$, then $g \in H$.*

(ii) *If $f \notin H$, then either $g = 0$ or $g = f$.*

Proof. If $p = 0 = q$, then $g = 0$ and we are done. Thus, we assume that either $p \neq 0$ or $q \neq 0$.

Suppose that $p = 0$ and $q \neq 0$. If follows that $g = q \in \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \setminus \{0\}$. Since $l, m \neq 0$, we have $f = l+m \notin \mathrm{Ann}_{R^\flat}(q) = \mathrm{Ann}_{R^\flat}(g)$, contradicting the assumption $fg = 0$.

If $p \neq 0$ and $q = 0$, then we arrive at a similar contradiction. Thus, we assume for the rest of the proof that $p, q \neq 0$.

In $R^\flat$, we have

$$0 = fg = lp + lq + mp + mq = lp + lq + mp$$

since $(\mathbf{Y})^2 = 0$. It follows that

$$lp + lq + mp \in \mathrm{Span}_{\mathbb{F}_2}(X_i X_j + X_i Y_j + X_j Y_i \mid 1 \leq i < j \leq t) \subseteq R[\mathbf{Y}].$$

It follows that there are elements $\gamma_{i,j} \in \mathbb{F}_2$ such that

$$lp + lq + mp = \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_i X_j + X_i Y_j + X_j Y_i) \tag{3.10.1}$$

in $R[\mathbf{Y}]$. Fix elements $a_i, \alpha_i, b_j, \beta_j \in \mathbb{F}_2$ such that

$$l = \sum_{i=1}^{d} \alpha_i X_i, \quad p = \sum_{i=1}^{d} a_i X_i, \quad m = \sum_{j=1}^{e} \beta_j Y_j, \quad q = \sum_{j=1}^{e} b_j Y_j.$$

Substituting these expressions into (3.10.1) and collecting homogeneous components, we obtain the following:

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_i a_j X_i X_j = \sum_{1\le i<j\le t}\gamma_{i,j} X_i X_j, \tag{3.10.2}$$

$$\sum_{i=1}^{d}\sum_{j=1}^{e}(\alpha_i b_j + a_i\beta_j)X_i Y_j = \sum_{1\le i<j\le t}\gamma_{i,j}(X_i Y_j + X_j Y_i). \tag{3.10.3}$$

The set $\{X_i Y_j \mid 1 \le i \le d, 1 \le j \le e\}$ is linearly independent over $\mathbb{F}_2$ since the X_i's are linearly independent. Thus, equation (3.10.3) yields the following system of equations in $\mathbb{F}_2$:

$$\alpha_i b_j + a_i\beta_j = \begin{cases}\gamma_{i,j} = \alpha_j b_i + a_j\beta_i & \text{for } 1 \le i < j \le t,\\ 0 & \text{otherwise.}\end{cases} \tag{3.10.4}$$

The assumption $l, m \neq 0$ implies that there are indices i_0, j_0 such that $\alpha_{i_0} = 1 = \beta_{j_0}$. Assume that i_0 and j_0 are the largest such indices.

Case 1: $i_0 > t$. (This is a special case of the case $f \notin H$.) In this case, equation (3.10.4) implies that $b_j = a_{i_0}\beta_j$ for $j = 1, \ldots, e$. It follows that $q = a_{i_0}m$, and thus $g = p + a_{i_0}m$. Consider the element

$$g + a_{i_0} f = p + a_{i_0} l \in R_1.$$

Since $fg = 0 = a_{i_0} f^2$, we have $f(g + a_{i_0} f) = 0$. Since $f \neq 0$, Proposition 3.11 (i) implies that $g + a_{i_0} f = 0$, that is, that $g = a_{i_0} f$. Since $a_{i_0} \in \mathbb{F}_2$ and $g \neq 0$, it follows that $g = f$, as desired.

Case 2: $j_0 \ge t$. (This is another special case of the case $f \notin H$.) As in Case 1, it follows that $g = f$, as desired.

Case 3: For all $i, j > t$ we have $\alpha_i = 0 = \beta_j$, and $f \in H$. If there is an index $i_1 > t$ such that $a_{i_1} = 1$, then we conclude that $f = g$ as in Case 1. Thus, we assume that $a_i = 0$ for all $i > t$ and, similarly, that $b_j = 0$ for all $j > t$.

Case 3A: $f \in H$. The condition $f \in H$ implies that $\beta_i = \alpha_i$ for $i = 1, \ldots, t$; see Remark 3.7. To show that $g \in H$, we need to show that $b_i = a_i$ for $i = 1, \ldots, t$. For each i, equation (3.10.4) implies that

$$\alpha_i b_i = a_i\beta_i = a_i\alpha_i.$$

If $\alpha_i = 1$, then it follows that $b_i = a_i$, as desired. Assume that $\alpha_i = 0$. It follows that $\beta_i = \alpha_i = 0$ and $\beta_{i_0} = \alpha_{i_0} = 1$, so equation (3.10.4) implies that

$$a_i = \alpha_i b_{i_0} + a_i\beta_{i_0} = \alpha_{i_0} b_i + a_{i_0}\beta_i = b_i$$

as desired.

Case 3B: $f \notin H$. It follows that there is an index $i_2 \leq t$ such that $\alpha_{i_2} \neq \beta_{i_2}$.

Case 3Bi: $\alpha_{i_2} = 1$ and $\beta_{i_2} = 0$. Equation (3.10.4) with $j = i_2$ yields

$$b_{i_2} = \alpha_{i_2} b_{i_2} = a_{i_2}\beta_{i_2} = 0.$$

For $j \neq i_2$, equation (3.10.4) implies that

$$b_j + a_{i_2}\beta_j = \alpha_{i_2} b_j + a_{i_2}\beta_j = 0.$$

Combining these displays, we find that $b_j = a_{i_2}\beta_j$ for $j = i, \ldots, t$. It follows that $q = a_{i_2} m$, and we deduce that $g = f$ as in Case 1.

Case 3Bii: $\alpha_{i_2} = 0$ and $\beta_{i_2} = 1$. Argue as in Case 3Bi to conclude that $g = f$. □

Proposition 3.11. *Continue with the assumptions and notation of Construction* 3.1*, Remark* 3.7*, and Notation* 3.9*. For all $l \in R_1 \setminus 0$ and $m \in \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y}) \setminus 0$, we have*

(i) $\operatorname{Ann}_{R^\flat}(l) = 0 \oplus [\operatorname{Ann}_R(l)_1 \oplus 0] \oplus R^\flat_2$

(ii) $\operatorname{Ann}_{R^\flat}(m) = 0 \oplus [0 \oplus \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})] \oplus R^\flat_2$

(iii) $\operatorname{Ann}_{R^\flat}(l+m) = \begin{cases} 0 \oplus [\operatorname{Span}_{\mathbb{F}_2}(l+m)] \oplus R^\flat_2 & \text{if } l+m \notin H, \\ 0 \oplus H \oplus R^\flat_2 & \text{if } l+m \in H. \end{cases}$

Proof. (i) Since l is homogeneous of degree 1, the ideal $\operatorname{Ann}_{R^\flat}(l)$ is also $\mathbb{Z}$-graded. Thus, we need only check the equality $\operatorname{Ann}_{R^\flat}(l) = 0 \oplus [\operatorname{Ann}_R(l) \oplus 0] \oplus R^\flat_2$ for graded pieces. Since $R^\flat_0$ consists of units of $R^\flat$, the assumption $l \neq 0$ implies that $\operatorname{Ann}_{R^\flat}(l)_0 = 0$. Since $\operatorname{Soc}(R^\flat) = R^\flat_2$, it is straightforward to show that $\operatorname{Ann}_{R^\flat}(l)_2 = R^\flat_2$. In degree 1, the containment $\operatorname{Ann}_{R^\flat}(l)_1 \supseteq \operatorname{Ann}_R(l)_1$ is straightforward. For the reverse containment $\operatorname{Ann}_{R^\flat}(l)_1 \subseteq \operatorname{Ann}_R(l)_1$, let

$$f \in \operatorname{Ann}_{R^\flat}(l)_1 \subseteq R^\flat_1 = R_1 \oplus \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$$

and fix $p \in R_1$ and $q \in \operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$ such that $f = p + q$.

In the polynomial ring $R[\mathbf{Y}]$ we have $lf \in R[\mathbf{Y}]_{(2,0)} \oplus R[\mathbf{Y}]_{(1,1)}$. In light of the relations used to create $R^\flat$, it follows that

$$lf \in \operatorname{Span}_{\mathbb{F}_2}(X_iX_j + X_iY_j + X_jY_i \mid 1 \leq i < j \leq t) \subseteq R[\mathbf{Y}].$$

It follows that there are elements $\gamma_{i,j} \in \mathbb{F}_2$ such that

$$lf = \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iX_j + X_iY_j + X_jY_i),$$
$$lp + lq = \sum_{1 \leq i < j \leq t} \gamma_{i,j}X_iX_j + \sum_{1 \leq i < j \leq t} \gamma_{i,j}(X_iY_j + X_jY_i)$$

in $R[\mathbf{Y}]$. The elements lp and $\sum_{1\leq i<j\leq t}\gamma_{i,j}X_iX_j$ are in $R[\mathbf{Y}]_{(2,0)}$ and the elements lq and $\sum_{1\leq i<j\leq t}\gamma_{i,j}(X_iY_j+X_jY_i)$ are in $R[\mathbf{Y}]_{(1,1)}$. It follows that

$$lp=\sum_{1\leq i<j\leq t}\gamma_{i,j}X_iX_j, \tag{3.11.1}$$

$$lq=\sum_{1\leq i<j\leq t}\gamma_{i,j}(X_iY_j+X_jY_i) \tag{3.11.2}$$

Since $l\neq 0$, equation (3.11.2) implies that $q=0$ by Lemma 3.6. Thus, we have $f=p\in R_1\cap\operatorname{Ann}_{R^\flat}(l)=\operatorname{Ann}_R(l)_1$, as desired.

(ii) Again, it suffices to show that $\operatorname{Ann}_{R^\flat}(m)_1\subseteq\operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$. Let $p\in R_1$ and $q\in\operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$ such that $f=p+q\in\operatorname{Ann}_{R^\flat}(m)$. Since $(\mathbf{Y})^2=0$ in $R^\flat$, we have

$$0=fm=pm+qm=pm$$

so Lemma 3.6 implies that $p=0$. It follows that $f=q\in\operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})$, as desired.

(iii) Since $\operatorname{Ann}_{R^\flat}(l+m)$ is a $\mathbb{Z}$-graded ideal, it suffices to verify the equalities degree-by-degree. The degree 0 and degree 2 parts are straightforward since $l+m\neq 0$ and $\operatorname{Soc}(R^\flat)=R^\flat_2$. Since $H^2=0=(l+h)^2$ by Proposition 3.5 (iii), (iv), the degree 1 parts follow from Lemma 3.10. □

Theorem 3.12. *Continue with the assumptions and notation of Construction* 3.1*, Remark* 3.7*, and Notation* 3.9*. Assume that* $\operatorname{Soc}(R)=R_2$.

(i) *The number of distinct vertices in* $\Gamma_E(R^\flat)$ *is* $c^\flat=c+v+2$.

(ii) $\Gamma_E(R^\flat)$ *is formed from* $\Gamma_E(R)$ *by adding* $v+2$ *vertices* $[f_1],\dots,[f_v],[X_1+Y_1]$, $[Y_1]$*, and* $v+2$ *edges* $[z]-[Y_1],[z]-[f_1],\dots,[z]-[f_v],[z]-[X_1+Y_1]$.

(iii) $\Gamma_E(R^\flat)$ *is a star if and only if* $\Gamma_E(R)$ *is a star.*

Proof. (i) Proposition 3.8 (ii) says that $\operatorname{Soc}(R^\flat)=R^\flat_2$. As $z\in\operatorname{Soc}(R)=R_2\subseteq R^\flat_2=\operatorname{Soc}(R^\flat)$, it follows that the vertex set of $\Gamma_E(R^\flat)$ consists of the socle vertex $[z]$ and the vertices $[u]$ where u ranges through the elements of $R^\flat_1$ with distinct annihilators.

The non-zero elements of $R^\flat_1$ are of the form l, m, and $l+m$ where $l\in R_1\setminus 0$ and $m\in\operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})\setminus 0$; see Proposition 3.5 (i). Using Proposition 3.11, we see that:

(i) the vertices $[l]$, $[m]$, and $[l+m]$ are all distinct in $\Gamma_E(R^\flat)$ since the annihilator ideals are all different;

(ii) given another element $l'\in R_1\setminus 0$, we have $\operatorname{Ann}_{R^\flat}(l)=\operatorname{Ann}_{R^\flat}(l')$ if and only if $\operatorname{Ann}_R(l)=\operatorname{Ann}_R(l')$, so $[l]=[l']$ in $\Gamma_E(R^\flat)$ if and only if $[l]=[l']$ in $\Gamma_E(R)$;

(iii) given another element $m'\in\operatorname{Span}_{\mathbb{F}_2}(\mathbf{Y})\setminus 0$, we have $\operatorname{Ann}_{R^\flat}(m)=\operatorname{Ann}_{R^\flat}(m')$, so $[m]=[m']$ in $\Gamma_E(R^\flat)$;

(iv) If $l+m\notin H$, then $\operatorname{Ann}_{R^\flat}(l+m)=\operatorname{Ann}_{R^\flat}(l'+m')$ if and only if $l+m=l'+m'$, so $[l+m]=[l'+m']$ in $\Gamma_E(R^\flat)$ if and only if $l+m=l'+m'$; and

(v) If $l + m \in H$, then $\mathrm{Ann}_{R^\flat}(l + m) = \mathrm{Ann}_{R^\flat}(l' + m')$ if and only if $l' + m' \in H \setminus \{0\}$.

Thus, the distinct vertices of $\Gamma_E(R^\flat)$ are the following:

(i′) $[z]$;

(ii′) $[l]$ where l ranges through the distinct vertices $[l] \neq [z]$ of $\Gamma_E(R)$;

(iii′) $[Y_1]$;

(iv′) $[l + m]$ where l and m range through $R_1 \setminus 0$ and $\mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \setminus 0$, respectively, with $l + m \notin H$; and

(v′) $[X_1 + Y_1]$

accounting for all the elements of H. In particular, the number of vertices in $\Gamma_E(R^\flat)$ is

$$
\begin{aligned}
c^\flat &= 1 + (c - 1) + 1 + \left|[R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y})] \setminus [R_1 \cup \mathrm{Span}_{\mathbb{F}_2}(\mathbf{Y}) \cup H]\right| + 1 \\
&= c + 2 + 2^{d+e} - 2^d - 2^e - 2^t + 2 \\
&= c + v + 2
\end{aligned}
$$

by Proposition 3.8 (iii).

(ii) Using the fact that $z \in \mathrm{Soc}(R^\flat)$ with Proposition 3.11, we see the following:

(i″) the vertices $[z]$ and $[f]$ are adjacent in $\Gamma_E(R^\flat)$ for every $0 \neq f \in R_1^\flat$;

(ii″) given $l, l' \in R_1 \setminus 0$, the vertices $[l]$ and $[l']$ are adjacent in $\Gamma_E(R^\flat)$ if and only if they are adjacent in $\Gamma_E(R)$; and

(iii″) the vertices $[Y_1]$, $[l + m]$, and $[X_1 + Y_1]$ are only adjacent to $[z]$ in $\Gamma_E(R^\flat)$.

Thus, the graph $\Gamma_E(R^\flat)$ has the desired form.

(iii) This follows from part (ii). □

Next, we deal with the case where $\mathrm{Soc}(R) \neq R_2$, that is, when $\mathrm{Soc}(R) \supsetneq R_2$. This case is slightly more complicated, but part (iii) gives us many more stars, using Theorem 3.12.

Theorem 3.13. *Continue with the assumptions and notation of Construction* 3.1*, Remark* 3.7*, and Notation* 3.9*. Assume that* $\mathrm{Soc}(R) \neq R_2$*, and assume that* $z \in \mathrm{Soc}(R) \setminus R_2$*. Let* $[l_1], \dots, [l_{c-1}], [z]$ *be the distinct vertices of* $\Gamma_E(R)$*.*

(i) *The number of distinct vertices in* $\Gamma_E(R^\flat)$ *is* $c^\flat = c + v + 3$*.*

(ii) *The graph* $\Gamma_E(R^\flat)$ *is obtained from* $\Gamma_E(R)$ *by adding the following* $v+3$ *vertices* $[Y_1], [f_1], \dots, [f_v], [zY_1], [X_1 + Y_1]$ *and the* $v + c + 2$ *edges* $[zY_1] - [X_1 + Y_1]$, $[zY_1] - [Y_1], [zY_1] - [f_1], \dots, [zY_1] - [f_v], [zY_1] - [z], [zY_1] - [l_1], \dots, [zY_1] - [l_{c-1}]$.

(iii) *The following conditions are equivalent:*

(a) $\Gamma_E(R^\flat)$ *is a star;*

(b) $\Gamma_E(R)$ *is a single vertex;*

(c) $R_2 = 0$*;*

(d) $\mathrm{Soc}(R) = R_1$*; and*

(e) $c = 1$.

Proof. Parts (i) and (ii) are proved like the corresponding parts of Theorem 3.12. The only real difference is in the fact that $z \in R_1$ implies that $z \notin R_2^\flat = \mathrm{Soc}(R^\flat)$; see Proposition 3.8 (ii). Thus, the graph $\Gamma_E(R^\flat)$ has a new socle vertex $[zY_1]$.

(iii) We first show that (e) $\Rightarrow$ (n) for $n =$ a–d. Assume that $c = 1$. It follows by definition that $[z]$ is the only vertex in $\Gamma_E(R)$. In other words, every non-unit $f \in R \setminus \{0\}$ has $\mathrm{Ann}_R(f) = \mathrm{Ann}_R(z) = \mathfrak{m}$ where $\mathfrak{m} = 0 \oplus R_1 \oplus R_2$. It follows that $R_2 = \mathfrak{m}^2 = 0$ and hence $\mathrm{Soc}(R) = \mathfrak{m} = R_1$. Finally, the description of $\Gamma_E(R^\flat)$ from part (ii) shows that $\Gamma_E(R^\flat)$ is a star in this case. This gives the desired implications.

Next, we show (n) $\Rightarrow$ (e) for $n =$ a–d. We argue by contrapositive, so assume that $c \geq 2$. It follows by definition that $\Gamma_E(R)$ is not a single vertex. Thus, there is an element $f \in \mathfrak{m}$ such that $\mathrm{Ann}_R(f) \neq \mathrm{Ann}_R(z) = \mathfrak{m}$. That is, $\mathfrak{m} f \neq 0$, so $R_2 = \mathfrak{m}^2 \neq 0$. It follows that $\mathrm{Soc}(R) \neq R_1$ since $0 \neq R_2 \subseteq \mathrm{Soc}(R) \setminus R_1$. Finally, the description of $\Gamma_E(R^\flat)$ in part (ii) shows that $\Gamma_E(R^\flat)$ contains the cycle $[z] - [f] - [zY_1] - [z]$ so $\Gamma_E(R^\flat)$ is not a star in this case. This gives the desired implications. $\square$

The example below uses the computations from Theorems 3.12 and 3.13.

Example 3.14. Start with $R = \mathbb{F}_2[\mathbf{X}]/(\mathbf{X})^2$ where $\mathbf{X} = X_1, \ldots, X_{d_1}$ is a sequence of indeterminates with $d = d_1 \geq 1$. Then $R_2 = 0$, $\mathrm{Soc}(R) = R_1$, and $\Gamma_E(R)$ is a single vertex; see Example 3.3 (i). Thus, Theorem 3.13 (iii) applies, and the graph $\Gamma_E(R^\flat)$ obtained using $e = e_1$ and $t = t_1$ is a star with number of vertices

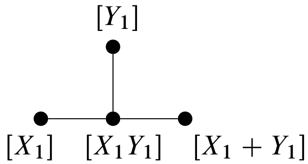
$$c^\flat = 1 + v_1 + 3 = 2^{d_1+e_1} - 2^{d_1} - 2^{e_1} - 2^{t_1} + 6.$$

For instance, when $e_1 = 1$ we must have $t_1 = 1$ and

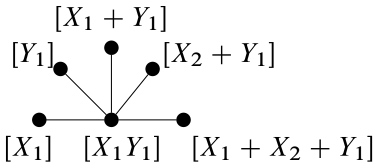
$$c^\flat = 2^{d_1+1} - 2^{d_1} - 2^1 - 2^1 + 6 = 2^{d_1} + 2. \tag{3.14.1}$$

Some example graphs in this case are the following:

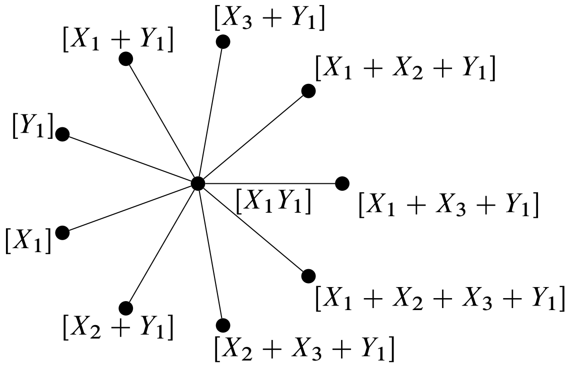
$e_1 = 1 = t_1, \quad d_1 = 1$:



$d_1 = 2$:



$d_1 = 3$:



For instance, when $e_1 = 2$ we must have $t_1 = 1$ or 2, and in this case we have

$$c^\flat = \begin{cases} 2^{d_1} \cdot 3 & \text{if } t_1 = 1, \\ 2^{d_1} \cdot 3 - 2 & \text{if } t_1 = 2. \end{cases} \tag{3.14.2}$$

Note that the case $t_1 = 2$ above requires that $d_1 \geq 2$. However, when $d_1 = 1$, the formula yields $c^\flat = 4$, which is covered by the case $d_1 = e_1 = t_1 = 1$. Similarly, the values of (3.14.1) and (3.14.2) for $d_1 = 0$ can also be found.

The following table includes the values of $c^\flat$ for star graphs we can construct using this method with $c^\flat < 100$:

d_1	e_1	t_1	$c^\flat$
1	1	1	4
1	2	1	6
1	3	1	10
1	4	1	18
1	5	1	34
1	6	1	66

d_1	e_1	t_1	$c^\flat$
2	2	1	12
2	2	2	10
2	3	1	24
2	3	2	22
2	4	1	48
2	4	2	46

d_1	e_1	t_1	$c^\flat$
2	5	1	96
2	5	2	94
3	3	1	52
3	3	2	50
3	3	3	46

Proposition 3.8 (ii) implies that $\mathrm{Soc}(R^\flat) = R^\flat_2$, so we can apply Theorem 3.12 to conclude that $\Gamma_E(R^{\flat\flat})$ is a star with $c^{\flat\flat}$ vertices where

$$\begin{aligned} c^{\flat\flat} &= c^\flat + v_2 + 2, \\ &= [2^{d_1+e_1} - 2^{d_1} - 2^{e_1} - 2^{t_1} + 6] + [2^{d_2+e_2} - 2^{d_2} - 2^{e_2} - 2^{t_2} + 2] + 2, \\ &= [2^{d_1+e_1} - 2^{d_1} - 2^{e_1} - 2^{t_1} + 6] + [2^{d_1+e_1+e_2} - 2^{d_1+e_1} - 2^{e_2} - 2^{t_2} + 2] + 2, \\ &= 2^{d_1+e_1+e_2} - 2^{d_1} - 2^{e_1} - 2^{t_1} - 2^{e_2} - 2^{t_2} + 10. \end{aligned}$$

The tables in Appendix A include the values of $c^{\flat\flat}$ for star graphs we can construct using this method with $c^{\flat\flat} < 100$. The next display gives some special cases of this formula:

$$c^{\flat\flat} = \begin{cases} 2^{d_1} \cdot 7 & \text{if } t_1 = e_1 = t_2 = 1 \text{ and } e_2 = 2, \\ 2^{d_1} \cdot 7 - 2 & \text{if } t_1 = e_1 = 1 \text{ and } t_2 = e_2 = 2, \\ 2^{d_1} \cdot 15 - 4 & \text{if } t_1 = 1 \text{ and } e_1 = t_1 = e_2 = 2, \\ 2^{d_1} \cdot 15 - 6 & \text{if } t_1 = e_1 = t_2 = e_2 = 2. \end{cases} \tag{3.14.3}$$

The case $t_1 = e_1 = t_2 = e_2 = 1$ is not included, as it repeats a previous formula.

This process can be repeated ad nauseum, but the number of vertices grows very quickly, even if one uses only one new variable at each iteration. See Appendix A.

For ease of reference, the list of even c-values with $c < 100$ we can produce with this method is 2–18, 22–34, 46–66, 94–98. (Note that $c = 2$ is achieved from the ring $\mathbb{F}_2[X]/(X^3)$.)

Example 3.15. Fix an integer $d = d_1 \geq 2$, and let $\mathbf{X} = X_1, \ldots, X_{d_1}$ be indeterminates. Consider the ring $R = \mathbb{F}_2[\mathbf{X}]/((X_1^2, \ldots, X_{d_1}^2) + (\mathbf{X})^3)$ from Example 3.3 (ii). The graph $\Gamma_E(R)$ is a star with 2^{d_1} vertices, and $\operatorname{Soc}(R) = R_2$, so we may apply Construction 3.1 as in Example 3.14 to find that $\Gamma_E(R^\flat)$ is a star with number of vertices

$$c^\flat = 2^{d_1+e_1} - 2^{e_1} - 2^{t_1} + 4.$$

Some special cases of this are listed next:

$$c^\flat = \begin{cases} 2^{d_1+1} & \text{if } t_1 = e_1 = 1, \\ 2^{d_1+2} - 2 & \text{if } t_1 = 1 \text{ and } e_1 = 2, \\ 2^{d_1+2} - 4 & \text{if } t_1 = e_1 = 2, \\ 2^{e_1} \cdot 3 + 2 & \text{if } d_1 = 2 \text{ and } t_1 = 1, \\ 2^{e_1} \cdot 3 & \text{if } d_1 = 2 \text{ and } t_1 = 2, \\ 2^{e_1} \cdot 7 + 2 & \text{if } d_1 = 3 \text{ and } t_1 = 1, \\ 2^{e_1} \cdot 7 & \text{if } d_1 = 3 \text{ and } t_1 = 2, \\ 2^{e_1} \cdot 7 - 4 & \text{if } d_1 = 3 \text{ and } t_1 = 3, \\ 2^{e_1} \cdot 15 + 2 & \text{if } d_1 = 4 \text{ and } t_1 = 1, \\ 2^{e_1} \cdot 15 & \text{if } d_1 = 4 \text{ and } t_1 = 2, \\ 2^{e_1} \cdot 15 - 2 & \text{if } d_1 = 4 \text{ and } t_1 = 3, \\ 2^{e_1} \cdot 15 - 4 & \text{if } d_1 = 4 \text{ and } t_1 = 4, \end{cases} \tag{3.15.1}$$

This yields a few more c-values to add from the list from Example 3.14:

$$c^\flat = \begin{cases} 20 & \text{if } d_1 = 2, e_1 = 3, \text{ and } t_1 = 1, \\ 44 & \text{if } d_1 = 2, e_1 = 4, \text{ and } t_1 = 1, \\ 42 & \text{if } d_1 = 2, e_1 = 4, \text{ and } t_1 = 2, \\ 92 & \text{if } d_1 = 2, e_1 = 5, \text{ and } t_1 = 1, \\ 90 & \text{if } d_1 = 2, e_1 = 5, \text{ and } t_1 = 2. \end{cases} \tag{3.15.2}$$

Thus, the list of even c-values with $c < 100$ we can produce with this method (combined with the values from Example 3.14 is 2–34, 42–66, 90–98. At this time, we do not know how to obtain the values 36–40, 68–88.

Next, we show how to build some star graphs with odd numbers of vertices.

Construction 3.16. Let R be a $\mathbb{Z}$-graded ring $R = \mathbb{F}_2 \oplus R_1 \oplus R_2$ generated over $\mathbb{F}_2$ by R_1. Let Y be an indeterminate, and set

$$R^\# = R[Y]/((\mathrm{Soc}(R)Y) + (R_1 Y^2) + (Y^3)).$$

Remark 3.17. Continue with the assumptions and notation of Construction 3.16. Once again, the ring R is local with maximal ideal $\mathfrak{m} = R_+ = 0 \oplus R_1 \oplus R_2$. Note that we are not assuming that $r^2 = 0$ for all $r \in R_1$.

The next result is proved like Propositions 3.4 and 3.5.

Proposition 3.18. *Continue with the assumptions and notation of Construction* 3.16.

(i) *The ring $R^\#$ is $\mathbb{Z}^2$-graded with*

$$\begin{aligned} R^\# &= R^\#_{(0,0)} \oplus [R^\#_{(1,0)} \oplus R^\#_{(0,1)}] \oplus [R^\#_{(2,0)} \oplus R^\#_{(1,1)} \oplus R^\#_{(0,2)}] \\ &\cong \mathbb{F}_2 \oplus [R_1 \oplus \mathrm{Span}_{\mathbb{F}_2}(Y)] \oplus \left[R_2 \oplus \frac{R_1 \, \mathrm{Span}_{\mathbb{F}_2}(Y)}{\mathrm{Soc}(R)_1 \, \mathrm{Span}_{\mathbb{F}_2}(Y)} \oplus \mathrm{Span}_{\mathbb{F}_2}(Y^2) \right]. \end{aligned}$$

(ii) *The ring $R^\#$ is local with maximal ideal*

$$\mathfrak{m}^\# = R^\#_+ = 0 \oplus [R^\#_{(1,0)} \oplus R^\#_{(0,1)}] \oplus [R^\#_{(2,0)} \oplus R^\#_{(1,1)} \oplus R^\#_{(0,2)}].$$

The element $Y \in R^\#$ is a non-unit such that $Y^2 \neq 0$. Contrast this with the behavior of the non-units in the rings R, R', and $R^\flat$.

The next results are proved like Propositions 3.8 and 3.11, and Theorem 3.12.

Proposition 3.19. *Continue with the assumptions and notation of Construction* 3.16.

(i) *For all $l \in R_1 \setminus \mathrm{Soc}(R)$ and $m \in \mathrm{Span}_{\mathbb{F}_2}(Y) \setminus \{0\}$, we have $lm \neq 0$ in $R^\#$.*

(ii) $\mathrm{Soc}(R^\#) = 0 \oplus [\mathrm{Soc}(R)_1 \oplus 0] \oplus R^\#_2$.

Proposition 3.20. *Continue with the assumptions and notation of Construction* 3.16. *For all $l \in R_1 \setminus 0$, we have*

(i) $\mathrm{Ann}_{R^\#}(l) = \begin{cases} \mathfrak{m}^\# = R^\#_+ & \text{if } l \in \mathrm{Soc}(R), \\ 0 \oplus [\mathrm{Ann}_R(l)_1 \oplus 0] \oplus R^\#_2 & \text{if } l \notin \mathrm{Soc}(R). \end{cases}$

(ii) $\mathrm{Ann}_{R^\#}(Y) = 0 \oplus [\mathrm{Soc}(R)_1 \oplus 0] \oplus R^\#_2 = \mathrm{Ann}_{R^\#}(l + Y)$.

We break the description of $\Gamma_E(R^\#)$ into three results.

Theorem 3.21. *Continue with the assumptions and notation of Construction* 3.16. *Let c denote the number of vertices in $\Gamma_E(R)$. Assume that there is an element $l \in R_1$ such that* $\mathrm{Ann}_R(l) = \mathrm{Soc}(R)$.

(i) *The number of distinct vertices in $\Gamma_E(R^\#)$ is $c^\# = c$.*

(ii) *$\Gamma_E(R^\#)$ is graph isomorphic to $\Gamma_E(R)$.*

(iii) *$\Gamma_E(R^\#)$ is a star if and only if $\Gamma_E(R)$ is a star.*

Theorem 3.22. *Continue with the assumptions and notation of Construction* 3.16. *Let c denote the number of vertices in $\Gamma_E(R)$, and fix an element $z \in \mathrm{Soc}(R) \setminus 0$. Assume that $\mathrm{Ann}_R(l) \neq \mathrm{Soc}(R)$ for all $l \in R_1$.*

(i) *The number of distinct vertices in $\Gamma_E(R^\#)$ is $c^\# = c + 1$.*

(ii) *$\Gamma_E(R^\#)$ is formed from $\Gamma_E(R)$ by adding one vertex $[Y]$ and one edge $[z] - [Y]$.*

(iii) *$\Gamma_E(R^\#)$ is a star if and only if $\Gamma_E(R)$ is a star.*

Corollary 3.23. *Continue with the assumptions and notation of Construction* 3.16. *Let c denote the number of vertices in $\Gamma_E(R)$, and fix an element $z \in \mathrm{Soc}(R) \setminus 0$. Assume that $\mathfrak{m}^2 \neq 0$ and that $r^2 = 0$ for all non-units $r \in R$.*

(i) *$\mathrm{Ann}_R(l) \neq \mathrm{Soc}(R)$ for all $l \in R_1$.*

(ii) *The number of distinct vertices in $\Gamma_E(R^\#)$ is $c^\# = c + 1$.*

(iii) *$\Gamma_E(R^\#)$ is formed from $\Gamma_E(R)$ by adding one vertex $[Y]$ and one edge $[z] - [Y]$.*

(iv) *$\Gamma_E(R^\#)$ is a star if and only if $\Gamma_E(R)$ is a star.*

Proof. By Theorem 3.22, it suffices to show that $\mathrm{Ann}_R(l) \neq \mathrm{Soc}(R)$ for all $l \in R_1$. So, let $l \in R_1$ be given, and suppose that $\mathrm{Ann}_R(l) = \mathrm{Soc}(R)$. By assumption, we have $l \in \mathrm{Ann}_R(l) = \mathrm{Soc}(R)$, so $l\mathfrak{m} = 0$. This implies that $\mathfrak{m} \subseteq \mathrm{Ann}_R(l)$. The condition $\mathrm{Ann}_R(l) = \mathrm{Soc}(R)$ implies that $l \neq 0$, so we have $\mathfrak{m} = \mathrm{Ann}_R(l) = \mathrm{Soc}(R)$. It follows that $\mathfrak{m}^2 = 0$, a contradiction. □

Example 3.24. We return now to the assumptions and notation of Construction 3.1. Proposition 3.4 implies that $(\mathfrak{m}^\flat)^2 \neq 0$ and that $r^2 = 0$ for all non-units $r \in R^\flat$. Theorems 3.12–3.13 and Corollary 3.23 imply that $\Gamma_E(R^{\flat\#})$ is a star with $c + 1$ vertices.

Thus, for each star graph with an even number of vertices $c \geq 2$ we constructed in Examples 3.14 and 3.15, we obtain a star graph with an odd number of vertices $c+1$. (Of course, the graph of the ring $\mathbb{F}_2[X]/(X^2)$ is a degenerate star with one vertex.) The list of odd c-values with $c < 100$ we can produce with this method is 1–35, 43–67, 91–99, while we do not know how to produce 37–41, 69–89.

Next, we indicate how we obtain the list of stars from the introduction.

Example 3.25. Let n be a non-negative integer.

$c = 2^n - 4$: If $n \geq 4$, then we may use equation (3.15.1) with $t_1 = e_1 = 2$ and $n = d_1 + 2$ to find a ring $R^\flat$ such that $\Gamma_E(R^\flat)$ is a star with $2^n - 4$ vertices. For the remaining values of n, the cases $n = 0, 1, 2$ imply that $2^n - 4 \leq 0$ (so we do not consider these); and the case $n = 3$ gives $c = 2^n - 4 = 4$ which we obtained explicitly in Example 3.14.

$c = 2^n - 3$: The rings produced in the previous paragraph all satisfy the hypotheses of Theorems 3.12–3.13 and Corollary 3.23, as in Example 3.24, so we obtain rings $R^{\flat\sharp}$ such that $\Gamma_E(R^{\flat\sharp})$ is a star with $2^n - 4 + 1$ vertices, when $n \geq 3$. For the remaining values of n, the cases $n = 0, 1$ imply that $2^n - 3 \leq 0$ (so we do not consider these); and the case $n = 2$ gives $c = 2^n - 3 = 1$ which we obtained explicitly in Example 3.24.

The remaining cases are derived similarly using the numbered equations from Examples 3.14 and 3.15, in conjunction with Example 3.24.

Example 3.26. Continue with the assumptions and notation of Construction 3.16. In the ring $R^\#$, the element Y satisfies $\mathrm{Ann}_{R^\#}(Y) = 0 \oplus [\mathrm{Soc}(R)_1 \oplus 0] \oplus R_2^\# = \mathrm{Soc}(R^\#)$; see Propositions 3.19 (ii) and 3.20 (ii). Thus, Theorem 3.21 (ii) implies that $\Gamma_E(R^{\#\#})$ is graph isomorphic to $\Gamma_E(R^\#)$. Thus, one can not simply iterate this process to create rings with star graphs of any size.

Remark 3.27. Because they are needed for the proof of Proposition 5.8 we include three more examples of star graphs. These examples are instrumental to our argument that rings of length less than five must have a finite clique number. See Section 5.

Example 3.28. Let k be a field of characteristic 2, and set $R = k[X,Y]/(X^2, Y^2)$ where X and Y are indeterminates. Then R is a local ring with length 4 and maximal ideal $\mathfrak{m} = (x, y)R$ where x and y are the residues of X and Y in R.

We claim that $\Gamma_E(R)$ is a star with number of vertices equal to $2+|k|$. The argument is similar to the proofs above, so we only outline the steps. We have the following:

$$\mathrm{Soc}(R) = \mathfrak{m}^2 = (xy)R,$$

$$\mathrm{Ann}_R(y) = (y)R + \mathfrak{m}^2 = (y)R,$$

$$\mathrm{Ann}_R(x + ay) = (x + ay)R + \mathfrak{m}^2 = (x + ay)R \quad \text{for all } a \in k.$$

It follows that for all $b, c, d \in k$ such that $bx + cy + dxy \neq 0$, we have

$$[bx + cy + dxy] = \begin{cases} [xy] & \text{if } b = 0 = c, \\ [y] & \text{if } b = 0 \text{ and } c \neq 0, \\ [x + b^{-1}cy] & \text{if } b \neq 0, \end{cases}$$

and the distinct vertices in $\Gamma_E(R)$ are $[xy]$, $[y]$, and all classes of the form $[x + ay]$ where $a \in k$. No distinct vertices of the form $[y]$, $[x + ay]$, and $[x + a'y]$ are adjacent in $\Gamma_E(R)$, and all such vertices are adjacent to $[xy]$.

Example 3.29. Let k be a field of characteristic 2, and let $(Q, 2Q, k)$ be a discrete valuation ring. (Note that such a ring Q exists, e.g., as a ring of Witt vectors.) We set $R = Q[Y]/(4, Y^2)$ where Y is an indeterminate. Then R is a local ring with length 4 and maximal ideal $\mathfrak{m} = (2, y)R$ where y is the residue of Y in R. Since $4 = 0 = y^2$ in R, it follows that $\mathfrak{m}^2 = (2y)R$.

We claim that $\Gamma_E(R)$ is a star with number of vertices equal to $2 + |k|$. Because most of our previous examples contain a field, we include more details for this one. First, we observe that $R \cong (Q/4Q)[Y]/(Y^2)$, so R is a free $Q/4Q$-module of rank 2 with basis $1, y$. Units in R are of the form $\overline{q_0} + \overline{q_1}y$ where $q_0 \in Q \setminus 2Q$ and $q_1 \in Q$; here, for each element $q \in Q$, we write $\overline{q}$ for the residue of q in $Q/4Q$. Non-units in R are of the form $2\overline{q_0} + \overline{q_1}y$ where $q_0, q_1 \in Q$. Moreover, it is not difficult to see that every non-unit $r \in R$ satisfies $r^2 = 0$.

We claim that $\operatorname{Soc}(R) = \mathfrak{m}^2$. The containment $\operatorname{Soc}(R) \supseteq \mathfrak{m}^2$ follows from the fact that $\mathfrak{m}^3 = 0$. For the reverse containment, let $r \in \operatorname{Soc}(R)$, say $r = 2\overline{q_0} + \overline{q_1}y$ for elements $q_0, q_1 \in Q$. The equalities $0 = 2r = 4\overline{q_0} + 2\overline{q_1}y = 2\overline{q_1}y$ imply that $2\overline{q_1} = 0$ in $Q/4Q$; this uses the fact that R is free of rank 2 over $Q/4Q$ with basis $1, y$. Since Q is a discrete valuation ring with maximal ideal generated by 2, it follows that $q_1 \in 2Q$, so we have $q_1 = 2q_1'$ for some $q_1' \in Q$. Similarly, the equation $0 = yr$ implies that $q_0 = 2q_0'$ for some $q_0' \in Q$. Thus, we have

$$r = 2\overline{q_0} + \overline{q_1}y = 4\overline{q_0'} + 2\overline{q_1'}y = 2\overline{q_1'}y \in \mathfrak{m}^2$$

as desired.

Next, we claim that

$$\operatorname{Ann}_R(y) = (y)R + \mathfrak{m}^2 = (y)R.$$

The containment $(y)R + \mathfrak{m}^2 \subseteq (y)R$ follows from the fact that $\mathfrak{m}^2 = (2y)R \subseteq (y)R$ and the reverse containment is routine. The containment $\operatorname{Ann}_R(y) \supseteq (y)R$ is from the fact that $y^2 = 0$. For the reverse containment, let $r \in \operatorname{Ann}_R(y)$, where, since r is a non-unit, $r = 2\overline{q_0} + \overline{q_1}y$ for elements $q_0, q_1 \in Q$. Again, the equation $ry = 0$ implies that $\overline{q_0} = 2\overline{q_0'}$ for some $q_0' \in Q$. From this we have $r = \overline{q_1}y \in (y)R$, as desired.

Next, we claim that

$$\operatorname{Ann}_R(2 + \overline{q}y) = (2 + \overline{q}y)R + \mathfrak{m}^2 = (2 + \overline{q}y)R \tag{3.29.1}$$

for all $\overline{q} \in Q/4Q$. The equality $(2+\overline{q}y)R + \mathfrak{m}^2 = (2+\overline{q}y)R$ follows as in the previous paragraph because $2y = y(2+\overline{q}y)$. The containment $\operatorname{Ann}_R(2+\overline{q}y) \supseteq (2+\overline{q}y)R+\mathfrak{m}^2$ is straightforward; thus, let $r \in \operatorname{Ann}_R(2+\overline{q}y)$. Since r is a non-unit we have $r = 2\overline{q_0} + \overline{q_1}y$ for elements $q_0, q_1 \in Q$. Then we have

$$\begin{aligned} 0 = r(2+\overline{q}y) &= (2\overline{q_0} + \overline{q_1}y)(2+\overline{q}y) \\ &= 4\overline{q_0} + (2\overline{q_0}\,\overline{q} + 2\overline{q_1})y + \overline{q_1}\,\overline{q}y^2 = (2\overline{q_0}\,\overline{q} + 2\overline{q_1})y. \end{aligned}$$

As before, this implies that $2\overline{q_0}\,\overline{q} + 2\overline{q_1} = 0$ in $Q/4Q$, and it follows that $-2\overline{q_0}\,\overline{q} + 2\overline{q_1} = 0$ in $Q/4Q$. Thus, we have $2\overline{q_1} = 2\overline{q_0}\,\overline{q}$ in $Q/4Q$, and it follows that $\overline{q_1} = \overline{q_0}\,\overline{q} + 2q_0'$ for some $q_0' \in Q$. From this we have

$$r = 2\overline{q_0} + \overline{q_1}y = 2\overline{q_0} + (\overline{q_0}\,\overline{q} + 2\overline{q_0'})y = \overline{q_0}(2+\overline{q}y) + 2\overline{q_0'}y \in (2+\overline{q}y)R + \mathfrak{m}^2$$

as desired.

Next we claim that every vertex in $\Gamma_E(R)$ is of the form $[y]$, $[2y]$ or $[2+\overline{q}y]$ for some $q \in Q$. Let $0 \neq r \in \mathfrak{m}$. If $r \in \mathfrak{m}^2 = \operatorname{Soc}(R)$, then $[r] = [2y]$ since $0 \neq 2y \in \mathfrak{m}^2 = \operatorname{Soc}(R)$. Assume then that $r \notin \mathfrak{m}^2$, and fix $q_0, q_1 \in Q$ such that $r = 2\overline{q_0} + \overline{q_1}y$. If $q_0 \in 2Q$, then $r = \overline{q_1}y$, and the fact that $r \notin \mathfrak{m}^2$ implies that $q_1 \in Q \backslash 2Q$; thus $\overline{q_1}$ is a unit in R and $[r] = [y]$. Assume then that $q_0 \notin 2Q$. It follows that q_0 is a unit in R, so $[r] = [2\overline{q_0} + \overline{q_1}y] = [2 + (\overline{q_0})^{-1}\overline{q_1}y]$ which is of the form $[2+\overline{q}y]$, as desired.

The vertices $[2]$ and $[2y]$ are distinct since $y \in \operatorname{Ann}_R(2y)$, but $y \notin \operatorname{Ann}_R(2)$. Likewise, the vertices $[2y]$ and $[2+\overline{q}y]$ are distinct for all $q \in Q$. The vertices $[y]$ and $[2+\overline{q}y]$ are distinct for all $q \in Q$ since $y \in \operatorname{Ann}_R(y)$, but $y \notin \operatorname{Ann}_R(2+\overline{q}y)$.

We claim that the vertices $[2+\overline{q}y]$ and $[2+\overline{q'}y]$ are equal if and only if $q - q' \in 2Q$, i.e., if and only if q and q' represent the same element in the field $Q/2Q \cong k$. For the first implication, assume that $q - q' \in 2Q$ and write $q - q' = 2q''$ where $q'' \in Q$. Then

$$2 + \overline{q}y = 2 + \overline{q'}y + 2\overline{q''}y$$

in R. Since $2\overline{q''}y$ is in $\mathfrak{m}^2$, by (3.29.1) it follows that $[2+\overline{q}y] = [2+\overline{q'}y]$.

For the converse, assume that $[2+\overline{q}y] = [2+\overline{q'}y]$ in $\Gamma_E(R)$. It follows that $2+\overline{q}y \in \operatorname{Ann}_R(2+\overline{q'}y) = (2+\overline{q'}y)R$, so there are elements $q_0, q_1 \in Q$ where

$$2+\overline{q}y = (2+\overline{q'}y)(\overline{q_0} + \overline{q_1}y) = 2\overline{q_0} + (\overline{q'}\overline{q_0} + 2\overline{q_1})y. \tag{3.29.2}$$

It follows that $2 = 2\overline{q_0}$ in $Q/4Q$, so $1 - q_0 = 2q''$ for some $q'' \in Q$. From the y coefficients in (3.29.2), we have

$$\overline{q} = \overline{q'}\overline{q_0} + 2\overline{q_1} = \overline{q'}(\overline{1-2q''}) + 2\overline{q_1} = \overline{q'} + 2(\overline{q_1} - \overline{q'q''})$$

in $Q/4Q$. It follows that $q - q' \in 2Q$, as desired. This ends the proof of the claim.

From the above claims, we conclude that $\Gamma_E(R)$ is a star with central vertex $[2y]$ and distinct ends $[y]$ and $[2+\overline{q}y]$ where q ranges through a set of representatives of $Q/2Q \cong k$ in $Q/4Q$. Thus, the graph $\Gamma_E(R)$ is a star with $|k|+2$ vertices.

Example 3.30. Let k be a field of characteristic 2, and let $(Q, 2Q, k)$ be a discrete valuation ring. We let X and Y be indeterminates and set

$$\begin{aligned} R &= Q[\![X, Y]\!]/(4, X^2, Y^2, 2X, 2Y, 2 - XY) \\ &\cong Q[X, Y]/(4, X^2, Y^2, 2X, 2Y, 2 - XY). \end{aligned}$$

We show that the graph $\Gamma_E(R)$ is a star with $|k| + 2$ vertices.

The ring R is local with maximal ideal $\mathfrak{m} = (2, x, y)R = (x, y)R$ where x and y are the residues of X and Y in R; this is from the equation $2 = xy$ in R. Given the other equations determining R, we have $\mathfrak{m}^2 = (xy)R$. (Note that the element 4 in the ideal defining R is redundant since $4 = 2(2 - XY) + X(2Y)$ in $Q[\![X, Y]\!]$. We include it explicitly so that it is clear that $4 = 0$ in R.) Since $x^2 = 0 = y^2$ and $xy = 2$ in R, the elements of R all have the form $\overline{s} + \overline{p}x + \overline{q}y$, with $p, q, s \in Q$.

We claim that $\mathrm{len}(R) = 4$. To see this, note that the ring $Q[\![X, Y]\!]/(4, X^2, Y^2)$ has length 8, being a quotient of a regular local ring by the squares of the elements of a regular system of parameters. Modding out by a non-zero element of the socle reduces the length by 1. Thus, the ring $Q[\![X, Y]\!]/(4, X^2, Y^2, 2XY)$ has length 7, the ring $Q[\![X, Y]\!]/(4, X^2, Y^2, 2X)$ has length 6, the ring $Q[\![X, Y]\!]/(4, X^2, Y^2, 2X, 2Y)$ has length 5, and $Q[\![X, Y]\!]/(4, X^2, Y^2, 2X, 2Y, 2 - XY)$ has length 4.

Consider an element $r = \overline{s} + \overline{p}x + \overline{q}y \in R$, with $p, q, s \in Q$. We claim that $r = 0$ if and only if $p, q \in 2Q$ and $s \in 4Q$. One implication follows from the equalities $4 = 0 = 2x = 2y$ in R. For the converse, assume that $r = 0$. It follows that there are elements $f, f', g, g', h \in Q[\![X, Y]\!]$ such that

$$s + pX + qY = X^2 f + Y^2 g + 2Xf' + 2Yg' + (2 - XY)h. \tag{3.30.1}$$

Write $f = \sum_{i,j \geq 0} f_{i,j} X^i Y^j$ with $f_{i,j} \in Q$, and similarly for f', g, g', h. Comparing constant terms and coefficients for x and y in this equation, we have

$$s = 2h_{0,0} \in 2Q, \quad p = 2f'_{0,0} + 2h_{1,0} \in 2Q, \quad q = 2g'_{0,0} + 2h_{0,1} \in 2Q$$

so it remains to show that $s \in 4Q$. To this end, compare coefficients for XY in (3.30.1) to find that

$$0 = 2f'_{0,0} + 2g'_{0,0} - h_{0,0} + 2h_{1,1}.$$

It follows that $h_{0,0} \in 2Q$, so $s = 2h_{0,0} \in 4Q$, as desired.

As a consequence of the previous paragraph, we find that the kernel of the natural map $Q \to R$ is precisely $4Q$.

Next, we note that an element $r = \overline{s} + \overline{p}x + \overline{q}y \in R$, with $p, q, s \in Q$, is a unit in R if and only if $s \notin 2Q$. Indeed, if $s \in 2Q$, then $r \in (2, x, y)R = \mathfrak{m}$. For the converse, assume that $s \notin 2Q$, that is, that s is a unit in Q. It follows that $s + pX + qY$ is invertible in $Q[\![X, Y]\!]$, say with inverse $\sum_{i,j \geq 0} a_{i,j} X^i Y^j$. Given the fact that $x^2 = 0 = y^2$ in R, it follows readily that $r^{-1} = \sum_{i=0}^{1} \sum_{j=0}^{1} \overline{a_{i,j}} x^i y^j$.

Given what we have shown, the following facts are readily verified. First, one has

$$\begin{aligned}
\mathrm{Soc}(R) &= \mathfrak{m}^2,\\
\mathrm{Ann}_R(x) &= (x)R + \mathfrak{m}^2 = (x)R,\\
\mathrm{Ann}_R(y) &= (y)R + \mathfrak{m}^2 = (y)R,\\
\mathrm{Ann}_R(x+\overline{q}y) &= (x+\overline{q}y)R + \mathfrak{m}^2 = (x+\overline{q}y)R \quad \text{for all } q \in Q.
\end{aligned}$$

We have $2 \in \mathrm{Soc}(R)$, so every vertex $v \neq [2]$ in $\Gamma_E(R)$ has the form $[x]$, $[y]$ or $[x+\overline{q}y]$ for some unit $q \in Q$. Furthermore, we have $[x] \neq [y]$ and $[x] \neq [x+\overline{q}y] \neq [y]$ for each unit $q \in Q$. Also, given units $q, q' \in Q$, we have $[x+\overline{q}y] = [x+\overline{q'}y]$ if and only if $q - q' \in 2Q$ if and only if q and q' represent the same element in $Q/2Q \cong k$. From this, it follows that $\Gamma_E(R)$ is a star with central vertex $[2]$ and with $2 + |k|$ vertices.

4 Graph Homomorphisms and Graphs Associated to Modules

In this section we study *graph homomorphisms* $\Gamma_E(R) \to \Gamma_E(S)$ induced by ring homomorphisms $R \to S$. This allows us to produce an Artinian ring R of length 4 such that $\Gamma_E(R)$ is an infinite star. In addition, we introduce and study a "torsion graph" associated to an R-module M, which is used in Section 5 to produce an Artinian ring R of length 6 such that $\Gamma_E(R)$ has infinite clique number.

Proposition 4.1. *Given a flat ring monomorphism $\varphi\colon R \hookrightarrow S$, the graph $\Gamma_E(R)$ is isomorphic to an* induced subgraph *of $\Gamma_E(S)$.*

Proof. Let r be a zero divisor in R. Then $rr' = 0$ for some zero divisor $r' \in R$. Since φ is a ring monomorphism, it follows that $\varphi(r)$ and $\varphi(r')$ are non-zero elements of S such that $\varphi(r)\varphi(r') = 0$. Thus, $Z^*(R)$ maps into $Z^*(S)$. Moreover, if $r, r' \in Z^*(R)$ are equivalent, then $\varphi(r)$ and $\varphi(s)$ are equivalent in $Z^*(S)$: the equivalence of r and r' means that $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$, so the flatness of φ implies that $\mathrm{Ann}_S(\varphi(r)) = \mathrm{Ann}_R(r)S = \mathrm{Ann}_R(r')S = \mathrm{Ann}_S(\varphi(r'))$, as desired; see [28, Theorem 7.4 (iii)].

Moreover, if $\mathrm{Ann}(r_1) \neq \mathrm{Ann}(r_2)$, then $\mathrm{Ann}(\varphi(r_1)) \neq \mathrm{Ann}(\varphi(r_2))$ since $x \in \mathrm{Ann}(r_1) \setminus \mathrm{Ann}(r_2)$ implies $\varphi(x) \in \mathrm{Ann}(\varphi(r_1))$, but $\varphi(x)\varphi(r_2) \neq 0$ since $xr_2 \neq 0$. Thus, φ preserves equivalence classes. Also, we have already seen that edges in $\Gamma_E(R)$ correspond to edges in $\Gamma_E(S)$. This means that $\Gamma_E(R)$ is a subgraph of $\Gamma_E(S)$. Finally, to see that $\Gamma_E(R)$ is an induced subgraph, note that if there is an edge between $[\varphi(r_1)]$ and $[\varphi(r_2)]$ in $\Gamma_E(S)$, then there is an edge between $[r_1]$ and $[r_2]$ in $\Gamma_E(R)$ since $0 = \varphi(r_1)\varphi(r_2) = \varphi(r_1 r_2)$ and φ is injective. □

A special case of the next result is stated without proof in [29, page 3552, lines 4–5]. It is worth noting that the analogous result for $\Gamma(R)$ is proved in [9, Theorem 2.2].

Proposition 4.2. *Let U be a multiplicatively closed subset, consisting of non-zero divisors, in the ring R. Then $\Gamma_E(R)$ and $\Gamma_E(U^{-1}R)$ are isomorphic as graphs.*

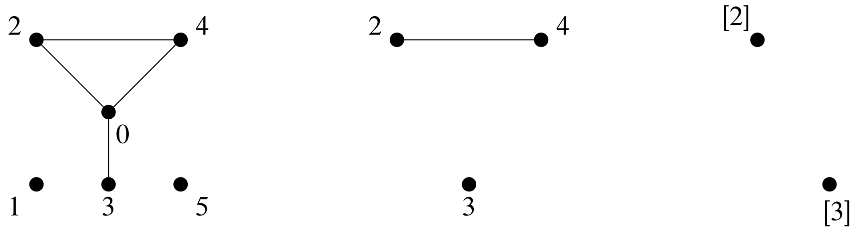
Proof. Let $\varphi : R \hookrightarrow U^{-1}R$. By Proposition 4.1, $\Gamma_E(R)$ is an induced subgraph of $\Gamma_E(U^{-1}R)$, therefore, we only need to show that there is a one-to-one correspondence between the vertices of $\Gamma_E(R)$ and those of $\Gamma_E(U^{-1}R)$. Let $[r/u] \in \Gamma_E(U^{-1}R)$. Then there exists a non-zero element $r'/u' \in U^{-1}R$ such that $rr'/uu' = 0$; i.e., there exists u'' such that $u''rr' = 0$ in R. However, since u'' is a non-zero divisor, we must have $rr' = 0$; so $r \in Z^*(R)$. Moreover, this shows that $r'/u' \in \mathrm{Ann}_{U^{-1}R}(r/u)$ if and only if $r' \in \mathrm{Ann}_R(r)$. It follows that $[r/u] = [r/1]$ in $\Gamma_E(U^{-1}R)$. $\square$

Corollary 4.3. *If* $Z^*(R) \cup \{0\} = \mathrm{nil}(R)$*, then* $\mathrm{nil}(R)$ *is a prime ideal and hence* $\Gamma_E(R) \cong \Gamma_E(R_{\mathrm{nil}(R)})$*; if R is also Noetherian, then* $\Gamma_E(R) \cong \Gamma_E(A)$ *where A is an Artinian local ring.*

As a tool for studying the clique numbers of $\Gamma_E(R)$, we introduce the "torsion graph" of a finitely generated R-module M. Note that our definition is different from those recently appearing in the literature [23].

Definition 4.4. To the pair R, M, we associate a **torsion graph**: (i) let $G^R(M)$ be the graph where every element of M is a distinct vertex; (ii) let $\Gamma^R(M)$ be the graph where each non-zero torsion element of M is represented by a distinct vertex; and (iii) let $\Gamma_E^R(M)$ be the graph whose vertices $[m]$ are the equivalence classes of non-zero torsion elements, where $m, n \in M$ are equivalent provided that $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(n)$. For each of these graphs, join an edge between each pair of distinct vertices if and only if their annihilator ideals have a non-trivial intersection; i.e., if and only if $\mathrm{Ann}_R(m) \cap \mathrm{Ann}_R(n) \neq (0)$.

Example 4.5. Let $R = M = \mathbb{Z}/6\mathbb{Z}$. The graphs associated to M are shown below:



$G^R(M)$ $\Gamma^R(M)$ $\Gamma_E^R(M)$

On the other hand, if we let $M = \mathbb{Z}/6\mathbb{Z}$, but change the ring to $R = \mathbb{Z}$, then $G^{\mathbb{Z}}(M) = K_6$, $\Gamma^{\mathbb{Z}}(M) = K_5$ (since 0 is omitted), and $\Gamma_E^{\mathbb{Z}}(M) = K_3$. For the last graph, note that there are now three classes determined by $\mathrm{Ann}_R(\overline{1}) = 6\mathbb{Z}$, $\mathrm{Ann}_R(\overline{2}) = 3\mathbb{Z}$, and $\mathrm{Ann}_R(\overline{3}) = 2\mathbb{Z}$.

Proposition 4.6. *If R is a domain or* $\mathrm{Ann}_R(M) \neq (0)$, *then* $\Gamma_E^R(M)$ *is complete.*

Proof. If R is a domain and M is torsion free, then the result holds vacuously; likewise, if the elements in the torsion submodule $\mathrm{Tor}(M)$ form a single equivalence class, then the result holds trivially. Thus, suppose $\mathrm{Tor}(M)$ contains at least two distinct classes $[m]$ and $[m']$. If R is a domain, there are $0 \neq r \in \mathrm{Ann}_R(m)$ and $0 \neq r' \in \mathrm{Ann}_R(m')$ such that $0 \neq rr' \in \mathrm{Ann}_R(m) \cap \mathrm{Ann}_R(m')$. Likewise, if $0 \neq r \in \mathrm{Ann}_R(M)$. □

Remark 4.7. These examples and results show that many standard results for zero divisor graphs of rings do not hold for torsion graphs associated to modules, regardless of how the vertices are chosen. For example, all previous zero divisor graphs have been connected. Moreover, considering $\Gamma_E(R)$, cycle graphs and complete graphs with more than two vertices are not possible.

From a different perspective, given a ring R and R-module M, one can consider the graph $\Gamma_E(R \ltimes M)$, where $R \ltimes M$ is the **trivial extension** of R by M, also known as the "idealization" of M. (Another notation for this construction is $R(+)M$, as in the survey [5].) As an additive Abelian group, one has $R \ltimes M = R \oplus M$. The multiplication $(r, m) \cdot (r', m') = (rr', rm' + r'm)$ makes $R \ltimes M$ into a ring. The following results will be useful in studying infinite cliques in the next section.

Fact 4.8. Set $S = R \ltimes M$ and let $r \in R$ and $m \in M$. Then:

(i) $\mathrm{Ann}_S(r, 0) = \mathrm{Ann}_R(r) \oplus \mathrm{Ann}_M(r)$, where $\mathrm{Ann}_M(r) = \{m \in M | rm = 0\}$;

(ii) $\mathrm{Ann}_S(0, m) = \mathrm{Ann}_R(m) \oplus M$;

(iii) $\mathrm{Ann}_S(r, m) = \{(s, n) | s \in \mathrm{Ann}_R(r), n \in M \text{ with } rn + sm = 0\}$;

(iv) If R is a domain and $r \neq 0$, then $\mathrm{Ann}_S(r, 0) = \mathrm{Ann}_S(r, m) = 0 \oplus \mathrm{Ann}_M(r)$.

Proposition 4.9. *For any ring R and any R-module M, the graph $\Gamma_E^R(M)$ is a subgraph of $\Gamma_E(R \ltimes M)$; it is an induced subgraph if and only if $\Gamma_E^R(M)$ is complete.*

Proof. Fact 4.8 (ii) shows the following: (1) if m is a non-zero element of M, then $(0, m) \in R \ltimes M$ is a non-zero torsion element; and (2) two non-zero torsion elements $m, m' \in M$ are equivalent if and only if the elements $(0, m)$ and $(0, m')$ in $R \ltimes M$ are equivalent. This shows that $\Gamma_E^R(M)$ is a subgraph of $\Gamma_E(R \ltimes M)$. However, the classes $[(0, m)]$ in $\Gamma_E(R \ltimes M)$ form a complete subgraph, therefore $\Gamma_E^R(M)$ would need to be complete in order to be induced. □

It is natural to ask whether the natural ring homomorphism $R \to R \ltimes M$ induces a well-defined graph homomorphism $\Gamma_E(R) \to \Gamma_E(R \ltimes M)$. Fact 4.8 (i) shows that this is equivalent to the following: for all $r, r' \in R$ if $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$, then $\mathrm{Ann}_M(r) = \mathrm{Ann}_M(r')$. Our next result gives a criterion guaranteeing that this condition is satisfied.

Proposition 4.10. *Assume that M satisfies one of the following conditions:*

(i) *M is an R-submodule of* $\mathrm{Hom}_R(N, R)$ *for some R-module N.*

(ii) *The natural "biduality" map $\delta_M\colon M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is injective.*

(iii) *M is a submodule of a finite rank free R-module.*

Then the natural ring homomorphism $R \to R \ltimes M$ induces a well-defined graph monomorphism $\Gamma_E(R) \to \Gamma_E(R \ltimes M)$ making $\Gamma_E(R)$ into an induced subgraph of $\Gamma_E(R \ltimes M)$.

Proof. It is straightforward to show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), so we assume that M satisfies condition (i). Let $r, r' \in R$ such that $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$. We claim that $\mathrm{Ann}_M(r) = \mathrm{Ann}_M(r')$. By symmetry, it suffices to show that $\mathrm{Ann}_M(r) \subseteq \mathrm{Ann}_M(r')$, so let $f \in \mathrm{Ann}_M(r)$. Then f is an R-module homomorphism $N \to R$ such that $rf = 0$, that is, such that $rf(n) = 0$ for all $n \in N$. It follows that $\mathrm{Im}(f) \subseteq \mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$, and similar reasoning implies that $f \in \mathrm{Ann}_M(r')$.

Using the claim, Fact 4.8 (i) implies that for all $r, r' \in R$ if $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$, then $\mathrm{Ann}_{R\ltimes M}(r, 0) = \mathrm{Ann}_{R\ltimes M}(r', 0)$. Thus, the rule of assignment $[r] \mapsto [(r, 0)]$ describes a well-defined function from the vertex set of $\Gamma_E(R)$ to the vertex set of $\Gamma_R(R \ltimes M)$. Moreover, Fact 4.8 (i) implies that for all $r, r' \in R$ if $\mathrm{Ann}_{R\ltimes M}(r, 0) = \mathrm{Ann}_{R\ltimes M}(r', 0)$, then $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(r')$; hence this map is injective. Finally, it is straightforward to show that $(r, 0)(r', 0) = 0$ in $R \ltimes M$ if and only if $rr' = 0$ in R, so $\Gamma_E(R)$ is an induced subgraph of $\Gamma_E(R \ltimes M)$. $\square$

The next example shows that the natural ring homomorphism $R \to R \ltimes M$ does not necessarily induce a well-defined graph homomorphism $\Gamma_E(R) \to \Gamma_E(R \ltimes M)$. See also Example 5.2.

Example 4.11. Let $R = k[X, Y]/(X, Y)^2$ where k is a field, and set $M = R/XR$. Then we have $\mathrm{Ann}_R(X) = (X, Y)R = \mathrm{Ann}_R(Y)$, but $\mathrm{Ann}_M(X) = M \supsetneq YM = \mathrm{Ann}_M(Y)$. Thus, we have $[X] = [Y]$ in $\Gamma_E(R)$, but $[(X, 0)] \neq [(Y, 0)]$ in $\Gamma_E(R \ltimes M)$. It follows that the rule of assignment $[r] \mapsto [(r, 0)]$ does not describe a well-defined function from the vertex set of $\Gamma_E(R)$ to the vertex set of $\Gamma_R(R \ltimes M)$.

5 Cliques

One of our main motivations is the question of how pathological the behavior of $\Gamma_E(R)$ can be, and how one might avoid such pathologies by imposing mild conditions on R. As mentioned in the introduction, assuming that the ring is Noetherian or Artinian is not enough to ensure that the associated graph is finite or even has finite clique number. To construct a "small" ring R such that $\omega(\Gamma_E(R)) = \infty$, we begin by considering direct products of rings and reduce to the local case. Ultimately our construction yields an Artinian ring of length 6 with an infinite clique, but the question of whether such an example exists in length 5 is open.

Proposition 5.1. *Let R and S be rings.*

(i) *$\Gamma_E(R \times S)$ is infinite if and only if $\Gamma_E(R)$ or $\Gamma_E(S)$ is infinite.*

(ii) *$\Gamma_E(R \times S)$ has an infinite clique if and only if $\Gamma_E(R)$ or $\Gamma_E(S)$ has an infinite clique.*

(iii) *$\omega(\Gamma_E(R \times S)) = \infty$ if and only if $\omega(\Gamma_E(R)) = \infty$ or $\omega(\Gamma_E(S)) = \infty$.*

Proof. (i) The distinct vertices of $\Gamma_E(R \times S)$ have the form $[(1,0)]$, $[(0,1)]$, $[(r,1)]$, $[(1,s)]$, $[(r,0)]$, $[(0,s)]$, and $[(r,s)]$ where $r \in Z^*(R)$ and $s \in Z^*(S)$. (This holds even when R or S is a domain.) Thus, it is routine to show that

$$|\Gamma_E(R \times S)| = 2 + 2|\Gamma_E(R)| + 2|\Gamma_E(S)| + |\Gamma_E(R)||\Gamma_E(S)|,$$

and the result follows immediately.

(ii) If $\Gamma_E(R)$ has an infinite clique, then $\Gamma_E(R \times S)$ has an infinite clique with vertices of the form $[(r,0)]$; and similarly if $\Gamma_E(S)$ has an infinite clique.

Conversely, assume that $\Gamma_E(R \times S)$ has an infinite clique. Given the form of the vertices of $\Gamma_E(R \times S)$, it follows that $\Gamma_E(R \times S)$ has an infinite clique containing only vertices of one of the following forms: $[(r,1)]$, $[(1,s)]$, $[(r,0)]$, $[(0,s)]$, or $[(r,s)]$ where $r \in Z^*(R)$ and $s \in Z^*(S)$. No two vertices of the form $[(r,1)]$ or $[(1,s)]$ are adjacent, so the infinite clique must contain only vertices of one of the following forms: $[(r,0)]$, $[(0,s)]$, or $[(r,s)]$. If there is an infinite clique in $\Gamma_E(R \times S)$ with vertices of the form $[(r,0)]$, then the r-values for this clique yield an infinite clique in $\Gamma_E(R)$; and similarly if there is an infinite clique in $\Gamma_E(R \times S)$ with vertices of the form $[(0,s)]$.

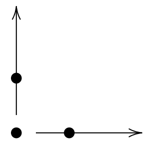
Thus, we assume that $\Gamma_E(R \times S)$ contains a clique with infinitely many vertices of the form $[(r_i,s_i)]$ where $i = 1,2,3,\ldots$. Since these vertices are distinct, the ideals $\mathrm{Ann}_R(r_i) \oplus \mathrm{Ann}_S(s_i) = \mathrm{Ann}_{R\times S}(r_i,s_i)$ must be distinct. Thus, there are either infinitely many distinct ideals in the set $\{\mathrm{Ann}_R(r_i)\}_i$ or in the set $\{\mathrm{Ann}_S(s_i)\}_i$. Since $r_ir_j = 0$ and $s_is_j = 0$ for all $i \neq j$, it follows that either the $[r_i]$ form an infinite clique in $\Gamma_E(R)$ or the $[s_i]$ form an infinite clique in $\Gamma_E(S)$.

(iii) Argue as in the proof of part (ii), showing that $\Gamma_E(R \times S)$ has arbitrarily large cliques if and only if $\Gamma_E(R)$ or $\Gamma_E(S)$ has arbitrarily large cliques. □

With the next example we show that even the Artinian condition on a ring is not enough to prevent its graph of zero divisors from having an infinite clique.

Example 5.2. Let k be a field and set $R = k[X,Y]/(X,Y)^2$. Let x and y be the residues in R of the variables X and Y. As a k-vector space, the ring R has rank 3 with basis $1, x, y$. Moreover, this basis imposes a $\mathbb{Z}^2$-graded structure on R that is represented by the following diagram.

R

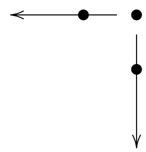


Set $M = E_R(k)$, the injective hull of k, in other words, the graded canonical module of R, which is given as $M = \mathrm{Hom}_{k[X,Y]}(R, k[X, Y])$. As a k-vector space, the module M has rank 3 with basis e, x^{-1}, y^{-1}, where the R-module structure is described according to the following rules:

$$x \cdot e = 0, \quad x \cdot x^{-1} = e, \quad x \cdot y^{-1} = 0,$$
$$y \cdot e = 0, \quad y \cdot y^{-1} = e, \quad y \cdot x^{-1} = 0.$$

This basis imposes a $\mathbb{Z}^2$-graded structure on M, represented by the next diagram.

M



It is straightforward to verify the following computations where $a \in k$:

$$\mathrm{Ann}_R(e) = (x, y)R, \quad \mathrm{Ann}_R(y^{-1}) = xR, \quad \mathrm{Ann}_R(x^{-1} + ay^{-1}) = (ax - y)R.$$

In particular, the graph $\Gamma_E^R(M)$ has at least $|k| + 2$ distinct vertices. (In fact, these are exactly the distinct vertices of $\Gamma_E^R(M)$.) Thus, Proposition 4.9 implies that the graph $\Gamma_E(R \ltimes M)$ contains a clique with $|k|+2$ distinct vertices. In particular, if k is infinite, then $\Gamma_E(R \ltimes M)$ contains an infinite clique. It is straightforward to show that $R \ltimes M$ has length 6.

We conclude the example by showing that the rule of assignment $[r] \mapsto [(r, 0)]$ does not describe a well-defined function from the vertex set of $\Gamma_E(R)$ to the vertex set of $\Gamma_R(R \ltimes M)$. (Contrast this with Proposition 4.10.) As in Example 4.11, this follows from the next equalities

$$\mathrm{Ann}_R(x) = (x, y)R = \mathrm{Ann}_R(y)$$
$$\mathrm{Ann}_M(x) = (e, y^{-1})R \neq (e, x^{-1})R = \mathrm{Ann}_M(y)$$

by Fact 4.8 (i).

It is natural to ask whether the previous example is minimal, that is, if one can construct a ring R of length 5 or less such that $\Gamma_E(R)$ contains an infinite clique. Accordingly, we next characterize all graphs $\Gamma_E(R)$ such that $\mathrm{len}(R) \leq 3$, and we show that $\Gamma_E(R)$ cannot contain an infinite clique if $\mathrm{len}(R) \leq 4$. The length 5 case is still open; see Question 5.9 and Remark 5.10.

Proposition 5.3. *If* $\mathrm{len}(R) \leq 3$, *then* $\Gamma_E(R)$ *is finite. In particular:*

(i) *If* $\mathrm{len}(R) = 1$, *then* $\Gamma_E(R) = \emptyset$.

(ii) *If* $\mathrm{len}(R) = 2$, *then* $\Gamma_E(R)$ *is either a single edge or a single vertex.*

(iii) *If* $\mathrm{len}(R) = 3$, *then* $|\Gamma_E(R)| \leq 6$*; specifically, the graph* $\Gamma_E(R)$ *is*

  (a) *a triangle with three ends if R is a product of three fields;*

  (b) *a* path *of length* 3 *if R is a product of a field and a local ring of length* 2, *or*

  (c) *either a single edge or a single vertex if R is local.*

Proof. If $\mathrm{len}(R) = 1$, then R is a field and has no zero divisors. If $\mathrm{len}(R) = 2$, then R is either a product of two fields, or a local ring of length 2. If $R \cong K_1 \times K_2$, then $\Gamma_E(R)$ is the edge $[(1,0)] - [(0,1)]$. Otherwise, there is a complete discrete valuation ring (Q, τ) such that $R \cong Q/(\tau^2)Q$; in this case, every non-unit of R is a unit multiple of $\overline{\tau}$, so $\Gamma_E(R)$ is the single vertex $[\overline{\tau}]$.

Suppose $\mathrm{len}(R) = 3$. Then R is isomorphic to one of the following: (1) a product $K_1 \times K_2 \times K_3$ of three fields, (2) a product of a field K_1 with a local ring R_2 of length 2, or (3) a local ring.

(1) If $R \cong K_1 \times K_2 \times K_3$, then R has exactly eight ideals, including 0 and R itself, namely, the products of copies of 0 and the K_i's. Each of the six non-trivial ideals is the annihilator of an element (a, b, c) with $a, b, c = 0$ or 1. It is not difficult to see that the three vertices [(1,0,0)], [(0,1,0)], and [(0,0,1)] are all adjacent, with ends [(0,1,1)], [(1,0,1)], and [(1,1,0)], respectively; i.e., the graph is as described.

(2) Assume that $R \cong K_1 \times R_2$, where R_2 has length 2. Then there is a complete discrete valuation ring (Q, τ) such that $R \cong Q/(\tau^2)Q$, and $\Gamma_E(R)$ is the graph $[(0,1)] - [(1,0)] - [(0,\overline{\tau})] - [(1,\overline{\tau})]$.

(3) Assume that $(R, \mathfrak{m})$ is local of length 3, and let e be the embedding dimension of R, that is, the minimal number of generators of $\mathfrak{m}$. Since $\mathrm{len}(R) = 3$, we have $e = 1$ or $e = 2$. If $e = 1$, then there is a complete discrete valuation ring (Q, τ) such that $R \cong Q/(\tau^3)Q$, and $\Gamma_E(R)$ is the edge $[\overline{\tau}] - [\overline{\tau}^2]$. If $e = 2$, then the equalities $3 = \mathrm{len}(R) = \mathrm{len}(R/\mathfrak{m}) + \mathrm{len}(\mathfrak{m}/\mathfrak{m}^2) + \mathrm{len}(\mathfrak{m}^2)$ imply that $\mathfrak{m}^2 = 0$; it follows that $\mathrm{Soc}(R) = \mathfrak{m}$, and $\Gamma_E(R)$ is the single vertex $[x]$ for any $0 \neq x \in \mathfrak{m}$. □

Remark 5.4. In general, if R is a product of n fields, then $\Gamma_E(R)$ will have $2^n - 2$ vertices and clique number n, corresponding to the n primes $K_1 \times \cdots \times \widehat{K_i} \times \cdots \times K_n$.

Proposition 5.5 (See [7, Theorem 3.8].). *If R is a reduced Noetherian ring, then* $\omega(\Gamma_E(R)) = |\mathrm{Ass}(R)| < \infty$.

Proof. Let R be a reduced Noetherian ring and let U be the set of non-zero divisors of R. Then $U^{-1}R$ is Noetherian and a finite product of fields $K_1 \times \cdots \times K_n$. Proposition 4.2 implies that $\Gamma_E(R) \cong \Gamma_E(U^{-1}R)$, so Remark 5.4 implies that $\omega(\Gamma_E(R)) = \omega(\Gamma_E(U^{-1}R)) = n = |\operatorname{Ass}(U^{-1}R)| = |\operatorname{Ass}(R)|$. □

Noting that a reduced ring contains no self-annihilating elements, we consider a relationship between clique size and self-annihilating elements.

Proposition 5.6. *Let S be clique in $\Gamma_E(R)$, and set $N = \{[r] \in S : r^2 \neq 0\}$ and $A = \{[a] \in S : a^2 = 0\}$. If $|S| \geq 3$ and $|N| \geq 2$, then $|\Gamma_E(R)| \geq |A| + 2^{|N|} - 1$.*

Proof. Suppose S is a clique in $\Gamma_E(R)$, and $X = \{x_1, \dots x_k\} \subsetneq S$ with $k \geq 2$. Consider the element $x = x_1 + \cdots + x_k$. For any $y \in S - X$ we have $xy = \sum_{i=1}^{k} x_i y = 0$, while for $x_i \in X$, we have $xx_i = (x_1 + \cdots + x_k)x_i = x_i^2$, since $x_i x_j = 0$ for $i \neq j$. If no element of X is self-annihilating, then $[x]$ is not adjacent to any vertex of X, while $[x]$ is adjacent to all other vertices of S. So each such choice of X determines a unique vertex $[x]$ such that $[x] \notin S$. Letting $|N| = n$, there are $2^n - n - 1$ subsets of N with at least two elements, so we have at least $2^n - n - 1$ vertices of $\Gamma_E(R)$ which are not elements of S.

We now see

$$|\Gamma_E(R)| \geq |S| + (2^n - n - 1) = |A| + |N| + (2^n - n - 1) = |A| + 2^n - 1. \qquad \square$$

Corollary 5.7. *If a clique of $\Gamma_E(R)$ contains infinitely many elements which are not self-annihilating, then R is uncountable.*

The next result uses the following facts: an Artinian ring R is isomorphic to a finite product of Artinian local rings, and the length of R is the product of the lengths of the factors. We note that, except for the case (v) (c), all the graphs turn out to be finite. Also, by the notation edim we mean the embedding dimension.

Proposition 5.8. *Let R be a ring of length* 4.

(i) *If R is a product of four fields, then $\omega(\Gamma_E(R)) = 4$.*

(ii) *If R is a product of two fields and a local ring of length* 2*, then $\omega(\Gamma_E(R)) = 3$.*

(iii) *If R is a product of two local rings of length* 2*, then $\omega(\Gamma_E(R)) = 3$.*

(iv) *If R is a product of a field and a local ring R_2 of length* 3*, then:*

 (a) *If* $\operatorname{edim}(R_2) = 1$*, then $\omega(\Gamma_E(R)) = 3$; and*

 (b) *If* $\operatorname{edim}(R_2) = 2$*, then $\Gamma_E(R)$ is a $K_{1,2}$ star graph and has clique number* 2.

(v) *If R is local, we have the following:*

(a) *If* $\operatorname{edim}(R) = 1$*, then* $\Gamma_E(R)$ *is a* $K_{1,2}$ *star graph and has clique number* 2*;*

(b) *If* $\operatorname{edim}(R) = 3$*, then* $\Gamma_E(R)$ *is a single vertex; and*

(c) *If* $\operatorname{edim}(R) = 2$*, then* $\Gamma_E(R)$ *can be infinite, but* $\omega(\Gamma_E(R)) \le 3$.

Proof. (i) This is from Remark 5.4.

(ii) Assume that $R \cong K_1 \times K_2 \times R_3$ where K_1 and K_2 are fields and R_3 is a local ring of length 2. Then there is a complete discrete valuation ring (Q, τ) such that $R_3 \cong Q/(\tau^2)Q$. Note that the only ideals of R_3 are R_3, $\overline{\tau}R_3$, and 0.

It is straightforward to see that $\Gamma_E(R)$ contains a clique with vertices $[(1,0,0)]$, $[(0,1,0)]$, and $[(0,0,1)]$. (There are several other cliques on three vertices.) To show that $\Gamma_E(R)$ does not have a clique on four vertices, it suffices to show that it does not have four vertices of degree at least 4.

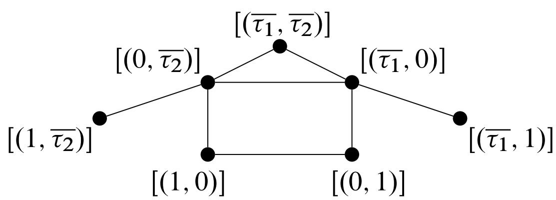
We claim that this graph consists of the following:

- two vertices of degree 5, namely $[(1,0,0)]$ and $[(0,1,0)]$;
- four vertices of degree 3, namely $[(0,0,1)]$, $[(0,0,\overline{\tau})]$, $[(1,0,\overline{\tau})]$, $[(0,1,\overline{\tau})]$;
- one vertex of degree 2, namely $[(1,1,0)]$; and
- three vertices of degree 1, namely $[(1,0,1)]$, $[(0,1,1)]$, and $[(1,1,\overline{\tau})]$.

For instance, to check that all the vertices of $\Gamma_E(R)$ are listed, observe that R has exactly twelve ideals, each of the form $I_1 \times I_2 \times I_3$, where I_j is an ideal of R_j for $j = 1,2,3$. Each of the non-trivial ideals is the annihilator ideal of a zero-divisor. For instance we have $0 \times K_2 \times \overline{\tau}R_3 = \operatorname{Ann}_R(1,0,\overline{\tau})$. Checking that each vertex has the given degree is tedious but not difficult. For instance, the vertices adjacent to $[(1,0,\overline{\tau})]$ are $[(0,1,0)]$, $[(0,0,\overline{\tau})]$, and $[(0,1,\overline{\tau})]$.

(iii) Assume that $R \cong R_1 \times R_2$ where R_1 and R_2 are local rings of length 2. Then there are complete discrete valuation rings (Q_i, τ_i) such that $R_i \cong Q_i/(\tau_i^2)Q$ for $i = 1,2$. Each ring R_i has exactly three ideals, namely R_i, $\overline{\tau_i}R_i$, and 0. This implies that R has exactly nine ideals. Each of the seven non-trivial ideals is the annihilator ideal of a zero-divisor.

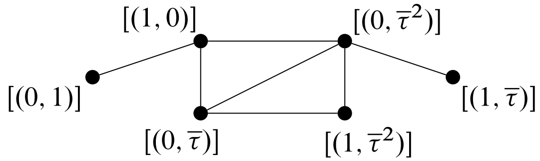
It is straightforward to see that $\Gamma_E(R)$ contains a clique with vertices $[(0,\overline{\tau_2})]$, $[(\overline{\tau_1},\overline{\tau_2})]$, and $[(\overline{\tau_1},0)]$. This is actually the only clique on three vertices, and there are no cliques on four vertices, as one can see from the following sketch of $\Gamma_E(R)$:



It follows readily that $\omega(\Gamma_E(R)) = 3$.

(iv) Assume that $R \cong K_1 \times R_2$ where K_1 is a field and R_2 is a local ring of length 3. Let e_2 be the embedding dimension of R_2, that is, the minimal number of generators of $\mathfrak{m}_2$. Since $\operatorname{len}(R_2) = 3$, we have $e_2 = 1$ or $e_2 = 2$.

If $e_2 = 1$, then there is a complete discrete valuation ring (Q, τ) such that $R_2 \cong Q/(\tau^3)Q$, and $\Gamma_E(R)$ is the following graph with $\omega(\Gamma_E(R)) = 3$:



If $e_2 = 2$, then we have $\mathfrak{m}_2^2 = 0$, as in the proof of Proposition 5.3. It follows that $\operatorname{Soc}(R_2) = \mathfrak{m}_2$, so the graph $\Gamma_E(R)$ has the form $[(1,0)] - [(0,x)] - [(1,x)]$ for some (equivalently, any) $0 \neq x \in \mathfrak{m}_2$.

(v) Assume that R is local with $e = \operatorname{edim}(R)$. Since $\operatorname{len}(R) = 4$, we can have $e = 1$, 2, or 3. If $e = 1$, then $R \cong Q/(\tau^4)$ where (Q, τ) is a discrete valuation ring; in this case, the graph $\Gamma_E(R)$ is $[\overline{\tau}] - [\overline{\tau}^3] - [\overline{\tau}^2]$. If $e = 3$, then $\mathfrak{m}^2 = 0$, as in the proof of Proposition 5.3; in this case, it follows that $\operatorname{Soc}(R) = \mathfrak{m}$ so $\Gamma_E(R)$ is the single vertex $[x]$ for some (equivalently, any) $0 \neq x \in \mathfrak{m}$.

Assume now that $e = 2$. The fact that $4 = \operatorname{len}(R) = \operatorname{len}(R/\mathfrak{m}) + \operatorname{len}(\mathfrak{m}/\mathfrak{m}^2) + \operatorname{len}(\mathfrak{m}^2/\mathfrak{m}^3) + \operatorname{len}(\mathfrak{m}^3)$ implies that $\mathfrak{m}^2$ is principal (and nonzero) and $\mathfrak{m}^3 = 0$.

We argue by cases.

Case 1: There is a generating sequence x, y of $\mathfrak{m}$ such that $x^2 \neq 0$, and $xy = y^2 = 0$. In this case, it follows readily that $\Gamma_E(R)$ is a single edge $[x] - [x^2]$, which has clique number 2.

Case 2: There is a generating sequence x, y of $\mathfrak{m}$ such that $x^2 \neq 0$, $y^2 \neq 0$, and $xy = 0$. In this case, we show that $\Gamma_E(R)$ can be infinite, but that $\omega(\Gamma_E(R)) = 3$.

Since $\mathfrak{m}^3 = 0$ and x^2, y^2 are non-zero, we conclude that x^2 and y^2 are each generators for the principal ideal $\mathfrak{m}^2$. It follows that there is a unit $u \in R$ such that $y^2 = ux^2$.

We claim that

$$\operatorname{Ann}_R(x^2) = \operatorname{Ann}_R(y^2) = \mathfrak{m}, \tag{5.8.1}$$

$$\operatorname{Ann}_R(x) = (y)R + \mathfrak{m}^2 = (y)R, \tag{5.8.2}$$

$$\operatorname{Ann}_R(y) = (x)R + \mathfrak{m}^2 = (x)R, \tag{5.8.3}$$

$$\operatorname{Ann}_R(x + ay) = (x - u^{-1}a^{-1}y)R + \mathfrak{m}^2 = (x - u^{-1}a^{-1}y)R, \tag{5.8.4}$$

where a is a unit in R.

(5.8.1) This follows from the fact that x^2 and y^2 are non-zero elements of $\mathfrak{m}^2$ and that $\mathfrak{m}^3 = 0$.

(5.8.2) Since $xy = 0$ and $\mathfrak{m}^2 = (y^2)R \subseteq (y)R$, it suffices to show that $\operatorname{Ann}_R(x) \subseteq (y)R + \mathfrak{m}^2$. Let $r \in \operatorname{Ann}_R(x)$. Since r is a non-unit, there are elements $s, t \in R$ such that $r = sx + ty$. Then we have

$$0 = xr = x(sx + ty) = sx^2.$$

If s were in $R \backslash \mathfrak{m}$, then s would be a unit, so the display would imply that $x^2 = 0$, a contradiction. Thus, we have $s \in \mathfrak{m}$ and so

$$r = sx + ty \in \mathfrak{m}x + (y)R \subseteq \mathfrak{m}^2 + (y)R$$

as desired.

(5.8.3) This is analogous to (5.8.2).

(5.8.4) First, we observe that

$$(x + ay)(x - u^{-1}a^{-1}y) = x^2 - u^{-1}y^2 = 0$$

since $y^2 = ux^2$. Next, we note that $x(x - u^{-1}a^{-1}y) = x^2$; it follows that $\mathfrak{m}^2 = (x^2)R \subseteq (x - u^{-1}a^{-1}y)R$. Thus, it remains to show the reverse containment. Let $r \in \operatorname{Ann}_R(x + ay)$, and write $r = sx + ty$ for $s, t \in R$. Then we have

$$0 = (x + ay)r = (x + ay)(sx + ty) = sx^2 + aty^2 = sx^2 + atux^2 = (s + atu)x^2.$$

It follows that $s + atu \in \operatorname{Ann}_R(x^2) = \mathfrak{m} = (x, y)R$, so there are elements $b, c \in R$ such that $s + atu = bx + cy$. We then have

$$\begin{aligned} sx + ty &= -atux + bx^2 + cxy + ty = -atux + bx^2 + ty \\ &= -tau(x - a^{-1}u^{-1}y) + bx^2 \in (x - u^{-1}a^{-1}y)R + \mathfrak{m}^2 \end{aligned}$$

as desired.

Next, we observe that the vertices of $\Gamma_E(R)$ are of the following form: $[x^2]$, $[x]$, $[y]$, or $[x + ay]$ for some unit $a \in R$. Let r be a non-zero non-unit of R. Then r is of the form $r = sx + ty$ with $s, t \in R$. If $s, t \in \mathfrak{m}$, then $r \in \mathfrak{m}^2 = \operatorname{Soc}(R)$, so we have $[r] = [x^2]$. If $s \in \mathfrak{m}$ and $t \in R \backslash \mathfrak{m}$, then $sx \in \mathfrak{m}^2 = \operatorname{Soc}(R)$ so $[r] = [sx + ty] = [ty] = [y]$ since t is a unit in R. Similarly, if $s \notin \mathfrak{m}$ and $t \in \mathfrak{m}$, then $[r] = [x]$. The last remaining case has $s, t \in R \backslash \mathfrak{m}$, that is, they are both units, so we have $[r] = [sx + ty] = [x + s^{-1}ty]$.

Since $\mathfrak{m}^3 = 0$, we see that every vertex $v \neq [x^2]$ is adjacent to $[x^2]$. Also, the vertices $[x]$ and $[y]$ are adjacent, so the vertices $[x^2]$, $[x]$, and $[y]$ form a triangle in $\Gamma_E(R)$, i.e., a clique of size 3. Thus, we need to show that $\Gamma_E(R)$ does not have a clique of size 4. To this end, it suffices to note that the equations (5.8.1)–(5.8.4) show that every vertex $v \neq [x^2]$ in $\Gamma_E(R)$ has degree at most 3, so this graph can not have a clique of size 4. This concludes the proof in Case 2.

Case 3: There is a generating sequence x, y of $\mathfrak{m}$ such that $x^2 \neq 0$ and $xy \neq 0$. In this case, a change of variables reverts back to Case 1 or 2. Specifically, since $\mathfrak{m}^2$ is principal and x^2, xy are non-zero, we have $(x^2)R = \mathfrak{m} = (xy)$. Write $xy = vx^2$ for some unit $v \in R$, and set $\tilde{y} = y - vx$. Then $x^2 \neq 0$ and $x\tilde{y} = xy - vx^2 = 0$, so we revert back to Case 1 or 2, depending on whether or not $(\tilde{y})^2 = 0$.

Case 4: Since all the previous arguments deal with the case where an element in a generating sequence is not square zero, we are reduced to the assumption that any element in a generating sequence *is* square zero. Moreover, if an element in $\mathfrak{m}$ is not a minimal generator, then as a member of $\mathfrak{m}^2$, its square is also zero since $\mathfrak{m}^3 = 0$. Thus, in this case we assume that $s^2 = 0$ for all $s \in \mathfrak{m}$. We will show that $\Gamma_E(R)$ is a *non-degenerate star*, in which case $\omega(\Gamma_E(R)) = 2$. Our assumptions imply that for any generating sequence x, y of $\mathfrak{m}$, we have $x^2 = y^2 = 0$, but $xy \neq 0$ since $\mathfrak{m}^2 \neq 0$. It follows that $\mathfrak{m}^2 = (xy)R$. Note that

$$0 = (x + y)^2 = x^2 + 2xy + y^2 = 2xy.$$

If 2 is a unit, then $xy = 0$, contradicting the fact that $0 \neq \mathfrak{m}^2 = (xy)R$. Thus, $2 \in \mathfrak{m}$, and hence $4 = 2^2 = 0$ in R. There are the following two possibilities:

Case 4a: $2 = 0$ in R. In this case, the ring R has characteristic 2, so R contains a field of characteristic 2. Since R is Artinian, it is complete, so it contains a subfield $k_0 \subseteq R$ such that $k_0 \cong k$. Cohen's structure theorem implies that there is a ring epimorphism $\tau\colon k_0[\![X, Y]\!] \twoheadrightarrow R$ given by $X \mapsto x$ and $Y \mapsto y$. Since $x^2 = 0 = y^2$, we conclude that $X^2, Y^2 \in \operatorname{Ker}(\tau)$, so there is an induced epimorphism $\tau'\colon k_0[\![X, Y]\!]/(X^2, Y^2) \twoheadrightarrow R$. Since $\operatorname{len}(k_0[\![X, Y]\!]/(X^2, Y^2)) = 4 = \operatorname{len}(R)$, the map τ' is an isomorphism $R \cong k_0[\![X, Y]\!]/(X^2, Y^2) \cong k_0[X, Y]/(X^2, Y^2)$. Example 3.28 implies that $\Gamma_E(R)$ is a non-degenerate star.

Case 4b: $2 \neq 0$ in R.

Case 4b(i): 2 is a minimal generator for $\mathfrak{m}$. In this case, Cohen's structure theorem implies that there is a complete discrete valuation ring $(Q, 2Q, k)$ and an epimorphism $\tau\colon Q[\![Y]\!] \twoheadrightarrow R$ such that $\mathfrak{m} = (2, \tau(Y))R$. Since $2^2 = 0 = \tau(Y)^2$ in R, it follows that $4, Y^2 \in \operatorname{Ker}(\tau)$. Thus, the induced epimorphism $\tau'\colon Q[\![Y]\!]/(4, Y^2) \twoheadrightarrow R$ is an isomorphism, as $\operatorname{len}(R) = 4 = \operatorname{len}(Q[\![Y]\!]/(4, Y^2))$. It follows that we have $R \cong Q[\![Y]\!]/(4, Y^2) \cong Q[Y]/(4, Y^2)$. Thus, Example 3.29 implies that $\Gamma_E(R) \cong \Gamma_E(Q[Y]/(4, Y^2))$ is a non-degenerate star.

Case 4b(ii): 2 is not a minimal generator for $\mathfrak{m}$. In this case, we have $2 \in \mathfrak{m}^2$. Cohen's structure theorem implies that there is a complete discrete valuation ring $(Q, 2Q, k)$ and an epimorphism $\tau\colon Q[\![X, Y]\!] \twoheadrightarrow R$ such that $\mathfrak{m} = (\tau(X), \tau(Y))R$. Set $x = \tau(X)$ and $y = \tau(Y)$. Since $x^2 = 0 = y^2 = 4$ in R, it follows that $2 \in \mathfrak{m}^2 = (xy)R$. Writing $2 = axy$ for some element $a \in R$, we see that a must be a unit in R; if not, then $a \in \mathfrak{m}$ and so $2 = axy \in \mathfrak{m}^3 = 0$, a contradiction. Define $\tau'\colon Q[\![X, Y]\!] \to R$ by sending $X \mapsto ax$ and $Y \mapsto y$. Then τ' is also a ring epimorphism with the added advantage of satisfying $2 = \tau'(X)\tau'(Y)$ in R.

It follows that $2 - XY \in \mathrm{Ker}(\tau')$. As $\tau'(X)^2 = 0 = \tau'(Y)^2 = 4$ in R, we have $X^2, Y^2, 4 \in \mathrm{Ker}(\tau')$. The facts $2 \in \mathfrak{m}^2$ and $\mathfrak{m}^3 = 0$ imply that $2\tau'(X) = 0 = 2\tau'(Y)$, so $2X, 2Y \in \mathrm{Ker}(\tau')$. In summary, we have $(4, X^2, Y^2, 2X, 2Y, 2 - XY) \subseteq \mathrm{Ker}(\tau')$. We conclude that $R \cong Q[\![X,Y]\!]/(4, X^2, Y^2, 2X, 2Y, 2 - XY)$, because $\mathrm{len}(Q[\![X,Y]\!]/(4, X^2, Y^2, 2X, 2Y, 2-XY)) = 4$; see Example 3.30. From this example, we also know that $\Gamma_E(R)$ is a non-trivial star. $\square$

Question 5.9. Is there an Artinian ring R with $\mathrm{len}(R) = 5$ and $\omega(\Gamma_E(R)) = \infty$?

Remark 5.10. Working as in the proof of Proposition 5.8, one readily reduces Question 5.9 to the case where R is local with $\mathrm{edim}(R) = 2$ or 3.

6 Girth and Cut Vertices

Assume throughout this section that R is a Noetherian ring.

In this section we continue the investigation started in [35] into the graph theoretic properties satisfied by $\Gamma_E(R)$. One of our primary tools is the behavior of the associated primes of R as represented in $\Gamma_E(R)$; see Proposition 6.1 (ii). We prove that the *girth* of $\Gamma_E(R)$ is no more than 3 when finite. On the other hand, there is no similar bound on the *circumference* of $\Gamma_E(R)$. We also consider *cut vertices* in the graph.

6.1 Girth

We begin by listing several known results, some of which follow from direct proof on zero divisor graphs of rings, and others that are from results on semigroups. Some of the results in this section have a lot in common with work on Anderson and Livingston's graph $\Gamma(R)$. Therefore, where applicable, we point out the relevant papers.

Proposition 6.1.

(i) [17, Theorem 1]. *If $[x]$ and $[y]$ are non-adjacent vertices of $\Gamma_E(R)$, then the closed neighborhood of the vertex $[xy]$ contains the neighborhoods of $[x]$ and $[y]$; i.e., $\mathcal{N}([x]) \cup \mathcal{N}([y]) \subseteq \overline{\mathcal{N}([xy])}$.*

(ii) [35, Lemma 1.2]. *If $x, y \in R$ such that* $\mathrm{Ann}_R(x)$ *and* $\mathrm{Ann}_R(y)$ *are distinct associated primes of R, then $[x]$ and $[y]$ are adjacent in $\Gamma_E(R)$. If $[v]$ is a vertex of $\Gamma_E(R)$, then either* $\mathrm{Ann}_R(v) \in \mathrm{Ass}(R)$ *or there is a vertex $[w]$ adjacent to $[v]$ such that* $\mathrm{Ann}_R(w) \in \mathrm{Ass}(R)$.

(iii) [19, Theorem 1.6; 29, (2.4)]. *If $[v] \in \Gamma_E(R)$ is contained in a cycle, then it is contained in a cycle of length 3 or 4. In particular, if* $\mathrm{girth}(\Gamma_E(R))$ *is finite, then* $\mathrm{girth}(\Gamma_E(R)) \leq 4$.

(iv) [18, Theorem 1.5]. *If* $\mathrm{girth}(\Gamma_E(R)) < \infty$*, then each $[v] \in \Gamma_E(R)$ is either an end or is contained in some cycle.*

Proposition 6.2. (Compare to [1, Remark 2.7].) *Any vertex* $[x]$ *of* $\Gamma_E(R)$ *such that* $\mathrm{Ann}_R(x) \in \mathrm{Ass}(R)$ *dominates the edges of* $\Gamma_E(R)$. *That is, for every edge in* $\Gamma_E(R)$, *at least one end of that edge is in* $\overline{\mathcal{N}([x])}$. *Moreover, every vertex* $[y] \neq [x]$ *such that* y *is nilpotent is adjacent to* $[x]$.

Proof. This follows immediately from the fact that $yz = 0 \in \mathfrak{p}$ for all $\mathfrak{p} = \mathrm{Ann}(x) \in \mathrm{Ass}(R)$ forces y or z to be in $\mathrm{Ann}(x)$. □

Corollary 6.3. *If* $|\Gamma_E(R)| > 3$ *and the graph has at least one vertex* $[x]$ *with two or more ends, then*

(i) $\mathrm{Ass}(R) = \{\mathrm{Ann}_R(x)\}$;

(ii) *every vertex* $[y] \neq [x]$ *must be adjacent to* $[x]$*; in particular,*

  (a) $\deg([x]) = |\Gamma_E(R)| - 1$ *if* $|\Gamma_E(R)|$ *is finite, and*

  (b) *no vertex other than* $[x]$ *can have an end.*

Proof. Let $[x]$ be a vertex of $\Gamma_E(R)$ that has (at least) two ends $[y_1], [y_2]$. Since $\Gamma_E(R)$ has at least four vertices, it follows from [35, Proposition 3.2] that $\mathrm{Ann}_R(y_1)$ and $\mathrm{Ann}_R(y_2)$ are not prime. Proposition 6.1 (ii) implies that there is an associated prime $\mathfrak{p} = \mathrm{Ann}_R(z)$ such that $[z] \in \mathcal{N}([y_1])$; since $[y_1]$ is an end for $[x]$, we have $[x] = [z]$, so $\mathrm{Ann}_R(x) = \mathfrak{p} \in \mathrm{Ass}(R)$.

Since $[y_1]$ and $[y_2]$ are ends for $[x]$, we conclude that either $y_1^2 = 0$ or $y_2^2 = 0$; if not, we would have $\mathrm{Ann}_R(y_1) = \mathrm{Ann}_R(y_2)$, contradicting the assumption $[y_1] \neq [y_2]$. Assume by symmetry that $y_1^2 = 0$.

Let $\mathrm{Ann}(w) \in \mathrm{Ass}(R)$. Proposition 6.2 implies that $[y_1]$ is adjacent to $[w]$, so the fact that $[y_1]$ is an end for $[x]$ implies that $[x] = [w]$, and so $\mathrm{Ann}(x) = \mathrm{Ann}(w)$. That is, we have $\mathrm{Ass}(R) = \{\mathrm{Ann}(x)\}$. This explains (i), and Proposition 6.1 (ii) implies that every vertex $[v] \neq [x]$ must be adjacent to $[x]$. From this, part (ii) is immediate. □

Proposition 6.4 (Compare to [19, Theorem 1.12].). *If* $\Gamma_E(R)$ *is acyclic, then it is a star graph or a path of length* 3.

Proof. If $|\mathrm{Ass}(R)| \geq 3$ then the clique of associated primes contains a cycle; see Proposition 6.1 (ii). Thus, we have $|\mathrm{Ass}(R)| \leq 2$. Since the graph is acyclic and each vertex is adjacent to some associated prime, every vertex not in $\mathrm{Ass}(R)$ must be an end. In the case that one of the associated primes has two ends, then by Corollary 6.3, we have $|\mathrm{Ass}(R)| = 1$, and hence every vertex in the graph is adjacent to a single central vertex; i.e., the graph is a star. In the case that each associated prime has at most one end, $\Gamma_E(R)$ is a path. (Note that if $|\Gamma_E(R)| < 4$, the path is also a star.) □

Proposition 6.5 (Compare to [1, Theorems 2.6 and 2.8, Corollary 2.2].).

(i) *If* $|\mathrm{Ass}(R)| = 1$*, then either* $\mathrm{girth}(\Gamma_E(R)) = \infty$ *or* $\mathrm{girth}(\Gamma_E(R)) = 3$. *Moreover,* $\mathrm{diam}(\Gamma_E(R)) \leq 2$.

(ii) *If* $|\mathrm{Ass}(R)| \geq 3$*, then* $\mathrm{girth}(\Gamma_E(R)) = 3$.

Proof. Note that (ii) follows immediately from Proposition 6.1 (ii) since all elements in $\mathrm{Ass}(R)$ are adjacent. Thus, suppose $\mathfrak{p} = \mathrm{Ann}(x)$ is the unique associated prime of R. Every vertex $[v] \neq [x]$ is adjacent to $[x]$ by Proposition 6.1 (ii), hence the diameter is at most 2. Moreover, $\Gamma_E(R)$ is either a star graph, in which case $\mathrm{girth}(\Gamma_E(R)) = \infty$, or there is at least one additional edge, in which case $\mathrm{girth}(\Gamma_E(R)) = 3$. □

Theorem 6.6. *If the girth of* $\Gamma_E(R)$ *is finite, then it is* 3.

Proof. Applying Proposition 6.1 (iii), we assume that $\mathrm{girth}(\Gamma_E(R)) = 4$ and obtain a contradiction. We claim that we may assume $|\Gamma_E(R)| \geq 6$. Zero divisor graphs with exactly five vertices are the subject of [25], where it is shown that in the four realizable graphs the girth is either infinite or 3. Likewise, of the six connected graphs on exactly four vertices, only three can be realized as $\Gamma_E(R)$ by [35, Propositions 1.5 and 1.7], and the girth is either infinite or 3. Clearly no graph with less than four vertices can have girth 4. Thus, assume $|\Gamma_E(R)| \geq 6$.

By Proposition 6.5, there are exactly two associated primes, say $\mathfrak{p}_1 = \mathrm{Ann}_R(x_1)$ and $\mathfrak{p}_2 = \mathrm{Ann}_R(x_2)$. Proposition 6.1 (ii) implies that $[x_1]$ and $[x_2]$ are adjacent. Quite a bit of information can be deduced from Proposition 6.2 since we are assuming that $\Gamma_E(R)$ has no three cycles.

For instance, we claim that no vertex in $\Gamma_E(R)$, except possibly $[x_1]$ or $[x_2]$, is represented by a self-annihilating element. To see this, let $[y] \in \Gamma_E(R) \setminus \{[x_1], [x_2]\}$ and suppose that $y^2 = 0$. Then $y \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, i.e., $[y] \in \mathcal{N}([x_1]) \cap \mathcal{N}([x_2])$, resulting in the 3-cycle $[y] - [x_1] - [x_2] - [y]$.

Using similar reasoning, we show that no pair of distinct classes in $\mathcal{N}([x_i])$ can be adjacent. We argue for $i = 1$; the case $i = 2$ is by symmetry. Suppose that $[v]$ and $[w]$ are adjacent vertices in $\mathcal{N}([x_1])$. If $[v] \neq [x_2] \neq [w]$, then the condition $vw = 0$ implies that $v \in \mathfrak{p}_2$ or $w \in \mathfrak{p}_2$; hence, either $[v]$ or $[w]$ is in $\mathcal{N}([x_2])$, forming a 3-cycle with $[x_1]$ and $[x_2]$, a contradiction. Thus, either $[v]$ or $[w]$ is equal to $[x_2]$, say $[w] = [x_2]$. Then the condition $vw = 0$ implies that $v \in \mathrm{Ann}_R(w) = \mathrm{Ann}_R(x_2)$, so we have the 3-cycle $[v] - [x_1] - [x_2] - [v]$, another contradiction.

Next, we show that, given an edge $[a]-[b]$ such that $\{[a],[b]\} \cap \{[x_1],[x_2]\} = \emptyset$, we have a 4-cycle $[a]-[x_i]-[x_j]-[b]$. Since $[a] \notin \{[x_1],[x_2]\}$, Proposition 6.1 (ii) implies that $[a]$ is adjacent to $[x_i]$ for some i. Similarly $[b]$ is adjacent to either $[x_i]$ or $[x_j]$ where $j \neq i$. If $[b]$ is adjacent to $[x_i]$, then we would have the 3-cycle $[a] - [x_i] - [b]$, a contradiction. Thus $[b]$ is adjacent to $[x_j]$, and we have the 4-cycle $[a] - [x_i] - [x_j] - [b] - [a]$.

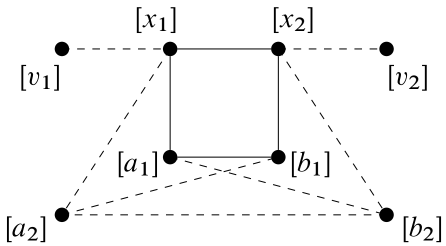
Next, we show that $\Gamma_E(R)$ has a 4-cycle containing $[x_1]$ and $[x_2]$. For this, it suffices to show that $\Gamma_E(R)$ contains an edge $[a] - [b]$ in $\Gamma_E(R)$ such that $\{[a],[b]\} \cap \{[x_1],[x_2]\} = \emptyset$, by the previous paragraph. Each $[x_i]$ has at most one end. Since $|\Gamma_E(R)| \geq 6$ this implies that there is a vertex $[a] \notin \{[x_1],[x_2]\}$ that is not an end. Proposition 6.1 (ii) implies that $[a]$ is adjacent to $[x_i]$ for some i. Since $[a]$ is not an end, there is a vertex $[b] \neq [x_i]$ adjacent to $[a]$. Note that $[b] \neq [x_j]$: if $[b] = [x_j]$, then we would have the 3-cycle $[a] - [x_i] - [x_j] = [b] - [a]$, a contradiction. Thus, the edge $[a] - [b]$ provides the desired 4-cycle.

Note also that Corollary 6.3 implies that each $[x_i]$ can have at most one end $[v_i]$, which (by the third paragraph of this proof) necessarily satisfies $v_i^2 \neq 0$.

Next, consider the relationship between $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and suppose $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$. If there is some $[y] \in \mathcal{N}([x_i]) \backslash \{[x_j]\}$, then there is the 3-cycle $[y] - [x_1] - [x_2] - [y]$. Thus, all the remaining vertices must be in $\mathcal{N}([x_j]) \backslash \mathcal{N}([x_i])$. But this is impossible, since if $[y]$ and $[z]$ are two such vertices, then they can not be adjacent to one another or $[x_i]$, and hence must both be ends of $[x_j]$, contradicting Corollary 6.3. Thus, both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are maximal elements in the family of annihilator ideals of R.

Next, we show that each vertex in $\mathcal{N}([x_i]) \backslash \{[x_j]\}$ that is not an end must be part of a 4-cycle with the edge $[x_1] - [x_2]$ and some vertex in $\mathcal{N}([x_j]) \backslash \{[x_i]\}$. Indeed, let $[v] \in \mathcal{N}([x_i]) \backslash \{[x_j]\}$ such that $[v]$ is not an end. Since $[v]$ is adjacent to $[x_i]$, and $[v]$ is not an end, there is another vertex $[w] \neq [x_i]$ adjacent to $[v]$. If $[w] = [x_j]$, then we have a 3-cycle $[v] - [w] = [x_j] - [x_i] - v$, a contradiction. So we have $[w] \neq [x_j]$. Thus there is an edge $[v] - [w]$ such that $\{[v], [w]\} \cap \{[x_1], [x_2]\} = \emptyset$, so the fifth paragraph of this proof provides a 4-cycle $[v] - [x_i] - [x_j] - [w] - [v]$.

The diagram below summarizes the paragraphs above and demonstrates what general form $\Gamma_E(R)$ must take: (1) $[x_1]$ and $[x_2]$ are adjacent and are part of a 4-cycle with every edge disjoint from the edge $[x_1] - [x_2]$; (2) each $[x_i]$ has at most one end; (3) any vertex in $\mathcal{N}([x_i]) \backslash \{[x_j]\}$ that is not an end must be part of a 4-cycle with $[x_1]$, $[x_2]$, and some vertex in $\mathcal{N}([x_j]) \backslash \{[x_i]\}$; and (4) there is at least one 4-cycle on the graph.



Let $\mathcal{A}$ be the set of all elements in $\mathcal{N}([x_1]) \backslash \{[x_2]\}$ which are not ends; i.e., each class $[a_i] \in \mathcal{A}$ is adjacent to some element(s) in $\mathcal{N}([x_2]) \backslash \{[x_1]\}$. Likewise, let $\mathcal{B}$ be the set of all elements $[b_j]$ in $\mathcal{N}([x_2]) \backslash \{[x_1]\}$ which are not ends.

Suppose there exist distinct classes $[a_1], [a_2] \in \mathcal{A}$. If $\mathcal{N}([a_1]) = \mathcal{N}([a_2])$, then a_1 and a_2 have the same annihilators, since $a_i^2 \neq 0$, so $[a_1] = [a_2]$, a contradiction. Assume by symmetry that we have $\mathcal{N}([a_2]) \not\subseteq \mathcal{N}([a_1])$, and let $[b_2] \in \mathcal{N}([a_2]) \backslash \mathcal{N}([a_1])$. Also, let $[b_1] \in \mathcal{N}([a_1])$. Thus, we have $[b_1], [b_2] \in \mathcal{B}$ such that $a_1 b_1 = 0 = a_2 b_2$ and $a_1 b_2 \neq 0$. Then $\mathcal{N}([a_1]) \cup \mathcal{N}([b_2]) \subseteq \overline{\mathcal{N}([a_1 b_2])} \subseteq \mathcal{N}([x_i])$ for some i, where the first containment is Proposition 6.1 (i), and the second follows from the fact that every annihilator ideal, in particular $\mathrm{Ann}_R(a_1 b_2)$, is contained in a maximal element of the family of annihilator ideals, hence either $\mathrm{Ann}_R(x_1)$ or $\mathrm{Ann}_R(x_2)$. Suppose $\mathrm{Ann}_R(a_1 b_2) \subseteq \mathrm{Ann}_R(x_1)$. Then $[b_1] \in \mathcal{N}([a_1]) \subseteq \mathcal{N}([x_1])$, translates into the 3-

cycle $[x_1]-[b_1]-[x_2]-[x_1]$, a contradiction. Likewise, if $\mathrm{Ann}_R(a_1b_2) \subseteq \mathrm{Ann}_R(x_2)$, then $[a_2] \in \mathcal{N}([b_2]) \subseteq \mathcal{N}([x_2])$ translates into the 3-cycle $[x_1]-[a_2]-[x_2]-[x_1]$. Thus, assuming that $|\mathcal{A}| \geq 2$, which forces $|\mathcal{B}| \geq 2$, leads to a contradiction. By symmetry, assuming that $|\mathcal{B}| \geq 2$ forces $|\mathcal{A}| \geq 2$ and leads to a contradiction. Therefore, we must have $|\mathcal{A}| = |\mathcal{B}| = 1$, say $\mathcal{A} = \{[a_1]\}$ and $\mathcal{B} = \{[b_1]\}$, where $a_1b_1 = 0$; i.e., there is exactly one 4-cycle in the graph. Next, recall that $|\Gamma_E(R)| \geq 6$. Based on the above arguments and assumptions, it follows that $|\Gamma_E(R)|$ must be exactly 6, and that $[x_1]$ and $[x_2]$ each have an end $[v_1]$, $[v_2]$, respectively.

Consider v_1v_2, which is a zero divisor annihilated by x_1 and x_2. If $[v_1v_2]$ is not $[x_1]$ or $[x_2]$, then the graph has the 3-cycle $[x_1]-[v_1v_2]-[x_2]-[x_1]$, a contradiction. Thus, we have $[v_1v_2] = [x_i]$ for some i. Without loss of generality, assume that $[v_1v_2] = [x_1]$. Since $x_1 \in \mathrm{Ann}_R(v_1v_2) = \mathrm{Ann}_R(x_1)$, it follows that $x_1^2 = 0$. Recall that $v_1^2, v_2^2 \neq 0$. In the graph we have $[v_1v_2] = [x_1]-[v_1]$, so $v_1^2v_2 = 0$ and $v_1^2v_2^2 = 0$. However, since $[x_1] = [v_1v_2]$ is not adjacent to $[v_2]$, we have $v_1v_2^2 \neq 0$. Since $[v_2]$ is an end for $[x_2]$ and $v_1^2v_2 = 0 \neq v_1^2$, we have $[v_1^2] = [x_2]$.

Now consider a_1+x_1, which is annihilated by x_1, but not by v_1, a_1, b_1, or x_2. Thus, $[a_1+x_1] = [v_1]$. Note that $(a_1+x_1)^2 = a_1^2$, hence $[a_1^2] = [v_1^2] = [x_2]$; this follows from the readily verified fact that $\mathrm{Ann}_R(a_1) = \mathrm{Ann}_R(v_1)$ implies that $\mathrm{Ann}_R(a_1^2) = \mathrm{Ann}_R(v_1^2)$. Thus, we have $a_1^2v_2 = 0$, so a_1v_2 is annihilated by a_1, x_1, x_2, and b_1. As there is no such class and a_1v_2 can not be zero, this is the final contradiction. □

Remark 6.7. To contrast the above results, note that Proposition 6.1 (iii) does not force every 5-cycle to have a chord. In fact, it does not preclude the existence of arbitrarily long cycles without chords. For example, for $n \geq 4$, the graph of the Noetherian ring $R = \mathbb{F}_2[X_1,\ldots,X_n]/(X_1X_2, X_2X_3,\ldots,X_{n-1}X_n, X_nX_1)$ has a C_n subgraph of $[X_1]-[X_2]-\cdots-[X_{n-1}]-[X_n]-[X_1]$ with no chord; i.e., circumference$(\Gamma_E(R)) \geq n$. On the other hand, girth$(\Gamma_E(R)) = 3$ by Theorem 6.6.

6.2 Cut Vertices

Cut vertices in Anderson and Livingston's graph $\Gamma(R)$ are investigated in [11].

Lemma 6.8. *If* $\mathrm{Ass}(R) = \{\mathrm{Ann}(v)\}$, *then* $v^2 = 0$.

Proof. The assumption $\mathrm{Ass}(R) = \{\mathrm{Ann}(v)\}$ implies that $\mathrm{Ann}(v) = Z^*(R) \cup \{0\}$, hence $v \in \mathrm{Ann}(v)$. □

In part (iii) of the next result, we employ the following terminology: Let A and B be disjoint sets of vertices of a graph G. We say that a vertex v of $G \setminus (A \cup B)$ *separates* A and B if for all $a \in A$ and all $b \in B$ every path from a to b in G passes through v.

Proposition 6.9. *Suppose* v *is a cut vertex in* $\Gamma_E(R)$. *Then* $\mathrm{Ann}(v)$ *is an associated prime of R. Moreover,*

(i) *if v has at least two ends, then* $\mathrm{Ass}(R) = \{\mathrm{Ann}(v)\}$*;*

(ii) *if v does not have an end, then* $\mathrm{Ass}(R) = \{\mathrm{Ann}(v)\}$*;*

(iii) *if v* separates *subsets* $\{[a], [a']\}$ *and* $\{[b], [b']\}$ *where* $[a], [a'], [b], [b']$ *are distinct vertices satisfying* $aa' = 0$ *and* $bb' = 0$*, then* $\mathrm{Ass}(R) = \{\mathrm{Ann}(v)\}$.

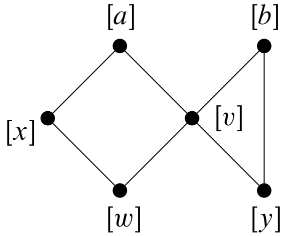
Proof. Let $[v]$ be a cut vertex in $\Gamma_E(R)$, in which case there are at least three vertices in the graph. If $[v]$ has an end, then $\mathrm{Ann}(v)$ is an associated prime by [35, Corollary 3.3], and (i) follows from Corollary 6.3. If $[v]$ does not have an end, then it is straightforward to show that the hypotheses of part (iii) are satisfied. Thus, it remains to prove part (iii).

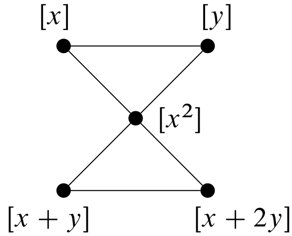
(iii) Assume that v separates subsets $\{[a], [a']\}$ and $\{[b], [b']\}$ where $[a], [a'], [b], [b']$ are distinct vertices satisfying $aa' = 0$ and $bb' = 0$. Suppose that there is an associated prime $\mathrm{Ann}(w) \neq \mathrm{Ann}(v)$. (Note that we do not yet know that $\mathrm{Ann}(v)$ is prime.) Since the elements of $\mathrm{Ass}(R)$ dominate the edges of the graph, at least one of $[a]$, $[a']$ is adjacent to $[w]$, and at least one of $[b]$, $[b']$ is adjacent to $[w]$. This provides a path from $[a]$ to $[b]$ via $[w]$ avoiding $[v]$, a contradiction. Thus, $\mathrm{Ann}(v)$ is the only ideal that might be an associated prime of R. Since R has an associated prime, the ideal $\mathrm{Ann}(v)$ is therefore the unique associated prime of R. □

Corollary 6.10. *If* $[v]$ *satisfies the hypotheses of Proposition* 6.9 (iii)*, and* $|\Gamma_E(R)| < \infty$*, then* $\deg([v]) > \deg([u])$ *for all* $[u] \in \Gamma_E(R)$.

Proof. If $\Gamma_E(R)$ is finite with $n + 1$ vertices, then Propositions 6.1 (ii) and 6.9 (iii) show that $\deg([v]) = n$. (Recall that no vertex is adjacent to itself.) Given any other $[u] \in \Gamma_E(R)$, the cut vertex $[v]$ must separate $[u]$ from some vertex $[w]$. So $[u]$ is not adjacent to all other vertices, and hence $\deg([u]) < n$. □

Example 6.11. The graph shown below on the left can not be the $\Gamma_E(R)$ for a ring R as per the Proposition; on the other hand, the graph on the right is $\Gamma_E(R)$ for $R = (\mathbb{Z}/3\mathbb{Z})[\![X, Y]\!]/(XY, X^3, Y^3, X^2 - Y^2)$, where lower case letters represent the cosets of the upper case letters in the quotient ring; see [35, Example 3.9].





The converse to Proposition 6.9 is not true; i.e., an associated prime need not be a cut vertex, even when R is finite, as the next example shows.

Example 6.12. In the ring $R = \mathbb{F}_2[X, Y, Z]/(X^2, Y^2, Z^2)$, the ideal $\mathrm{Ann}_R(xyz) = (x, y, z)R$ is the unique associated prime ideal, but not a cut vertex. To see this, note first that the only elements $v \in R$ such that $\mathrm{Ann}_R(v) = \mathrm{Ann}_R(xyz)$ are the non-zero scalar multiples of xyz. Write R as $A[Z]/(Z^2) \cong A \ltimes A$ where $A = \mathbb{F}_2[X,Y]/(X^2,Y^2)$. Let $r = f_0(x,y) + f_1(x,y)\cdot z$ and $s = g_0(x,y) + g_1(x,y)\cdot z$ be two non-equivalent zero divisors, where the constant coefficients of f_0, g_0 are necessarily zero. Assume that $[r] \neq [xyz] \neq [s]$. We show that $[r]$ and $[s]$ are connected by a path not containing $[xyz]$.

If both f_0 and g_0 have a linear term, then $[r]$ and $[s]$ are connected by the path $[r] - [f_0 z] - [g_0 z] - [s]$. Note that the assumptions on f_0 and g_0 imply that $[f_0 z] \neq [xyz] \neq [g_0 z]$. Also, the vertices $[f_0 z]$ and $[g_0 z]$ may be distinct or not, so this path has length 2 or 3.

If f_0 does not have a linear term (e.g., if $f_0 = 0$) and g_0 does have a linear term, then we use the path $[r] - [xz] - [g_0 z] - [s]$. If f_0 does have a linear term and g_0 does not have a linear term (e.g., if $g_0 = 0$), then we use the path $[r] - [f_0 z] - [xz] - [s]$. If f_0 and g_0 do not have linear terms, then we use the path $[r] - [xz] - [s]$.

7 Chromatic Numbers and Clique Numbers

Assume throughout this section that R is a Noetherian ring.

As mentioned in the survey, the origin of research in the theory of zero divisor graphs involved their chromatic numbers. It is important to note that D. D. Anderson and M. Naseer's [4] counterexample to Beck's conjecture that the chromatic and clique numbers of $G(R)$ are equal is not reduced and has clique number 5. When $\chi(G(R)) < 5$ or R is reduced and $\chi(G(R)) < \infty$, then Beck's conjecture is valid [12, Theorem 3.8 and §7]. In this section, we study smaller chromatic numbers for $\Gamma_E(R)$ as well as establish a version of Beck's conjecture when R is a reduced Noetherian ring, but ultimately prove that it does not hold in general, e.g., for the set of non-reduced rings or rings with clique number as small as 3.

We begin with a simple upper bound on the chromatic number:

Lemma 7.1. *Let Δ be the maximum degree of a vertex in $\Gamma_E(R)$. Then we have $\chi(\Gamma_E(R)) \leq \Delta$, except for the case $\Delta = 1$ where $\chi(\Gamma_E(R)) = 1$ or 2.*

Proof. By Brook's Theorem [22, Theorem 5.2.4], $\chi(\Gamma_E(R)) \leq \Delta$, unless the graph is complete or an odd cycle. However, when $\Gamma_E(R)$ has at least three vertices, it is never complete or a cycle, as per [35, Propositions 1.5 and 1.8]. The only exception is $|\Gamma_E(R)| = 2$, in which case $\chi(\Gamma_E(R)) = 2$, but $\Delta = 1$. □

Proposition 7.2. *If R is a reduced ring, then $\chi(\Gamma_E(R)) = \omega(\Gamma_E(R)) = |\mathrm{Ass}(R)|$.*

Proof. Recall Proposition 5.5, by which $|\mathrm{Ass}(R)| = \omega(\Gamma_E(R)) \leq \chi(\Gamma_E(R))$. Let $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. The fact that R is reduced implies that for each $x \in Z^*(R)$ there are indices i and j such that $x \in \mathfrak{p}_i$ and $x \notin \mathfrak{p}_j$. Define a coloring by $f([x]) = \min\{i : x \notin \mathfrak{p}_i\}$. If $f([x]) = k+1$, then $x \in \mathfrak{p}_i$ for $1 \leq i \leq k$, but $x \notin \mathfrak{p}_{k+1}$.

If $[x]$ and $[y]$ are adjacent, then $y \in \mathfrak{p}_{k+1}$, by Proposition 6.2. Thus, $[x]$ and $[y]$ are assigned different colors. Hence, we have $\chi(\Gamma_E(R)) \leq |\mathrm{Min}(R)| = |\mathrm{Ass}(R)|$. $\square$

Following Beck's lead, we establish some results for rings with small chromatic numbers for $\Gamma_E(R)$. We note that D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston [7, Section 3] have considered similar ideas for the Anderson and Livingston graph $\Gamma(R)$.

7.1 Chromatic/Clique Number 1

Since $\Gamma_E(R)$ is connected, we have $\chi(\Gamma_E(R)) = 1$ if and only if $\omega(\Gamma_E(R)) = 1$ if and only if $\Gamma_E(R)$ consists of a single vertex. Hence, when these conditions are satisfied, we have $xy = 0$ for every x, y in $Z^*(R)$. Thus, Anderson and Livingston's graph $\Gamma(R)$ is complete; see [10, Theorem 2.8]. We have the following characterization.

Proposition 7.3. *We have* $\chi(\Gamma_E(R)) = 1$ *if and only if* $\mathrm{Ass}(R) = \{\mathfrak{p}\}$ *such that* $\mathfrak{p}^2 = 0$. *When these conditions are satisfied, we have* $Q(R) = R_{\mathfrak{p}} \cong R_0 \ltimes V$, *where* $(R_0, \mathfrak{m}_0)$ *is a local ring such that* $\mathfrak{m}_0 = pR_0$ *where* $p = \mathrm{char}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ *satisfies* $\mathfrak{m}_0^2 = 0$ *and* V *is a finite-dimensional vector space over* $R_0/\mathfrak{m}_0$.

Proof. Assume first that $\chi(\Gamma_E(R)) = 1$. Then

$$1 \leq |\mathrm{Min}(R)| \leq |\mathrm{Ass}(R)| \leq \chi(\Gamma_E(R)) = 1$$

so $\mathrm{Ass}(R) = \{\mathfrak{p}\}$ for some prime $\mathfrak{p}$. It follows that $Z^*(R) = \mathfrak{p} - \{0\}$. As we noted above, we have $xy = 0$ for every x, y in $Z^*(R) = \mathfrak{p} - \{0\}$, so $\mathfrak{p}^2 = 0$.

Conversely, assume that $\mathrm{Ass}(R) = \{\mathfrak{p}\}$ such that $\mathfrak{p}^2 = 0$. It follows that $xy = 0$ for every x, y in $Z^*(R) = \mathfrak{p} - \{0\}$, so $\Gamma_E(R)$ is a single vertex, hence $\chi(\Gamma_E(R)) = 1$.

Continue to assume that $\mathrm{Ass}(R) = \{\mathfrak{p}\}$ such that $\mathfrak{p}^2 = 0$. It follows that $Z^*(R) = \mathfrak{p} - \{0\}$, so $Q(R) = R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ such that $(\mathfrak{p}R_{\mathfrak{p}})^2 = 0$. The fact that $R_{\mathfrak{p}}$ has the desired form is probably well known; however, we do not know of an appropriate reference, so we include a proof here.

Replace R by $R_{\mathfrak{p}}$ to assume that R is a local ring with unique maximal ideal $\mathfrak{p}$ such that $\mathfrak{p}^2 = 0$. In particular R is a complete local ring. Set $k = R/\mathfrak{p}$.

If R contains a field, then Cohen's structure theorem provides a monomorphism $k \to R$ such that the composition $k \to R \to R/\mathfrak{p} = k$ is an isomorphism. It follows that R is the internal direct sum $R = k \oplus \mathfrak{p}$ as a k-vector space. From this, it is straightforward to show that $R \cong k \ltimes \mathfrak{p}$. Since $\mathfrak{p}$ is finitely generated such that $\mathfrak{p}^2 = 0$, we conclude that $\mathfrak{p}$ is a finite dimensional vector space over k.

Assume that R does not contain a field, and set $p = \mathrm{char}(k)$. In this case, Cohen's structure theorem provides a complete discrete valuation ring (A, pA, k) and a ring homomorphism $f : A \to R$ such that the induced map $k = A/pA \to R/\mathfrak{p} = k$ is an isomorphism. Since R does not contain a field and $\mathfrak{p}^2 = 0$, we conclude that $\mathrm{Ker}(f) = p^2A$. In R we have $p \neq 0$ since R does not contain a field, and $p^2 = 0$

since $p^2 \in \mathfrak{p}^2 = 0$. In particular, we have $p \in \mathfrak{p} - \mathfrak{p}^2$, so $\mathfrak{p}$ has a minimal generating sequence of the form $p, x_1, \ldots, x_n$. The map $F\colon A[\![X_1, \ldots, X_n]\!] \to R$ given by $X_i \mapsto x_i$ is a well-defined ring epimorphism. Since we have chosen a minimal generating sequence for $\mathfrak{p}$, the fact that $\mathfrak{p}^2 = 0$ implies that $\mathrm{Ker}(F) = (p, X_1, \ldots, X_n)^2$. Thus, we have

$$R \cong A[\![X_1, \ldots, X_n]\!]/(p, X_1, \ldots, X_n)^2 \cong (A/p^2 A)[\![X_1, \ldots, X_n]\!]/(pX_i, X_i X_j).$$

From this description, it follows readily that $R \cong (A/p^2 A) \ltimes V$ where V is the finite dimensional vector space $V = (x_1, \ldots, x_n)R$ over $A/pA = k$. □

Corollary 7.4. *If R is a finite ring such that $\chi(\Gamma_E(R)) = 1$, then R is local with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$ and $\mathrm{char}(R) = p$ or p^2 where $p = \mathrm{char}(R/\mathfrak{m})$. Moreover, $R \cong R_0 \ltimes V$, where $(R_0, \mathfrak{m}_0)$ is either isomorphic to the finite field $R/\mathfrak{m}$ or a ring of order $|R/\mathfrak{m}|^2$ such that $\mathfrak{m}_0 = pR_0$ satisfies $\mathfrak{m}_0^2 = 0$, and V is a finite-dimensional vector space over $R_0/\mathfrak{m}_0$.*

7.2 Chromatic/Clique Number 2

The non-trivial star graphs in Section 3 have chromatic number 2, as does a path of length 3. Our next result says that these are the only ways to get chromatic number 2.

Proposition 7.5. *The following are equivalent:*

(i) $\omega(\Gamma_E(R)) = 2$;

(ii) $\chi(\Gamma_E(R)) = 2$;

(iii) $\Gamma_E(R)$ *is acyclic, with at least two vertices; and*

(iv) $\Gamma_E(R)$ *is a non-degenerate star or a path of length* 3.

Proof. If $\Gamma_E(R)$ is a single vertex, then $\omega(\Gamma_E(R)) = \chi(\Gamma_E(R)) = 1$, and all the conditions (i)–(iv) are false. Assume $\Gamma_E(R)$ has at least two vertices. Since it is connected, we have $\chi(\Gamma_E(R)) \geq \omega(\Gamma_E(R)) \geq 2$, hence (ii) $\Rightarrow$ (i). We have $\omega(\Gamma_E(R)) = 2$ if and only if $\Gamma_E(R)$ contains no cycle of length 3, since such a cycle is also a clique of size 3. By Theorem 6.6, $\Gamma_E(R)$ contains no cycle of length 3 if and only if it is acyclic. So (i) $\Leftrightarrow$ (iii). By [22, Proposition 1.6.1], we have $\chi(\Gamma_E(R)) = 2$ if and only if $\Gamma_E(R)$ contains no odd cycle. Thus, we have (iii) $\Rightarrow$ (ii). The equivalence (iv) $\Leftrightarrow$ (iii) is from Proposition 6.4. □

7.3 Chromatic/Clique Number 3

Proposition 7.6. *If $\chi(\Gamma_E(R)) = 3$, then $\omega(\Gamma_E(R)) = 3$.*

Proof. If $\chi(\Gamma_E(R)) = 3$, then $\omega(\Gamma_E(R)) \leq 3$ and $\Gamma_E(R)$ has an odd cycle by [22, Proposition 1.6.1]. Consequently, $\omega(\Gamma_E(R)) = 3$ by Theorem 6.6. □

The next example provides a finite local ring R such that $\omega(\Gamma_E(R)) = 3$ and $\chi(\Gamma_E(R)) = 4$. In particular, the converse of the previous result is false, as our example provides a negative answer to the question, motivated by Beck's original work, of whether or not $\omega(\Gamma_E(R)) = \chi(\Gamma_E(R))$.

Example 7.7. Let F be a field. Consider a sequence $\mathbf{X} = X_1, \ldots, X_5$ of indeterminates, and set

$$R = \frac{F[\mathbf{X}]}{(X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_1X_5) + (\mathbf{X})^3}.$$

Note that R is local and Artinian with maximal ideal $\mathfrak{m} = (\mathbf{X})R$ such that $\mathfrak{m}^3 = 0$.

To simplify computations, we perform arithmetic on subscripts modulo 5. For instance, we occasionally write X_{i+2} in place of X_{i-3} when $i \geq 4$. This allows us to consider expressions like X_iX_{i+2} for $i = 1, \ldots, 5$ without worrying about separate cases for $i \leq 3$ and $i > 3$. For instance, this allows us to write $X_{i-1}X_i = 0 = X_iX_{i+1}$ in R for $i = 1, \ldots, 5$.

For $i = 1, \ldots, 5$ let $e_i \in \mathbb{Z}^5$ be the ith standard basis vector. The ring R is $\mathbb{Z}^5$-graded with $\deg(X_i) = e_i$ because the ideal defining R is a monomial ideal. It follows readily that

$$\operatorname{Soc}(R) = 0 \oplus 0 \oplus R_2 \tag{7.7.1}$$

where we use the naturally induced $\mathbb{Z}$-grading, and

$$\operatorname{Ann}_R(X_i) = 0 \oplus \operatorname{Span}_F(X_{i-1}, X_{i+1}) \oplus R_2. \tag{7.7.2}$$

It follows that

$$\operatorname{Ann}_R(X_1^2) = \operatorname{Soc}(R) = \operatorname{Ann}_R(q(\mathbf{X})) \tag{7.7.3}$$

for all nonzero quadratic forms $q(\mathbf{X}) \in R_2$.

Next, we claim that for $i = 1, \ldots, 5$ and for all non-zero elements $a, b \in F$ we have

$$\operatorname{Ann}_R(aX_i + bX_{i+2}) = 0 \oplus \operatorname{Span}_F(X_{i+1}) \oplus R_2. \tag{7.7.4}$$

The containment $\supseteq$ follows from the fact that $X_iX_{i+1} = 0 = X_{i+1}X_{i+2}$. For the reverse containment, let $l \in \operatorname{Ann}_R(aX_i + bX_{i+2})$. Since R is graded with $\operatorname{Soc}(R) = R_2$, we assume without loss of generality that $l \in \operatorname{Ann}_R(aX_i + bX_{i+2})_1$. There are (unique) elements $c_1, \ldots, c_5 \in F$ such that $l = \sum_{j=1}^{5} c_jX_j$. The condition

$l \in \operatorname{Ann}_R(aX_i + bX_{i+2})$ implies that

$$0 = l(aX_i + bX_{i+2}) = \Big(\sum_{j=1}^{5} c_j X_j\Big)(aX_i + bX_{i+2}).$$

The coefficient of X_i^2 in the right-hand expression is $c_i a$; the $\mathbb{Z}^5$ grading implies that $c_i a = 0$, so $c_i = 0$ since a is a unit in the field F. (This uses the fact that X_i^2 is not in the ideal defining R.) Similarly, we have $c_{i+2} = 0$. The coefficient for $X_i X_{i+3}$ is $c_{i+3}a$, so the same reasoning implies that $c_{i+3} = 0$. Similarly, the $X_{i+2}X_{i+4}$ coefficient implies that $c_{i+4} = 0$. It follows that $l = c_{i+1}X_{i+1} \in \operatorname{Span}_F(X_{i+1})$, establishing the claim.

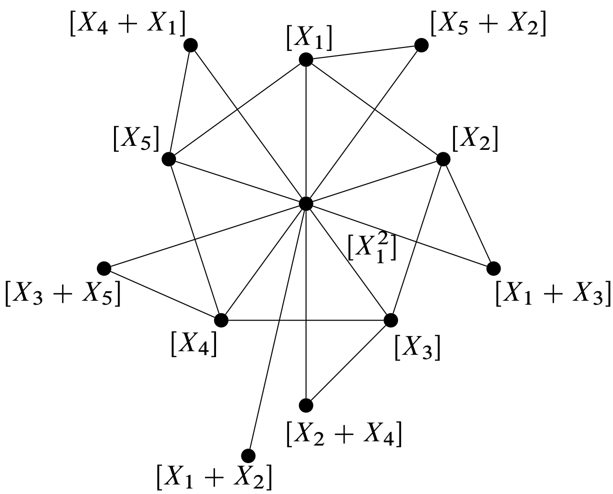
Note that the claim implies that

$$\operatorname{Ann}_R(aX_i + bX_{i+2}) = \operatorname{Ann}_R(X_i + X_{i+2}) \tag{7.7.5}$$

for $i = 1, \ldots, 5$ and for all non-zero elements $a, b \in F$. The same reason shows that all other linear forms have trivial annihilator; in other words:

$$l \in R_1 \setminus \bigcup_{i=1}^{5} (\operatorname{Span}_F(X_i) \cup \operatorname{Span}(X_i + X_{i+2})) \\ \Rightarrow \operatorname{Ann}_R(l) = R_2 = \operatorname{Ann}_R(X_1 + X_2).$$

Combining this with (7.7.2)–(7.7.5), we find that $\Gamma_E(R)$ is takes the form of a "pin-wheel" and one edge, as shown below.



From the graph, it is easy to see that $\omega(\Gamma_E(R)) = 3$. However, $\chi(\Gamma_E(R)) = 4$ since the 5-cycle on $[X_i], 1 \le i \le 5$ requires 3 colors [22, Theorem 1.6.1] and the vertex $[X_1^2]$ is adjacent to every $[X_i]$ and hence requires a fourth color.

A Tables for Example 3.14

The following table includes the values of $c^{\flat\flat}$ (with $c^{\flat\flat} < 100$) for star graphs we can construct using the method of Example 3.14.

d_1	e_1	t_1	d_2	e_2	t_2	$c^{\flat\flat}$
1	1	1	2	1	1	8
1	1	1	2	2	1	14
1	1	1	2	2	2	12
1	1	1	2	3	1	26
1	1	1	2	3	2	24
1	1	1	2	4	1	50
1	1	1	2	4	2	48
1	1	1	2	5	1	98
1	1	1	2	5	2	96
1	2	1	3	1	1	14
1	2	1	3	2	1	28
1	2	1	3	2	2	26
1	2	1	3	3	1	56
1	2	1	3	3	2	54
1	2	1	3	3	3	50
1	3	1	4	1	1	26
1	3	1	4	2	1	56
1	3	1	4	2	2	54
1	4	1	5	1	1	50
1	5	1	6	1	1	98
2	2	1	4	1	1	28
2	2	1	4	2	1	58
2	2	1	4	2	2	56
2	2	2	4	1	1	26
2	2	2	4	2	1	56
2	2	2	4	2	2	54
2	3	1	5	1	1	56
2	3	2	5	1	1	54

The next three iterations of the process are given below.

d_1	e_1	t_1	d_2	e_2	t_2	d_3	e_3	t_3	$c^{\flat\flat\flat}$
1	1	1	2	1	1	3	1	1	16
1	1	1	2	1	1	3	2	1	30
1	1	1	2	1	1	3	2	2	28
1	1	1	2	1	1	3	3	1	58
1	1	1	2	1	1	3	3	2	56
1	1	1	2	1	1	3	3	3	52
1	1	1	2	2	1	4	1	1	30
1	1	1	2	2	1	4	2	1	60
1	1	1	2	2	1	4	2	2	58
1	1	1	2	2	2	4	1	1	28
1	1	1	2	2	2	4	2	1	58
1	1	1	2	2	2	4	2	2	56
1	1	1	2	3	1	5	1	1	58
1	1	1	2	3	2	5	1	1	56
1	2	1	3	1	1	4	1	1	30
1	2	1	3	1	1	4	2	1	60
1	2	1	3	1	1	4	2	2	58
1	2	1	3	2	1	5	1	1	60
1	2	1	3	2	2	5	1	1	58
1	3	1	4	1	1	5	1	1	58
2	2	1	4	1	1	5	1	1	60
2	2	2	4	1	1	5	1	1	58

d_1	e_1	t_1	d_2	e_2	t_2	d_3	e_3	t_3	d_4	e_4	t_4	$c^{\flat\flat\flat\flat}$
1	1	1	2	1	1	3	1	1	4	1	1	32
1	1	1	2	1	1	3	1	1	4	2	1	62
1	1	1	2	1	1	3	1	1	4	2	2	60
1	1	1	2	1	1	3	2	1	5	1	1	62
1	1	1	2	1	1	3	2	2	5	1	1	60
1	1	1	2	2	1	4	1	1	5	1	1	62
1	1	1	2	2	2	4	1	1	5	1	1	60
1	2	1	3	1	1	4	1	1	5	1	1	62

d_1	e_1	t_1	d_2	e_2	t_2	d_3	e_3	t_3	d_4	e_4	t_4	d_5	e_5	t_5	$c^{\flat\flat\flat\flat\flat}$
1	1	1	2	1	1	3	1	1	4	1	1	5	1	1	64

B Graph Theory

Listed below are all the relevant definitions from Graph Theory. A good reference on the subject is [22] and for the material on zero divisor graphs, the papers [10], [2], and [1] provide a good background.

(i) A graph is **acyclic** if it contains no cycles.

(ii) A graph is **bipartite** (respectively, *r-partite*) if the vertices can be partitioned into two (resp., r) disjoint subsets so that every edge has one vertex in each subset (resp., every edge joins vertices in distinct subsets).

(iii) A graph is **chordal** if every cycle with four or more vertices has a *chord*, or edge joining two vertices of the cycle that are not adjacent.

(iv) The **circumference** of a graph is the maximum length of a cycle in the graph. If the graph is acyclic, then the circumference is zero.

(v) A **clique** in a graph is a subset of vertices of the graph that are all pairwise adjacent; i.e. a vertex set which induces a complete subgraph.

(vi) If a graph G contains a clique of size n and no clique has more than n elements, then the **clique number** of the graph is said to be n; if the clique size is unbounded, then the clique number is infinite. It is denoted by $\omega(G)$.

(vii) The **closure of a neighborhood** of a vertex v in a graph is the neighborhood of v along with v itself; i.e., $\mathcal{N}(v) \cup \{v\}$. It is denoted by $\overline{\mathcal{N}(v)}$.

(viii) The **chromatic number** or **coloring number** of a graph G, denoted $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G such that no pair of adjacent vertices has the same color.

(ix) A graph is **compact** if it is a simple connected graph satisfying the property that for every pair of non-adjacent vertices x and y, there is vertex z adjacent to every vertex adjacent to x and/or y.

(x) A graph is said to be **complete** if every vertex in the graph is adjacent to every other vertex in the graph. The notation for a complete graph on n vertices is K_n.

(xi) A **complete bipartite** is a bipartite graph such that every vertex in one partitioning subset is adjacent to every vertex in the other partitioning subset. If the subsets have cardinality m, and n, then this graph is denoted by $K_{m,n}$.

(xii) A **complete r-partite** graph is an r-partite graph such that every vertex in any partitioning subset is adjacent to every vertex in every other partitioning subset.

(xiii) A graph is said to be **connected** if there is a path between every pair of vertices of the graph.

(xiv) A **cut vertex** in a connected graph G is a vertex whose removal from the vertex set of G results in a disconnected graph; v is said to **separate** vertices a and b if every path between the two includes v.

(xv) A **cycle** in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex.

(xvi) A **cycle graph** is an n-gon for some integer $n \geq 3$.

(xvii) The **degree** of a vertex is the number of vertices adjacent to it.

(xviii) The **diameter** of a connected graph is the supremum of the distances between any two vertices.

(xix) A **directed graph** is a pair (V, E) of disjoint sets (of vertices and edges) together with two maps init: $E \to V$ and ter: $E \to V$ assigning to every edge e an initial vertex init(e) and a terminal vertex ter(e). The edge e is said to be directed from init(e) to ter(e).

(xx) The **distance** between two vertices v and w in a connected graph is the length of the shortest path between them; if no path exists between a pair of vertices, then the distance is defined to be infinite.

(xxi) A vertex is an **end** if it has degree 1.

(xxii) The **girth** of a graph is the length of the shortest cycle in the graph; it is infinite if the graph is acyclic.

(xxiii) A **graph** consists of a set of vertices, a set of edges, and an incident relation, describing which vertices are adjacent (i.e., joined by an edge) to which.

(xxiv) Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. A **homomorphism** $G \to G'$ is a function $\phi\colon V \to V'$ respecting adjacency, that is, such that for all $x, y \in V$ if $xy \in E$, then $\phi(x)\phi(y) \in E'$.

(xxv) An **induced subgraph** of a graph G is obtained by taking a subset U of the vertex set of G together with all edges which are incident in G only with vertices belonging to U.

(xxvi) Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. An **isomorphism** $G \xrightarrow{\cong} G'$ is a bijection $\phi\colon V \to V'$ with $xy \in E$ if and only if $\phi(x)\phi(y) \in E'$ for all $x, y \in V$.

(xxvii) The **neighborhood** of a vertex v in a graph is the set of all vertices adjacent to v. It is denoted by $\mathcal{N}(v)$. [Note that for simple graphs, $v \notin \mathcal{N}(v)$.]

(xxviii) A **non-degenerate star graph** is a star graph with at least two vertices.

(xxix) A **path of length** n between two vertices v and w is a finite sequence of vertices $u_0, u_1, \ldots, u_n$ such that $v = u_0$, $w = u_n$, and u_{i-1} and u_i are adjacent for all $1 \leq i \leq n$.

(xxx) A graph is **perfect** if for every induced subgraph, including the graph itself, the chromatic number and clique numbers agree.

(xxxi) A graph is **planar** if it can be drawn in the plane with no crossings of edges.

(xxxii) A **regular** graph is one in which all the vertices have the same degree.

(xxxiii) A vertex v in a graph G is said to **separate** vertices a and b if every path between a and b includes v.

(xxxiv) A **simple graph** is one with no loops on a vertex and no multiple edges between a pair of vertices.

(xxxv) A **star graph** is a complete bipartite graph in which one of the partitioning subsets is a singleton set. The notation for this graph is $K_{1,n}$.

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