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REFLEXIVITY AND RING HOMOMORPHISMS OF FINITE FLAT DIMENSION

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In this article we present a systematic study of the reflexivity properties of homologically finite complexes with respect to semidualizing complexes in the setting of nonlocal rings. One primary focus is the descent of these properties over ring homomorphisms of finite flat dimension, presented in terms of inequalities between generalized G-dimensions. Most of these results are new even when the ring homomorphism is local. The main tool for these analyses is a nonlocal version of the amplitude inequality of Iversen, Foxby, and Iyengar. We provide numerous examples demonstrating the need for certain hypotheses and the strictness of many inequalities.

Key Words: G-dimensions; Gorenstein dimensions; Ring homomorphisms; Semidualizing complexes; Semidualizing modules.

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INTRODUCTION

Grothendieck (1965) and Hartshorne (1966, 1967) introduced the notion of a dualizing complex as a tool for understanding cohomology theories in algebraic geometry and commutative algebra. The homological properties of these objects and the good behavior of rings admitting them are well documented and of continuing interest and application in these fields.

Semidualizing complexes arise in several contexts in commutative algebra as natural generalizations of dualizing complexes; see 1.2. A dualizing complex for R is semidualizing, as is a free R-module of rank 1. Such objects were introduced and studied in the abstract by Foxby (1972/1973) and Golod (1984) in the case where C is a module. The investigation of the general situation begins with the work

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Dedicated to the memory of Saunders Mac Lane.
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of Christensen (2001) and continues with, e.g., Araya et al. (2005), Frankild and Sather-Wagstaff (Preprint), Gerko (2001, 2004), and Sather-Wagstaff (Preprint).

The utility of these complexes was first demonstrated in the work of Avramov and Foxby (1997) where the dualizing complex D^{φ} of a local ring homomorphism $\varphi\colon R \to S$ of finite flat dimension (or more generally of finite G-dimension) is used as one way to relate the Bass series of R to that of S; see 1.8. When φ is module-finite, its dualizing complex is $\mathbf{R}\mathrm{Hom}_R(S, R)$, which is semidualizing for S. (For the general case, see Avramov and Foxby, 1997.) This provides another generalization of dualizing complexes: If R is Gorenstein, then D^{φ} is dualizing for S. It is believed that D^{φ} will give insight into the so-called *composition question* for homomorphisms of finite G-dimension.

A semidualizing complex C gives rise to the category of C-reflexive complexes equivalently, the category of complexes of finite G_C-dimension; see 1.4. When C is dualizing, every homologically finite complex X is C-reflexive (Hartshorne, 1966). On the other hand, a complex is R-reflexive exactly when it has finite G-dimension as defined by Auslander (1967) and Auslander and Bridger (1969) for modules, and Yassemi (1995) for complexes. This notion was introduced and studied in general by Foxby (1972/1973) and Golod (1984) when C and X are modules, and by Christensen (2001) in this generality.

The current article is part of our ongoing effort to increase the understanding of the semidualizing complexes and their corresponding reflexive complexes. More of our work in this direction is found in Frankild and Sather-Wagstaff (Preprint) and Sather-Wagstaff (Preprint) where we forward two new perspectives for this study. In Frankild and Sather-Wagstaff (Preprint) we endow the set of shift-isomorphism classes of semidualizing R-complexes with a nontrivial metric. Sather-Wagstaff (Preprint) investigates the consequences of the observation that, when R is a normal domain, the set of isomorphism classes of semidualizing R-modules is naturally a subset of the divisor class group of R. Each of these works relies heavily on the homological tools developed in the current article, which fall into roughly three categories.

First, we extend a number of results in Christensen (2001) from the setting of local rings and local ring homomorphisms to the nonlocal realm. This process is begun in Section 2 with an investigation of the behavior of these objects under localization, and it is continued in Section 3 where global statements are proved over a single ring.

The second advancement in this article is found in the descent results which populate Sections 4–6. Based in part on the ideas of Iyengar and Sather-Wagstaff (2004), we exploit the amplitude inequality of Iversen (1977) and Foxby and Iyengar (2003) in order to prove converses of a number of results from Christensen (2001). These results deal with the interactions between, on the one hand, semidualizing and reflexive complexes, and on the other hand, complexes and ring homomorphisms of finite flat dimension. Most of the results from Christensen (2001) that we focus on are stated there in the local setting, and the converses are new even there. However, our work in the earlier sections along with a nonlocal version of the amplitude inequality extend these converses and the original results to the global arena. Our version of the amplitude inequality is Theorem 4.2, wherein $\inf(X)$ and $\sup(X)$ are the infimum and supremum, respectively of the set $\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$ and $\mathrm{amp}(X) = \sup(X) - \inf(X)$.

Theorem I. *Let $\varphi\colon R \to S$ be a ring homomorphism and P a homologically finite S-complex with* $\mathrm{fd}_R(P)$ *finite and such that* $\varphi^*(\mathrm{Supp}_S(P))$ *contains* m-Spec(R). *For each homologically degree-wise finite R-complex X there are inequalities*

$$\inf\big(X \otimes^{\mathbf{L}}_R P\big) \le \inf(X) + \sup(P)$$
$$\sup\big(X \otimes^{\mathbf{L}}_R P\big) \ge \sup(X) + \inf(P)$$
$$\mathrm{amp}\big(X \otimes^{\mathbf{L}}_R P\big) \ge \mathrm{amp}(X) - \mathrm{amp}(P).$$

In particular,

(a) $X \simeq 0$ *if and only if* $X \otimes^{\mathbf{L}}_R P \simeq 0$;
(b) *X is homologically bounded if and only if $X \otimes^{\mathbf{L}}_R P$ is so*;
(c) *If* $\mathrm{amp}(P) = 0$, *e.g., if $P = S$, then* $\inf\big(X \otimes^{\mathbf{L}}_R P\big) = \inf(X) + \inf(P)$.

Section 4 deals for the most part with the behavior of the semidualizing and reflexive properties with respect to the derived functor $- \otimes^{\mathbf{L}}_R S$ where $\varphi\colon R \to S$ is a ring homomorphism of finite flat dimension, that is, with $\mathrm{fd}_R(S) < \infty$. As a sample, here is a summary of Theorems 4.5, 4.8, and 4.9.

Theorem II. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension and C, C', X homologically degree-wise finite R-complexes. Assume that every maximal ideal of R is contracted from S.*

(a) *The complex $C \otimes^{\mathbf{L}}_R S$ is S-semidualizing if and only if C is R-semidualizing.*
(b) *When C is semidualizing for R, there is an equality*

$$\mathrm{G}_C\text{-}\mathrm{dim}_R(X) = \mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\mathrm{dim}_S\big(X \otimes^{\mathbf{L}}_R S\big).$$

In particular, $X \otimes^{\mathbf{L}}_R S$ is $C \otimes^{\mathbf{L}}_R S$-reflexive if and only if X is C-reflexive.
(c) *If the induced map on Picard groups* $\mathrm{Pic}(R) \to \mathrm{Pic}(S)$ *is injective and C, C' are semidualizing R-complexes, then $C \otimes^{\mathbf{L}}_R S$ is isomorphic to $C' \otimes^{\mathbf{L}}_R S$ in* $\mathsf{D}(S)$ *if and only if C is isomorphic to C' in* $\mathsf{D}(R)$.

Section 5 is similarly devoted to the functor $\mathbf{R}\mathrm{Hom}_R(S, -)$ when $\varphi\colon R \to S$ is module-finite. The version of Theorem II for this context is contained in Theorems 5.5, 5.8, and 5.9. We highlight here the characterization of reflexivity of $\mathbf{R}\mathrm{Hom}_R(S, X)$ with respect to $C \otimes^{\mathbf{L}}_R S$ which is in Theorem 5.13.

Theorem III. *Let C, X be homologically finite R-complexes with C semidualizing. If φ is module-finite with* $\mathrm{fd}(\varphi) < \infty$ *and* m-Spec$(R) \subseteq \mathrm{Im}(\varphi^*)$, *then*

$$\begin{aligned}\mathrm{G}_C\text{-}\mathrm{dim}_R(X) - \mathrm{pd}_R(S) &\le \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\mathrm{dim}_S\big(X \otimes^{\mathbf{L}}_R S\big)\\ &\le \mathrm{G}_C\text{-}\mathrm{dim}_R(X) + \mathrm{pd}_R(S).\end{aligned}$$

Thus, $X \otimes^{\mathbf{L}}_R S$ is $\mathbf{R}\mathrm{Hom}_R(S, C)$-reflexive if and only if X is C-reflexive. If R is local or $\mathrm{amp}(C) = 0 = \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, R))$, *then*

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\mathrm{dim}_S\big(X \otimes^{\mathbf{L}}_R S\big) = \mathrm{G}_C\text{-}\mathrm{dim}_R(X).$$

In Section 6 we extend results of Section 5 to the case where φ is local and admits a Gorenstein factorization $R \to R' \to S$; see 6.2. To this end, we use a shift of the functor $\mathbf{R}\mathrm{Hom}_{R'}(S, - \otimes^{\mathbf{L}}_R R')$ in place of $\mathbf{R}\mathrm{Hom}_R(S, -)$. We prove in Theorem 6.5 that this is independent of the choice of Gorenstein factorization and, when φ is module-finite, agrees with $\mathbf{R}\mathrm{Hom}_R(S, -)$. The remainder of the section is spent documenting the translations of the results from Section 5 to this context.

The third focus of this article is found in the numerous examples within the text demonstrating that our results are, in a sense, optimal. These examples may be of independent interest, as the number of explicit computations in this area is somewhat limited. For this reason, and for ease of reference, we provide a resume of the more delicate examples here. Note that some of the rings constructed have connected prime spectra, and this makes the constructions a tad technical. We have taken this approach because rings with connected spectra can exhibit particularly nice local-global behavior and we wanted to make the point that the exemplified behavior can occur even when the spectra are connected.

Example 2.7 shows that one can have inequalities $\mathrm{G}_C\text{-dim}_R(X) < \sup(X)$ and $\mathrm{G}_C\text{-dim}_R(X) < \mathrm{G}_{C_\mathfrak{p}}\text{-dim}_{R_\mathfrak{p}}(X_\mathfrak{p})$, even when R is local. Thus, $\mathrm{G}_C\text{-dim}_R(X)$ cannot be computed as the length of a resolution of X, and the assumption $\mathrm{amp}(C) = 0$ is necessary in Lemma 2.1 and in the final statement of Lemma 2.4.

Example 2.13 provides a surjective ring homomorphism of finite flat dimension that is Cohen–Macaulay with nonconstant grade. Thus, $\mathrm{Spec}(S)$ must be connected in Corollary 2.12.

Example 3.8 shows that one can have $\mathrm{amp}(C) > 0$ when $\mathrm{amp}(C_\mathfrak{m}) = 0$ for each maximal ideal $\mathfrak{m}$. Furthermore, if C' is C-reflexive, the inequality $\mathrm{amp}(C) \leq \mathrm{amp}(C')$ from Corollary 3.7 can be strict, even when $\mathrm{amp}(C) = 0$. Thus, the connectedness of $\mathrm{Spec}(R)$ is needed in Proposition 2.10 and in Corollaries 3.5 and 3.7.

Example 3.10 provides a ring R with $\mathrm{Spec}(R)$ connected where

$$\begin{aligned} \inf(C) - \sup(C') &= \inf(\mathbf{R}\mathrm{Hom}_R(C', C)) < \inf(C) - \inf(C') \\ \inf(C') &< \mathrm{G}_C\text{-dim}_R(C') = \sup(C') \\ \mathrm{G}_A\text{-dim}_R(B) - \sup(B) &= \mathrm{G}_{B^{\dagger_C}}\text{-dim}_R(A^{\dagger_C}) - \inf(C) + \inf(A) \\ &< \mathrm{G}_A\text{-dim}_R(B) - \inf(B) \end{aligned}$$

showing that inequalities in Lemma 3.4 and Proposition 3.9 can be strict or not.

Example 5.11 shows that strictness can occur in each of the inequalities

$$\begin{aligned} \mathrm{G}_C\text{-dim}_R(S) &\leq \sup\{\mathrm{G}_{C_\mathfrak{m}}\text{-dim}_{R_\mathfrak{m}}(S_\mathfrak{m}) \mid \mathfrak{m} \in \text{m-Spec}(R)\} \\ \mathrm{G}_C\text{-dim}_R(S) &\leq \mathrm{pd}_R(S) \\ \inf(C) &\leq \inf(C \otimes^{\mathbf{L}}_R S) \\ \inf(C) - \mathrm{pd}_R(S) &\leq \inf(\mathbf{R}\mathrm{Hom}_R(S, C)) \\ \mathrm{G}_C\text{-dim}_R(S) &\leq \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-dim}_S(S) + \mathrm{pd}_R(S) \end{aligned}$$

from Propositions 2.9, 3.11, and 5.10 and from Theorems 4.5 and 5.5.

Example 5.14 pertains to Theorems 4.4, 4.8, 5.4, 5.8, and 5.13, showing that one can have $\mathrm{G}_C\text{-}\dim_R(X) = \infty$ even though each of the following is finite: $\mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(S, X))$, $\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(S, X))$, $\mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$, $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$, $\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$. Hence, the hypothesis on m-Spec(R) is necessary for each result.

As this introduction suggests, most of the results of this article are stated and proved in the framework of the derived category. We collect basic definitions and notations for the reader's convenience in Section 1.

1. COMPLEXES AND RING HOMOMORPHISMS

Throughout this work, R and S are commutative Noetherian rings and $\varphi\colon R \to S$ is a ring homomorphism.

This section consists of background material and includes most of the definitions and notational conventions used throughout the rest of this work.

1.1. We work in the derived category $\mathsf{D}(R)$ whose objects are the R-complexes, indexed homologically; references on the subject include Gelfand and Manin (1996), Hartshorne (1966), Neeman (2001), and Verdier (1977, 1996). For R-complexes X and Y, the left derived tensor product complex is denoted $X \otimes^{\mathbf{L}}_R Y$ and the right derived homomorphism complex is $\mathbf{R}\mathrm{Hom}_R(X, Y)$. For an integer n, the nth *shift* or *suspension* of X is denoted $\Sigma^n X$ where $(\Sigma^n X)_i = X_{i-n}$ and $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$. The symbol "$\simeq$" indicates an isomorphism in $\mathsf{D}(R)$ and "$\sim$" indicates an isomorphism up to shift.

The *infimum* and *supremum* of a complex X, denoted $\inf(X)$ and $\sup(X)$, are the infimum and supremum, respectively, of the set $\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$, and the *amplitude* of X is the difference $\mathrm{amp}(X) = \sup(X) - \inf(X)$. The complex X is *homologically finite*, respectively *homologically degree-wise finite*, if its total homology module $\mathrm{H}(X)$, respectively each individual homology module $\mathrm{H}_i(X)$, is a finite R-module. It is *homologically bounded above*, respectively *homologically bounded below* or *homologically bounded*, if $\sup(X) < \infty$, respectively $\inf(X) > -\infty$ or $\mathrm{amp}(X) < \infty$. The projective, injective, and flat dimensions of X are denoted $\mathrm{pd}_R(X)$, $\mathrm{id}_R(X)$, and $\mathrm{fd}_R(X)$, respectively; see Avramov and Foxby (1991).

The main objects of study in this article are the semidualizing complexes and their reflexive objects, introduced by Foxby (1972/1973), Golod (1984), and Christensen (2001).

1.2. A homologically finite R-complex C such that the homothety morphism

$$\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$$

is an isomorphism is *semidualizing*. Observe that the R-module R is semidualizing. An R-complex D is *dualizing* if it is semidualizing and has finite injective dimension; see Hartshorne (1966, Chapter V) and Foxby (In preparation, Chapter 15). Over local rings, dualizing complexes are unique up to shift-isomorphism.

The following result is proved like Jorgensen (to appear, (2.5.1)).

Lemma 1.3. *Let k be a field and R_1, R_2 local rings essentially of finite type over k and let R be a localization of $R_1 \otimes_k^{\mathbf{L}} R_2$. If D^i is a dualizing complex for R_i for $i = 1, 2$, then the complex $\left(D^1 \otimes_k^{\mathbf{L}} D^2\right) \otimes_{R_1 \otimes_k^{\mathbf{L}} R_2}^{\mathbf{L}} R$ is dualizing for R.*

1.4. Let C, X be homologically finite R-complexes with C semidualizing. If the complex $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded and the biduality morphism

$$\delta_X^C \colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$$

is an isomorphism, then X is *C-reflexive*. The complexes R and C are C-reflexive, and C is dualizing if and only if each homologically finite complex is C-reflexive by Hartshorne (1966, (V.2.1)). The G_C*-dimension* of a X is defined in Christensen (2001) as

$$\mathrm{G}_C\text{-}\dim_R(X) = \begin{cases} \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) & \text{when } X \text{ is C-reflexive} \\ \infty & \text{otherwise.} \end{cases}$$

When $C = R$ this is the G-*dimension* of Foxby, Auslander (1967), Auslander and Bridger (1969), and Yassemi (1995), denoted $\mathrm{G}\text{-}\dim_R(X)$; see also Christensen (2000). If $\mathrm{pd}_R(X)$ is finite, then so is $\mathrm{G}\text{-}\dim_R(X)$, and one has $\mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(X, R)) = -\inf(X)$ by Christensen (2001, (2.13)); if in addition R is local, then $\mathrm{G}\text{-}\dim_R(X) = \mathrm{pd}_R(X)$ by Christensen (2000, (2.3.10)). When C, X are modules and $\mathrm{G}_C\text{-}\dim_R(X) = 0$, one says X is *totally C-reflexive*.

Other invariants and formulas are available over a local ring.

1.5. When R is local with residue field k and X is homologically finite, the integers

$$\beta_i^R(X) = \mathrm{rank}_k(\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X, k))) \qquad \mu_R^i(X) = \mathrm{rank}_k(\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, X))$$

are the ith *Betti number* and *Bass number* of X. The formal Laurent series

$$P_X^R(t) = \sum_{i \in \mathbb{Z}} \beta_i^R(X) t^i \qquad I_R^X(t) = \sum_{i \in \mathbb{Z}} \mu_R^i(X) t^i$$

are the *Poincaré series* and *Bass series* of X. The *depth* of X is

$$\mathrm{depth}_R(X) = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X)).$$

When C is a semidualizing R-complex, and X is C-reflexive, the AB-formula reads

$$\mathrm{G}_C\text{-}\dim_R(X) = \mathrm{depth}(R) - \mathrm{depth}_R(X)$$

and the isomorphism $R \simeq \mathbf{R}\mathrm{Hom}_R(C, C)$ gives rise to a formal equality

$$P_C^R(t) I_R^C(t) = I_R^R(t)$$

by Christensen (2001, (3.14)) and Avramov and Foxby (1997, (1.5.3)). When D is dualizing for R, one has $I_R^D(t) = t^d$ for some integer d by Hartshorne (1966, (V.3.4)).

We say that D is *normalized* when $I_R^D(t) = 1$, that is, when $\inf(D) = \operatorname{depth}(R)$; see Avramov and Foxby (1997, (2.6)). In particular, a minimal injective resolution I of a normalized dualizing complex has $I_j \cong \oplus_{\mathfrak{p}} E_R(R/\mathfrak{p})$ where the sum is taken over the set of prime ideals $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) = j$.

We continue by recalling some standard morphisms.

1.6. Let X, Y, Z be R-complexes. For an R-algebra S, let U, V, W be S-complexes. We have cancellation, commutativity, associativity, and adjunction isomorphisms:

$$X \otimes_R^{\mathbf{L}} R \simeq X \tag{a}$$

$$X \otimes_R^{\mathbf{L}} Y \simeq Y \otimes_R^{\mathbf{L}} X \tag{b}$$

$$X \otimes_R^{\mathbf{L}} \left(Y \otimes_R^{\mathbf{L}} Z\right) \simeq \left(X \otimes_R^{\mathbf{L}} Y\right) \otimes_R^{\mathbf{L}} Z \tag{c}$$

$$\mathbf{R}\mathrm{Hom}_S\left(X \otimes_R^{\mathbf{L}} V, W\right) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_S(V, W)) \tag{d}$$

$$\mathbf{R}\mathrm{Hom}_R\left(U \otimes_S^{\mathbf{L}} V, Z\right) \simeq \mathbf{R}\mathrm{Hom}_S(U, \mathbf{R}\mathrm{Hom}_R(V, Z)). \tag{e}$$

Next, there are the tensor- and Hom-evaluation morphisms, respectively (Avramov and Foxby, 1991, (4.4)).

$$\omega_{XVW}\colon \mathbf{R}\mathrm{Hom}_R(X, V) \otimes_S^{\mathbf{L}} W \to \mathbf{R}\mathrm{Hom}_R\left(X, V \otimes_S^{\mathbf{L}} W\right) \tag{f}$$

$$\theta_{XVW}\colon X \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S(V, W) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X, V), W). \tag{g}$$

The morphism ω_{XVW} is an isomorphism when X is homologically finite, V is homologically bounded above, and either $\mathrm{fd}_S(W) < \infty$ or $\mathrm{pd}_R(X) < \infty$. The morphism θ_{XVW} is an isomorphism when X is homologically finite, V is homologically bounded, and either $\mathrm{id}_S(W) < \infty$ or $\mathrm{pd}_R(X) < \infty$.

1.7. Let C, P, V, W, Y be R-complexes with Y homologically bounded above, C semidualizing, and $\mathrm{pd}_R(P)$, $\mathrm{G}_C\text{-}\dim_R(W) < \infty$.

(a) Adjunction and C-reflexivity provide an isomorphism

$$\mathbf{R}\mathrm{Hom}_R(V, W) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(W, C), \mathbf{R}\mathrm{Hom}_R(V, C)).$$

(b) Since P is R-reflexive, Hom-evaluation gives an isomorphism

$$\mathbf{R}\mathrm{Hom}_R(P, Y) \simeq \mathbf{R}\mathrm{Hom}_R(P, R) \otimes_R^{\mathbf{L}} Y.$$

In this article we focus on several specific types of ring homomorphisms.

1.8. The ring homomorphism $\varphi\colon R \to S$ induces a natural map on prime spectra $\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. The flat dimension of φ is defined as $\mathrm{fd}(\varphi) = \mathrm{fd}_R(S)$.

Assume that φ is *local*, that is, the rings R and S are local with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively, and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. The *depth* of φ is $\operatorname{depth}(\varphi) = \operatorname{depth}(S) - \operatorname{depth}(R)$. When $\mathrm{fd}(\varphi)$ is finite, the *Bass series* of φ is the formal Laurent series with non-negative integer coefficients $I_\varphi(t)$ satisfying the formal equality $I_S^S(t) = I_R^R(t)I_\varphi(t)$ whose existence is given by Avramov et al. (1993, (5.1)) or Avramov and Foxby (1997, (7.1)). The homomorphism φ is *Gorenstein* at $\mathfrak{n}$ if $I_\varphi(t) = t^d$ for some integer d, in which case, $d = \operatorname{depth}(\varphi)$. When φ is module-finite, it is *Cohen–Macaulay* if S is perfect as an R-module, that is, when $\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(S,R)) = 0$.

When φ is surjective and has finite flat dimension (but is not necessarily local) it is *Cohen–Macaulay* if, for each prime ideal $\mathfrak{q} \subset S$, the localization $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \to S_\mathfrak{q}$ is Cohen–Macaulay where $\mathfrak{p} = \varphi^*(\mathfrak{q})$. In this event, φ is *Cohen–Macaulay of grade d* if one of the following equivalent conditions holds:

(i) S is a perfect R-module of grade d;
(ii) $d = \operatorname{grade}_{R_\mathfrak{p}} S_\mathfrak{q}$ for each prime ideal $\mathfrak{q} \subset S$;
(iii) $\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(S,R)) = 0$.

The map φ is *Gorenstein*[1] if it is Cohen–Macaulay and, for each prime ideal $\mathfrak{q} \subset S$, the $S_\mathfrak{q}$-module $\operatorname{Ext}_R^{d_\mathfrak{q}}(S,R)_\mathfrak{q}$ is cyclic for $d_\mathfrak{q} = \operatorname{grade}_{R_\mathfrak{p}}(S_\mathfrak{q})$ where $\mathfrak{p} = \varphi^*(\mathfrak{q})$.

Here are two more combinations of standard morphisms.

1.9. Assume that $\operatorname{fd}(\varphi)$ is finite and fix R-complexes W, X, Y, Z with W homologically bounded, X homologically finite, and Y homologically bounded above.

(a) Combining adjunction and tensor-evaluation yields an isomorphism

$$\mathbf{R}\mathrm{Hom}_S\big(X \otimes_R^{\mathbf{L}} S, Y \otimes_R^{\mathbf{L}} S\big) \simeq \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes_R^{\mathbf{L}} S.$$

(b) If φ is module-finite, then adjunction and Hom-evaluation provide

$$\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S,W), \mathbf{R}\mathrm{Hom}_R(S,Z)) \simeq S \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(W,Z).$$

(c) If φ is module-finite, then 1.7(b), tensor-evaluation, and adjunction yield

$$\mathbf{R}\mathrm{Hom}_R(X \otimes_R^{\mathbf{L}} S, \mathbf{R}\mathrm{Hom}_R(S,Y)) \simeq \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(S,R).$$

When X and Y are modules the next lemma is Grothendieck (1965, (2.5.8)). Example 4.10 demonstrates the necessity of flatness.

Lemma 1.10. *Let $\varphi\colon R \to S$ be flat and local such that the induced extension of residue fields is bijective. If X, Y are homologically degree-wise finite and bounded below R-complexes and $X \otimes_R^{\mathbf{L}} S \simeq Y \otimes_R^{\mathbf{L}} S$ in $\mathsf{D}(S)$, then $X \simeq Y$ in $\mathsf{D}(R)$.*

Proof. Consider minimal R-free resolutions $P \simeq X$ and $Q \simeq Y$. The S-complexes $P \otimes_R S$ and $Q \otimes_R S$ are minimal S-free resolutions of $X \otimes_R^{\mathbf{L}} S$ and $Y \otimes_R^{\mathbf{L}} S$,

[1]Avramov and Foxby (1992, 1998) originally used the terms *locally Cohen–Macaulay* and *locally Gorenstein* for these types of homomorphisms. As they have chosen to rename the second type *Gorenstein* (Avramov and Foxby, 1997, (8.1)), we have followed suit with the first type.

respectively. The first isomorphism in the following sequence follows from the flatness of φ

$$\begin{aligned}\mathrm{Hom}_R(P, Q) \otimes_R S &\simeq \mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_R S \\ &\simeq \mathbf{R}\mathrm{Hom}_S\big(X \otimes^{\mathbf{L}}_R S, Y \otimes^{\mathbf{L}}_R S\big) \\ &\simeq \mathrm{Hom}_S(P \otimes_R S, Q \otimes_R S)\end{aligned}$$

while the second is in 1.9(a) and the third is standard. This shows that the composition of tensor-evaluation and adjunction

$$f\colon \mathrm{Hom}_R(P, Q) \otimes_R S \xrightarrow{\simeq} \mathrm{Hom}_S\big(P \otimes_R S, Q \otimes_R S\big)$$

is a quasiisomorphism. The relevant definitions provide an equality

$$\partial_0^{\mathrm{Hom}_R(P,Q)\otimes_R S} = \partial_0^{\mathrm{Hom}_R(P,Q)} \otimes_R S$$

and the flatness of φ provides a natural isomorphism

$$\mathrm{Ker}\big(\partial_0^{\mathrm{Hom}_R(P,Q)}\big) \otimes_R S \cong \mathrm{Ker}\big(\partial_0^{\mathrm{Hom}_R(P,Q)\otimes_R S}\big). \tag{†}$$

Note that the set of chain maps from P to Q over R is exactly the set of cycles $Z_0^{\mathrm{Hom}_R(P,Q)} = \mathrm{Ker}\big(\partial_0^{\mathrm{Hom}_R(P,Q)}\big)$, and similarly for $P \otimes_R S$ and $Q \otimes_R S$.

The assumption $X \otimes^{\mathbf{L}}_R S \simeq Y \otimes^{\mathbf{L}}_R S$ provides an isomorphism in the category of S-complexes $\alpha\colon P \otimes^{\mathbf{L}}_R S \xrightarrow{\cong} Q \otimes^{\mathbf{L}}_R S$. Since f is a quasi-isomorphism, there exists a cycle $\alpha' \in \mathrm{Hom}_R(P, Q)_0 \otimes_R S$ such that the images of $f(\alpha')$ and α in $\mathrm{H}_0(\mathrm{Hom}_S(P \otimes_R S, Q \otimes_R S))$ are equal. In other words, the chain maps $f(\alpha')$ and α are homotopic. In particular, since $P \otimes_R S$ and $Q \otimes_R S$ are minimal and α is an isomorphism of complexes, the same is true of $f(\alpha')$.

The isomorphism (†) shows that $\alpha' = \sum_i \alpha'_i \otimes s_i$ for some $\alpha'_i \in \mathrm{Ker}\big(\partial_0^{\mathrm{Hom}_R(P,Q)}\big)$ and $s_i \in S$. For each i fix an $r_i \in R$ with the same residue as s_i in $k = R/\mathfrak{m} \cong S/\mathfrak{n}$. We shall show that the chain map $\alpha'' = \sum_i r_i\alpha'_i\colon P \to Q$ is an isomorphism of complexes. By construction, there is a commutative diagram

$$\begin{array}{ccc} (P \otimes_R S) \otimes_S k & \xrightarrow[\cong]{f(\alpha')\otimes_S k} & (Q \otimes_R S) \otimes_S k \\ \cong\big\downarrow 1.6(a)(c) & & \cong\big\downarrow 1.6(a)(c) \\ P \otimes_R k & \xrightarrow{\alpha''\otimes_R k} & Q \otimes_R k \end{array}$$

showing that $\alpha'' \otimes_R k$ is (degree-wise) surjective. Nakayama's Lemma then implies that α'' is degree-wise surjective, and the result follows from Matsumura (1989, (2.4)). □

The final background concept for this article is the Picard group.

1.11. The *Picard group* of R, denoted $\mathrm{Pic}(R)$, is the Abelian group of isomorphism classes of finitely generated locally free (i.e., projective) R-modules of rank 1 with operation given by tensor product. The assignment $M \mapsto M \otimes_R^{\mathbf{L}} S$ yields a well-defined group homomorphism $\mathrm{Pic}(\varphi)\colon \mathrm{Pic}(R) \to \mathrm{Pic}(S)$.

2. RESOLUTIONS AND LOCALIZATION

This section contains results used to globalize standard local results. We begin by observing that G_C-dimension can be measured by resolutions when C is a module. Example 2.7 shows that this fails when $\mathrm{amp}(C) > 0$ however, see Lemma 2.2.

Lemma 2.1. *Let X be a homologically finite R-complex and C a semidualizing R-module. Given an integer n, the following conditions are equivalent*:

(i) *There is an isomorphism $G \simeq X$ where G is a complex of totally C-reflexive modules with $G_i = 0$ for each $i > n$ and for each $i < \inf(X)$;*
(ii) *There is an inequality $\mathrm{G}_C\text{-}\dim_R(X) \leq n$;*
(iii) *One has $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ and $n \geq -\inf(\mathbf{R}\mathrm{Hom}_R(X, C))$;*
(iv) *$n \geq \sup(X)$ and in any bounded below complex G of totally C-reflexive modules with $G \simeq X$, the module $\mathrm{Coker}(\partial_{n+1}^G)$ is totally C-reflexive.*

In particular, there is an inequality $\sup(X) \leq \mathrm{G}_C\text{-}\dim_R(X)$.

Proof. The local case when X is a module is stated in Golod (1984, p. 68). For the general case, mimic the proof of Christensen (2000, (2.3.7)). □

The next result is Christensen (2001, (3.12)) which we state here for ease of reference. Example 2.5 shows that equality or strict inequality can occur.

Lemma 2.2. *If C, X are homologically finite R-complexes with C semidualizing, then* $\sup(X) - \mathrm{amp}(C) \leq \mathrm{G}_C\text{-}\dim_R(X)$.

Lemma 2.3. *If C is homologically finite, the following conditions are equivalent*:

(i) *C is R-semidualizing*;
(ii) *$S^{-1}C$ is $S^{-1}R$-semidualizing for each multiplicative subset $S \subset R$*;
(iii) *$C_{\mathfrak{m}}$ is a $R_{\mathfrak{m}}$-semidualizing for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. The implication (ii) $\Longrightarrow$ (iii) is trivial, while (i) $\Longrightarrow$ (ii) follows from the argument of Christensen (2001, (2.5)). For the remaining implication, condition (iii) implies that the natural map $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is locally an isomorphism, so it is an isomorphism and C is R-semidualizing. □

The proof of the next result is almost identical to that of Christensen (2001, (3.16)). Examples 2.5–2.7 show that the inequalities can be strict or not, that the converse of the second statement fails, and that the final inequality fails to hold if $\mathrm{amp}(C) > 0$.

Lemma 2.4. *Let C, X be homologically finite R-complexes with C semidualizing. For each multiplicative subset $S \subset R$, there is an inequality*

$$\mathrm{G}_{S^{-1}C}\text{-}\dim_{S^{-1}R}(S^{-1}X) \leq \mathrm{G}_C\text{-}\dim_R(X) + \inf(S^{-1}C) - \inf(C).$$

In particular, if X is C-reflexive, then $S^{-1}X$ is $S^{-1}C$-reflexive. Furthermore, if $\mathrm{amp}(C) = 0$, *then* $\mathrm{G}_{S^{-1}C}\text{-}\dim_{S^{-1}R}(S^{-1}X) \leq \mathrm{G}_C\text{-}\dim_R(X)$.

Example 2.5. When R is local and $\mathrm{amp}(C) = 0 = \mathrm{amp}(X)$, the inequalities in Lemmas 2.2 and 2.4 can be strict (if $0 \leq \mathrm{pd}_{S^{-1}R}(S^{-1}X) < \mathrm{pd}_R(X) < \infty$) or not (set $C = R = X$).

Example 2.6. The converse to the second statement in Lemma 2.4 can fail. Let $(R, \mathfrak{m})$ be a local non-Gorenstein ring with prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$. The module $\mathfrak{m}$ is not R-reflexive but the module $\mathfrak{m}_\mathfrak{p} \cong R_\mathfrak{p}$ is $R_\mathfrak{p}$-reflexive.

Example 2.7. The final inequalities in Lemmas 2.1 and 2.4 can fail if $\mathrm{amp}(C) > 0$. Note that Lemma 3.4 shows that X cannot be a semidualizing module.

Let k be a field and $R = k[\![Y, Z]\!]/(Y^2, YZ)$. Since R is complete local, it admits a dualizing complex D. With $\mathfrak{p} = (Y)R$ and $X = R/\mathfrak{p}$ the AB-formula implies

$$\mathrm{G}_D\text{-}\dim_R(X) = \mathrm{depth}(R) - \mathrm{depth}_R(X) = -1 < 0 = \sup(X)$$
$$\mathrm{G}_{D_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(X_\mathfrak{p}) = \mathrm{depth}(R_\mathfrak{p}) - \mathrm{depth}_{R_\mathfrak{p}}(X_\mathfrak{p}) = 0 > -1 = \mathrm{G}_D\text{-}\dim_R(X).$$

The next equalities are by definition, and the first inequality is by Christensen (2000, (A.8.6.1))

$$\mathrm{G}_D\text{-}\dim_R(D) = \inf(D) = \sup(D) - 1 < \sup(D)$$
$$\mathrm{G}_{D_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(D_\mathfrak{p}) = \inf(D_\mathfrak{p}) = \inf(D) + 1 > \inf(D) = \mathrm{G}_D\text{-}\dim_R(D)$$

while the second inequality follows from the arguments of Foxby (In preparation, Section 15).

We do not know if the extra hypotheses are necessary for the converses in the next result; they are not needed when G_C-dimension is replaced by projective dimension.

Proposition 2.8. *Let C, X be homologically finite R-complexes with C semidualizing. Consider the following conditions:*

(i) *X is C-reflexive;*
(ii) *$S^{-1}X$ is $S^{-1}C$-reflexive for each multiplicative subset $S \subset R$;*
(iii) *$X_\mathfrak{m}$ is $C_\mathfrak{m}$-reflexive for each maximal ideal $\mathfrak{m} \subset R$.*

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *always hold, and the converses hold when either* $\inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \geq -\infty$, $\dim(R) < \infty$, *or X is semidualizing.*

Proof. The implication (ii) $\Longrightarrow$ (iii) is trivial, while (i) $\Longrightarrow$ (ii) is in Lemma 2.4. So, assume that $X_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-reflexive for each maximal ideal $\mathfrak{m}$. The biduality map $\delta_X^C\colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ is locally an isomorphism, and so it is an isomorphism. It remains to show that $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded.

Assume first $\dim(R) < \infty$. For each maximal ideal $\mathfrak{m} \subset R$ the AB-formula provides the following equality while the inequality is (Foxby and Iyengar, 2003, (2.7))

$$\mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) = \operatorname{depth}(R_{\mathfrak{m}}) - \operatorname{depth}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \leq \dim(R_{\mathfrak{m}}) + \sup(X_{\mathfrak{m}}).$$

This explains the first inequality in the next sequence, while the equality is by definition and the second inequality is standard:

$$\begin{aligned}\inf(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}, C_{\mathfrak{m}})) &= \inf(C_{\mathfrak{m}}) - \mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}})\\ &\geq \inf(C_{\mathfrak{m}}) - \dim(R_{\mathfrak{m}}) - \sup(X_{\mathfrak{m}})\\ &\geq \inf(C) - \dim(R) - \sup(X).\end{aligned}$$

It follows that $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded because

$$\inf(\mathbf{R}\mathrm{Hom}_R(X, C)) = \inf\{\inf(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}, C_{\mathfrak{m}})) \mid \mathfrak{m} \in \text{m-Spec}(R)\}.$$

Assuming next that X is semidualizing, the AB-formula and Christensen (2001, (3.2.a)) provide the equality $\mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) = \inf(X_{\mathfrak{m}})$. As above one deduces

$$\inf(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}, C_{\mathfrak{m}})) \geq \inf(C) - \sup(X)$$

and the homological boundedness of $\mathbf{R}\mathrm{Hom}_R(X, C)$. □

For strictness in the next inequality, see Example 5.11 or argue as in Example 3.8.

Proposition 2.9. *If C is R-semidualizing, then there is an inequality*

$$\mathrm{G}_C\text{-}\dim_R(X) \leq \sup\left\{\mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{m-Spec}(R)\right\}$$

for each homologically finite R-complex X, with equality if $\operatorname{amp}(C) = 0$.

Proof. For the inequality, set $s = \sup\{\mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{m-Spec}(R)\}$ and $i = \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$, and assume $s < \infty$. For each maximal ideal $\mathfrak{m}$, one has

$$\mathrm{G}_{C_{\mathfrak{m}}}\text{-}\dim_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) + \inf(\mathbf{R}\mathrm{Hom}_R(X, C)_{\mathfrak{m}}) = \inf(C_{\mathfrak{m}}) \geq \inf(C).$$

It follows that $\mathbf{R}\mathrm{Hom}_R(X, C)$ is bounded because the previous sequence gives

$$i = \inf\{\inf(\mathbf{R}\mathrm{Hom}_R(X, C)_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{m-Spec}(R)\} \geq \inf(C) - s$$

so $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ by Proposition 2.8. With $\mathfrak{m} \in \mathrm{Supp}_R(\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(X, C)))$, the desired inequality is in the next sequence:

$$\mathrm{G}_C\text{-}\dim_R(X) = \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \leq \inf(C_\mathfrak{m}) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)_\mathfrak{m}) \leq s.$$

When $\mathrm{amp}(C) = 0$, equality follows from Lemma 2.4 since $\inf(C) = \inf(C_\mathfrak{m})$. □

Example 3.8 shows the need for the connectedness hypothesis in the next result. When R is Cohen–Macaulay, the condition $\mathrm{amp}(C_\mathfrak{m}) = 0$ is automatic by Christensen (2001, (3.4)).

Proposition 2.10. *Let C be a semidualizing R-complex and assume that* $\mathrm{Spec}(R)$ *is connected. If* $\mathrm{amp}(C_\mathfrak{m}) = 0$ *for each maximal ideal* $\mathfrak{m}$, *then* $\mathrm{amp}(C) = 0$.

Proof. If $\mathrm{amp}(C) > 0$, then $\mathrm{Spec}(R) = \mathrm{Supp}_R(C)$ is the disjoint union of the closed sets $\mathrm{Supp}_R(\mathrm{H}_{\inf(C)}(C)), \ldots, \mathrm{Supp}_R(\mathrm{H}_{\sup(C)}(C))$, contradicting connectedness. □

Question 2.11. If C is a semidualizing R-complex and $\mathrm{Spec}(R)$ is connected, must the inequality $\mathrm{amp}(C) = \sup\{\mathrm{amp}(C_\mathfrak{m}) \mid \mathfrak{m} \in \text{m-Spec}(R)\}$ hold?

Proposition 2.10 with $C = \mathbf{R}\mathrm{Hom}_R(S, R)$ yields the next local-global principle; see 1.8. Example 2.13 shows that this fails if $\mathrm{Spec}(S)$ is disconnected.

Corollary 2.12. *Let $\varphi\colon R \to S$ be a surjective Cohen–Macaulay ring homomorphism. If* $\mathrm{Spec}(S)$ *is connected, then φ is Cohen–Macaulay of constant grade.*

Example 2.13. We construct a a surjective Cohen–Macaulay ring homomorphism of nonconstant grade. Let k be a field and $R = k[Y, Z]$ a polynomial ring, and set

$$S = R/((Y, Z)R \cap (Y - 1)R)$$

with natural surjection $\varphi\colon R \to S$. Since R is regular, one has $\mathrm{pd}_R(S) < \infty$. The equality $(Y, Z)R + (Y - 1)R = R$ provides an isomorphism of R-algebras

$$S \cong R/(Y, Z)R \times R/(Y - 1)R.$$

In particular, the ring S is Cohen–Macaulay, and hence so is φ by Avramov and Foxby (1998, (8.10)). Set $\mathfrak{n}_1 = (Y, Z)S$ and $\mathfrak{n}_2 = (Y - 1, Z)S$. To prove that φ has nonconstant grade, it suffices by 1.8 to show that $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, R)) > 0$. For this we verify

$$\inf(\mathbf{R}\mathrm{Hom}_R(S, R)_{\mathfrak{n}_1}) = -2 \qquad \inf(\mathbf{R}\mathrm{Hom}_R(S, R)_{\mathfrak{n}_2}) = -1.$$

It is straightforward to verify that the localization $\varphi_{\mathfrak{n}_1}$ is equivalent to the natural surjection $R_{(Y,Z)} \to k$ which has projective dimension 2. Thus, one has

$$\inf\big(\mathbf{R}\mathrm{Hom}_R(S, R)_{\mathfrak{n}_1}\big) = \inf\big(\mathbf{R}\mathrm{Hom}_{R_{(Y,Z)}}(k, R_{(Y,Z)})\big) = -\mathrm{pd}_{R_{(Y,Z)}}(k) = -2$$

where the second equality is by Christensen (2001, (2.13)). This is the first desired equality; the second one follows similarly from the fact that the localization $\varphi_{\mathfrak{n}_2}$ is equivalent to the surjection $k[Y, Z]_{(Y-1,Z)} \to k[Z]_{(Z)}$ which has projective dimension 1.

Lemma 2.14. *Let $R = \coprod_{i\geq 0} R_i$ be a graded ring where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geq 1} R_i$ and let X, Y be homologically degree-wise finite complexes of graded R-module homomorphisms.*

(a) *For each integer i, one has $\mathrm{H}_i(X) = 0$ if and only if $\mathrm{H}_i(X_\mathfrak{m}) = 0$.*
(b) *There are equalities*

$$\inf(X) = \inf(X_\mathfrak{m}) \qquad \sup(X) = \sup(X_\mathfrak{m}) \qquad \mathrm{amp}(X) = \mathrm{amp}(X_\mathfrak{m})$$

so X is homologically bounded (respectively, homologically bounded above or homologically bounded below) if and only if the same is true of $X_\mathfrak{m}$.
(c) *If $\alpha\colon X \to Y$ is a graded homomorphism of complexes, then α is a quasi-isomorphism if and only if $\alpha_\mathfrak{m}$ is a quasi-isomorphism.*

Proof. Part (a) follows from Bruns and Herzog (1998, (1.5.15)) and the isomorphism $\mathrm{H}_i(X_\mathfrak{m}) \cong \mathrm{H}_i(X)_\mathfrak{m}$, and (b) is immediate from (a). For (c), apply (b) to the mapping cone of α. □

Proposition 2.15. *Let $R = \coprod_{i\geq 0} R_i$ be a graded ring where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geq 1} R_i$ and let C, X be homologically degree-wise finite complexes of graded R-module homomorphisms.*

(a) *The complex C is R-semidualizing if and only if $C_\mathfrak{m}$ is $R_\mathfrak{m}$-semidualizing.*
(b) *If C is R-semidualizing, then $\mathrm{G}_C\text{-}\dim_R(X) = \mathrm{G}_{C_\mathfrak{m}}\text{-}\dim(X_\mathfrak{m})$. Thus, the complex X is C-reflexive if and only if $X_\mathfrak{m}$ is $C_\mathfrak{m}$-reflexive.*

Proof. (a) One implication is contained in Lemma 2.3, so assume that $C_\mathfrak{m}$ is $R_\mathfrak{m}$-semidualizing. By Lemma 2.14, the R-complexes C and $\mathbf{R}\mathrm{Hom}_R(C, C)$ are homologically finite, and the homothety morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is a quasi-isomorphism. so C is semidualizing.

(b) It suffices to prove the final statement. Indeed, if X is C-reflexive and $X_\mathfrak{m}$ is $C_\mathfrak{m}$-reflexive, then the equality is a consequence of the following sequence:

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(X) &= \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \\ &= \inf(C_\mathfrak{m}) - \inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(X_\mathfrak{m}, C_\mathfrak{m})) \\ &= \mathrm{G}_{C_\mathfrak{m}}\text{-}\dim(X_\mathfrak{m})\end{aligned}$$

where the second equality is by Lemma 2.14(b), and the others are by definition.

For the final statement, one implication is in Lemma 2.4, so assume that $X_\mathfrak{m}$ is $C_\mathfrak{m}$-reflexive. Lemma 2.14 implies that X and $\mathbf{R}\mathrm{Hom}_R(X, C)$ are homologically finite and the biduality morphism $X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ is a quasi-isomorphism, so X is C-reflexive. □

3. DUALITY: GLOBAL RESULTS

This section is primarily devoted to reflexivity relations between semidualizing complexes in the nonlocal setting. We begin with a global version of Gerko (2004, (3.1), (3.4)).

Lemma 3.1. *Let C, C' be semidualizing R-complexes.*

(a) *If C' is C-reflexive, then $\mathbf{R}\mathrm{Hom}_R(C', C)$ is semidualizing and C-reflexive with* $\mathrm{G}_C\text{-}\dim(\mathbf{R}\mathrm{Hom}_R(C', C)) = \inf(C) - \inf(C')$.
(b) *If C' is C-reflexive, then the evaluation morphism $C' \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C', C) \to C$ is an isomorphism.*
(c) *If $C \otimes^{\mathbf{L}}_R C'$ is semidualizing, then C is $C \otimes^{\mathbf{L}}_R C'$-reflexive and the evaluation morphism $C \to \mathbf{R}\mathrm{Hom}_R(C', C \otimes^{\mathbf{L}}_R C')$ is an isomorphism.*

Proof. Part (a) is contained in Christensen (2001, (2.11)). For parts (b) and (c), observe that the maps are locally isomorphisms by Gerko (2004, (3.1), (3.4)) and are thus isomorphisms. □

The next result follows immediately from the local case; see Araya et al. (2005, (5.3)). Example 3.3 shows that $\mathbf{R}\mathrm{Hom}_R(C', C) \not\sim R$ in general; see also Example 3.8.

Lemma 3.2. *If C, C' are R-semidualizing, C' is C-reflexive, and C is C'-reflexive, then $\mathbf{R}\mathrm{Hom}_R(C', C)_\mathfrak{m} \sim R_\mathfrak{m}$ and $C'_\mathfrak{m} \sim C_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$.*

Example 3.3. One can have $\mathbf{R}\mathrm{Hom}_R(C', C) \not\sim R$ in Lemma 3.2. Assume that there exists $[L] \in \mathrm{Pic}(R)$ with $[L] \neq [R]$. If C is a semidualizing R-complex, then so is $C' = C \otimes^{\mathbf{L}}_R L$. Furthermore, C' is C-reflexive and C is C'-reflexive. However, one has $\mathbf{R}\mathrm{Hom}_R(C, C') \simeq L \not\sim R$ and $\mathbf{R}\mathrm{Hom}_R(C', C) \simeq \mathbf{R}\mathrm{Hom}_R(L, R) \not\sim R$.

The next lemma follows directly from Christensen (2001, (3.1), (3.2), (4.8.c)). To see that the second and third inequalities can be strict and that the others can be equalities, consult Example 3.10 or argue as in Example 3.8. For strictness in the first and last inequalities, let R be local and $\mathrm{amp}(C') > 0$, and use the guaranteed equalities.

Lemma 3.4. *If C, C' are R-semidualizing and C' is C-reflexive, then*

$$\inf(C) - \sup(C') \le \inf(\mathbf{R}\mathrm{Hom}_R(C', C)) \le \inf(C) - \inf(C')$$
$$\inf(C') \le \mathrm{G}_C\text{-}\dim_R(C') \le \sup(C')$$

with equality in the second and third inequalities if R is local or $\mathrm{amp}(C') = 0$. *In particular, if C, C' are both modules, then C' is totally C-reflexive.*

Example 3.8 shows the need for the connectedness hypothesis in the next result.

Corollary 3.5. *Let C, C' be semidualizing R-complexes with* $\mathrm{amp}(C) = 0$. *If* $\mathrm{Spec}(R)$ *is connected and C' is C-reflexive, then* $\mathrm{amp}(C') = 0$.

Proof. By Proposition 2.10 we may assume that R is local. The first inequality in the next sequence is in Lemma 2.1

$$\sup(C') \leq \mathrm{G}_C\text{-}\dim_R(C') = \inf(C') \leq \sup(C')$$

while the equality is in Christensen (2001, (3.1), (3.2)) and the last inequality is immediate. □

Question 3.6. If $\mathrm{Spec}(R)$ is connected and C, C' are semidualizing complexes such that C' is C-reflexive, does the inequality $\mathrm{amp}(C') \leq \mathrm{amp}(C)$ hold?

The answer is "yes" when R is local and C is dualizing for R by Christensen (2001, (3.4a)). The next result resolves the local case when C is not necessarily dualizing. The inequality can be strict (e.g., if $\mathrm{amp}(C) > 0$ and $C' = R$) or not (e.g., if R is Cohen–Macaulay). Consult Example 3.8 to see the need for connectedness.

Corollary 3.7. *Let R be local and C, C' semidualizing R-complexes. If C' is C-reflexive, then* $\mathrm{amp}(C') \leq \mathrm{amp}(C)$.

Proof. Since R is local, the equality in the following sequence is in Lemma 3.4

$$\inf(C') = \mathrm{G}_C\text{-}\dim_R(C') \geq \sup(C') - \mathrm{amp}(C)$$

while the inequality is in Lemma 2.2. □

Example 3.8. The conclusions of Proposition 2.10 and Corollaries 3.5 and 3.7 can fail if $\mathrm{Spec}(R)$ is not connected. Let k_1, k_2 be fields and set $R = k_1 \times k_2$. With $\mathfrak{m}_1 = 0 \times k_2$ and $\mathfrak{m}_2 = k_1 \times 0$, one has $\mathrm{Spec}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $R_{\mathfrak{m}_i} \cong k_i \cong R/\mathfrak{m}_i$ for $i = 1, 2$. Hence, R is Gorenstein, and an R-complex is dualizing if and only if it is semidualizing. If $pq \neq 0$, the next equality and isomorphism are easily verified:

$$\mathrm{amp}\big((\Sigma^a k_1^p) \times (\Sigma^b k_2^q)\big) = |a - b|$$
$$\mathbf{R}\mathrm{Hom}_R\big((\Sigma^a k_1^p) \times (\Sigma^b k_2^q), (\Sigma^c k_1^r) \times (\Sigma^d k_2^s)\big) \simeq \big(\Sigma^{c-a} k_1^{pr}\big) \times \big(\Sigma^{d-b} k_2^{qs}\big)$$

It follows that $(\Sigma^c k_1^r) \times (\Sigma^d k_2^s)$ is dualizing if and only if $r = s = 1$. So, the dualizing complex $C' = k_1 \times \Sigma k_2$ is R-reflexive, and the next computations are routine:

$$C'_{\mathfrak{m}_1} \simeq k_1 \qquad C'_{\mathfrak{m}_2} \simeq \Sigma k_2$$
$$\mathrm{amp}(C'_{\mathfrak{m}_i}) = 0 < 1 = \mathrm{amp}(C')$$
$$\mathrm{amp}(R) = 0 < 1 = \mathrm{amp}(C')$$

Here are the reflexivity relations between $\mathbf{R}\mathrm{Hom}_R(A, C)$ and $\mathbf{R}\mathrm{Hom}_R(B, C)$ when Lemma 3.1(a) guarantees that they are semidualizing. Example 3.10 shows that the first inequality can be an equality and the second one can be strict. The first one can also be strict: Use the guaranteed equality when R is local and $\mathrm{amp}(B) > 0$.

Proposition 3.9. *Let A, B, C be semidualizing R-complexes such that A and B are both C-reflexive and set $(-)^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(-, C)$. There are inequalities*

$$\begin{aligned} \mathrm{G}_A\text{-}\dim_R(B) - \sup(B) &\leq \mathrm{G}_{B^{\dagger_C}}\text{-}\dim_R(A^{\dagger_C}) - \inf(C) + \inf(A) \\ &\leq \mathrm{G}_A\text{-}\dim_R(B) - \inf(B) \end{aligned}$$

with equality at the second inequality when R is local or $\mathrm{amp}(B) = 0$. *In particular, B is A-reflexive if and only if $A^{\dagger_C}$ is $B^{\dagger_C}$-reflexive.*

Proof. It suffices to verify the final statement. Indeed, if B is A-reflexive and $A^{\dagger_C}$ is $B^{\dagger_C}$-reflexive, then Lemma 3.4 combined with 1.7(a) provide the desired inequalities and, when R is local or $\mathrm{amp}(B) = 0$, the equalities.

Assume that B is A-reflexive, and note that $A^{\dagger_C}$ is homologically finite since A is C-reflexive. Employ the isomorphism from 1.7(a) and the fact that B is C-reflexive to conclude that the complex $\mathbf{R}\mathrm{Hom}_R(A^{\dagger_C}, B^{\dagger_C})$ is homologically bounded. Next, consider the following commutative diagram of morphisms of complexes:

$$\begin{array}{ccc}
A^{\dagger_C} & \xrightarrow{\delta^{B^{\dagger_C}}_{A^{\dagger_C}}} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(A^{\dagger_C}, B^{\dagger_C}), B^{\dagger_C}) \\
{\scriptstyle 3.1(b)}\downarrow\simeq & & {\scriptstyle 1.7(a)}\downarrow\simeq \\
(B \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B, A))^{\dagger_C} & & \mathbf{R}\mathrm{Hom}_R(B^{\dagger_C\dagger_C}, \mathbf{R}\mathrm{Hom}_R(A^{\dagger_C}, B^{\dagger_C})^{\dagger_C}) \\
{\scriptstyle 1.6(d)}\uparrow\simeq & & \downarrow\simeq \\
\mathbf{R}\mathrm{Hom}_R(B, \mathbf{R}\mathrm{Hom}_R(B, A)^{\dagger_C}) & \underset{\simeq}{\xrightarrow{1.7(a)}} & \mathbf{R}\mathrm{Hom}_R(B, \mathbf{R}\mathrm{Hom}_R(A^{\dagger_C}, B^{\dagger_C})^{\dagger_C}).
\end{array}$$

The unmarked map $\mathbf{R}\mathrm{Hom}_R(\delta^C_B, \mathbf{R}\mathrm{Hom}_R(A^{\dagger_C}, B^{\dagger_C})^{\dagger_C})$ is an isomorphism since B is C-reflexive. Thus, $\delta^{B^{\dagger_C}}_{A^{\dagger_C}}$ is an isomorphism and $A^{\dagger_C}$ is $B^{\dagger_C}$-reflexive.

The converse follows from the isomorphisms $A \simeq A^{\dagger_C\dagger_C}$ and $B \simeq B^{\dagger_C\dagger_C}$. $\square$

Example 3.10. Here we construct a ring R with $\mathrm{Spec}(R)$ connected demonstrating the following: In Lemma 3.4, the first and fourth inequalities can be equalities and the other inequalities can be strict, and in Proposition 3.9 the first inequality can be an equality and the other inequality can be strict. Let k be a field and set

$$\begin{gathered} A_1 = k[X_1, Y_1]/(X_1^2, X_1Y_1) \qquad A_2 = k[X_2, Y_2]/(X_2^2, X_2Y_2) \\ A = A_1 \otimes_k A_2 \cong k[X_1, Y_1, X_2, Y_2]/(X_1^2, X_1Y_1, X_2^2, X_2Y_2). \end{gathered}$$

The natural maps $\varphi_i\colon A_i \to A$ are faithfully flat since they are obtained by applying $- \otimes_k A_i$ to the faithfully flat maps $k \to A_j$. For $i = 1, 2$ set $S_i = A_i \backslash (X_i, Y_i)A_i$. The local ring $R_i = S_i^{-1}A_i$ has maximal ideal $\mathfrak{m}_i = (X_i, Y_i)R_i$ and exactly one nonmaximal prime ideal $\mathfrak{p}_i = (X_i)R_i$. Let $S = A \backslash ((X_1, Y_1, X_2)A \cup (X_1, X_2, Y_2)A)$ and set $R = S^{-1}A$ which has exactly two maximal ideals $\mathfrak{n}_1 = (X_1, Y_1, X_2)R$ and $\mathfrak{n}_2 = (X_1, X_2, Y_2)R$ and exactly one nonmaximal prime ideal $\mathfrak{p} = (X_1, X_2)R$. As $\mathfrak{p} \subset \mathfrak{n}_1 \cap \mathfrak{n}_2$, $\mathrm{Spec}(R)$ is connected.

The containment $\varphi_i(S_i) \subset S$ provides faithfully flat maps ψ: $R_i \to R$. It is straightforward to verify that R is a localization of the tensor product $R_1 \otimes_k R_2$, and furthermore that ψ_i is the composition of the tensor product map $R_i \to R_1 \otimes_k R_2$ and the localization map $R_1 \otimes_k R_2 \to R$. Equally straightforward are the following:

$$\psi_1^*(\mathfrak{n}_1) = \mathfrak{m}_1 \qquad \psi_1^*(\mathfrak{n}_2) = \mathfrak{p}_1 \qquad \psi_1^*(\mathfrak{p}) = \mathfrak{p}_1$$
$$\psi_2^*(\mathfrak{n}_1) = \mathfrak{p}_2 \qquad \psi_2^*(\mathfrak{n}_2) = \mathfrak{m}_2 \qquad \psi_2^*(\mathfrak{p}) = \mathfrak{p}_1$$

In particular, if M_i is a nonzero R_i-module of finite length, then the R-module $M_i \otimes_{R_i} R$ is nonzero with finite length because $\mathrm{Supp}_R(M_i \otimes_{R_i} R) = \{\mathfrak{n}_i\}$.

Since R_i is essentially of finite type over k, it admits a normalized dualizing complex D^i. Hence, $\sup(D^i) = \dim(R_i) = 1$ and $\inf(D^i) = \mathrm{depth}(R_i) = 0$. From the structure of $\mathrm{Spec}(R_i)$, the minimal R_i-injective resolution of D^i is of the form

$$D^i \simeq 0 \to E_{R_i}(R_i/\mathfrak{p}_i) \to E_{R_i}(R_i/\mathfrak{m}_i) \to 0.$$

In particular, the R_i-module $\mathrm{H}_0(D^i)$ has nonzero finite length.

Set $C^i = D^i \otimes^{\mathbf{L}}_{R_i} R$ which is semidualizing for R by Theorem 4.5. By flatness, we have $\mathrm{H}_j(C^i) \cong \mathrm{H}_j(D^i) \otimes_{R_i} R$ for each integer j. In particular, since the R_i-module $\mathrm{H}_0(D^i)$ has nonzero finite length, the R-module $\mathrm{H}_0(C^i)$ has finite length and $\mathrm{Supp}_R(\mathrm{H}_0(C^i)) = \{\mathfrak{n}_i\}$. Nakayama's lemma implies $\mathrm{H}_0(C^1) \otimes_R \mathrm{H}_0(C^2) = 0$.

Using Lemma 1.3 and the isomorphism $C^1 \otimes^{\mathbf{L}}_R C^2 \simeq (D^1 \otimes_k D^2) \otimes^{\mathbf{L}}_{R_1 \otimes_k R_2} R$ we conclude that $C^1 \otimes^{\mathbf{L}}_R C^2$ is dualizing for R. Write $D = C^1 \otimes^{\mathbf{L}}_R C^2$. In particular, C^1, C^2 are D-reflexive, and Lemma 3.1 provides isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(C^1, D) \simeq C^2 \qquad \mathbf{R}\mathrm{Hom}_R(C^2, D) \simeq C^1.$$

We claim that $\inf(D) > 0$. Indeed, since $\inf(C^i) = 0$ for $i = 1, 2$ one has

$$\inf(D) = \inf\left(C^1 \otimes^{\mathbf{L}}_R C^2\right) > \inf(C^1) + \inf(C^2) = 0$$

where the inequality is Christensen (2000, (A.4.15)) using the last line of the previous paragraph.

We now show $\inf(D) = 1 = \mathrm{amp}(D)$. The Künneth formula $\mathrm{H}(D^1 \otimes_k D^2) = \mathrm{H}(D^1) \otimes_k \mathrm{H}(D^2)$ and the equalities $\sup(D^i) = 1$ provide the next equality

$$\sup(D) \le \sup(D^1 \otimes_k D^2) = 2$$

while the inequality is due to the fact that D is a localization of $D^1 \otimes_k D^2$. Since $\inf(D) \ge 1$, one has $0 \le \mathrm{amp}(D) \le 1$, and so it suffices to verify $\mathrm{amp}(D) \ge 1$. For this, note that the localizations $R_{\mathfrak{n}_i}$ are not Cohen–Macaulay and therefore one has $\mathrm{amp}(D) \ge \mathrm{amp}(D_{\mathfrak{n}_i}) \ge 1$. The desired computations now follow readily:

$$\inf(D) - \sup(C^1) = \inf(\mathbf{R}\mathrm{Hom}_R(C^1, D)) < \inf(D) - \inf(C^1)$$
$$\inf(C^1) < \mathrm{G}_D\text{-}\dim_R(C^1) = \sup(C^1)$$
$$\mathrm{G}_D\text{-}\dim_R(C^1) - \sup(C^1) = \mathrm{G}_{(C^1)^{\dagger_D}}\text{-}\dim_R(D^{\dagger_D}) - \inf(D) + \inf(D)$$
$$< \mathrm{G}_D\text{-}\dim_R(C^1) - \inf(C^1).$$

We next extend Christensen (2001, (2.9)). Example 5.11 shows that this inequality can be strict.

Proposition 3.11. *If C, X are homologically finite R-complexes with C semidualizing, then there is an inequality*

$$\mathrm{G}_C\text{-}\dim_R(X) \leq \mathrm{pd}_R(X)$$

with equality when $\mathrm{pd}_R(X)$ *is finite and either R is local or* $\mathrm{amp}(C) = 0$.

Proof. Assume that $\mathrm{pd}_R(X)$ is finite. The finiteness of $\mathrm{G}_C\text{-}\dim_R(X)$ is in Christensen (2001, (2.9)), and the local case of the equality is Christensen (2001, (3.5)). Proposition 2.9 provides the inequality in the following sequence:

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(X) &\leq \sup\{\mathrm{G}_{C_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{m-Spec}(R)\}\\ &= \sup\{\mathrm{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \in \text{m-Spec}(R)\}\\ &= \mathrm{pd}_R(X),\end{aligned}$$

while the first equality is by the local case and the second equality is classical.

Assume now that $\mathrm{amp}(C) = 0$. Let $P \simeq X$ be a projective resolution and set $g = \mathrm{G}_C\text{-}\dim_R(X)$. Lemma 2.1 implies that $G = \mathrm{Coker}(\partial^P_{g+1})$ is totally C-reflexive, and one checks locally (using the AB-formulas) that G is projective. □

Next we extend Christensen (2001, (3.17)). Example 4.6 shows that the inequalities can be strict. Partial converses of the final statement and conditions guaranteeing equality are in Theorems 4.4 and 5.4; to see that the converse can fail consult Example 5.14.

Proposition 3.12. *Let C, P, X be homologically finite R-complexes with C semidualizing and* $\mathrm{pd}_R(P)$ *finite. There are inequalities*

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R P\big) &\leq \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{pd}_R(P)\\ \mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(P, X)) &\leq \mathrm{G}_C\text{-}\dim_R(X) - \inf(P).\end{aligned}$$

In particular, if X is C-reflexive, then so are $X \otimes^{\mathbf{L}}_R P$ and $\mathbf{R}\mathrm{Hom}_R(P, X)$.

Proof. The final statement is proved as in Christensen (2001, (3.17)). For the inequalities, assume that the complexes X, $X \otimes^{\mathbf{L}}_R P$, and $\mathbf{R}\mathrm{Hom}_R(P, X)$ are C-reflexive. Since $\mathrm{pd}_R(P)$ is finite, adjunction and 1.7(b) yield an isomorphism

$$\mathbf{R}\mathrm{Hom}_R\big(X \otimes^{\mathbf{L}}_R P, C\big) \simeq \mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)$$

and so the following sequence provides the first inequality.

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(X \otimes^{\mathbf{L}}_R P) &= \inf(C) - \inf\big(\mathbf{R}\mathrm{Hom}_R\big(X \otimes^{\mathbf{L}}_R P, C\big)\big)\\ &= \inf(C) - \inf\big(\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big)\end{aligned}$$

$$\leq \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(P, R)) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$
$$= \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{pd}_R(P).$$

Similarly, the Hom-evaluation isomorphism gives a sequence of (in)equalities

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(P, X)) &= \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(P, X), C)) \\ &= \inf(C) - \inf\left(P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\right) \\ &\leq \inf(C) - \inf(P) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \\ &= \mathrm{G}_C\text{-}\dim_R(X) - \inf(P) \end{aligned}$$

providing the second inequality. □

4. RING HOMOMORPHISMS OF FINITE FLAT DIMENSION: BASE CHANGE

In this section we study the interaction between the semidualizing and reflexivity properties and the functor $-\otimes^{\mathbf{L}}_R S$ where φ: $R \to S$ is a ring homomorphism of finite flat dimension. We begin with a more general situation (Christensen, 2000, (A.4.15), (A.5.5)) wherein the inequalities may be strict (see Example 4.6) or not (use $P = R$).

4.1. If X, P are R-complexes such that $P \not\simeq 0$ is bounded and $\mathrm{fd}_R(P)$ is finite, then

$$\begin{aligned} \inf\left(X \otimes^{\mathbf{L}}_R P\right) &\geq \inf(X) + \inf(P) \\ \sup\left(X \otimes^{\mathbf{L}}_R P\right) &\leq \sup(X) + \mathrm{fd}_R(P) \\ \mathrm{amp}\left(X \otimes^{\mathbf{L}}_R P\right) &\leq \mathrm{amp}(X) + \mathrm{fd}_R(P) - \inf(P). \end{aligned}$$

Our nonlocal version of the amplitude inequality, based on Iversen (1977) and Foxby and Iyengar (2003, (3.1)), is next. It is Theorem I from the introduction and provides inequalities complimentary to those in 4.1. Example 4.6 shows that, without the hypothesis on m-Spec(R), bounds of this ilk and the nontrivial ensuing implications need not hold, and that the inequalities can be strict; to see that they may not be strict, use $P = R$.

Theorem 4.2. *Let P be a homologically finite S-complex with* $\mathrm{fd}_R(P)$ *finite and such that* $\varphi^*(\mathrm{Supp}_S(P))$ *contains* m-Spec(R). *For each homologically degree-wise finite R-complex X there are inequalities*

$$\begin{aligned} \inf\left(X \otimes^{\mathbf{L}}_R P\right) &\leq \inf(X) + \sup(P) \\ \sup\left(X \otimes^{\mathbf{L}}_R P\right) &\geq \sup(X) + \inf(P) \\ \mathrm{amp}\left(X \otimes^{\mathbf{L}}_R P\right) &\geq \mathrm{amp}(X) - \mathrm{amp}(P). \end{aligned}$$

In particular:

(a) $X \simeq 0$ *if and only if* $X \otimes^{\mathbf{L}}_R P \simeq 0$;

(b) *X is homologically bounded if and only if* $X \otimes_R^{\mathbf{L}} P$ *is so*;
(c) *If* $\mathrm{amp}(P) = 0$, *e.g., if* $P = S$, *then* $\inf(X \otimes_R^{\mathbf{L}} P) = \inf(X) + \inf(P)$.

Proof. For the first inequality, it suffices to verify the following implication: If $\mathrm{H}_n(X) \neq 0$, then $\inf(X \otimes_R^{\mathbf{L}} P) \leq n + \sup(P)$. Indeed, if $X \simeq 0$, that is, if $\inf(X) = \infty$, then the inequality is trivial. If $\inf(X)$ is finite, then using $n = \inf(X)$ gives the desired inequality. And if $\inf(X) = -\infty$, then taking the limit as $n \to -\infty$ gives the desired inequality.

Fix an integer n and assume that $\mathrm{H}_n(X) \neq 0$. Thus, there is a maximal ideal $\mathfrak{m} \in \mathrm{Supp}_R(\mathrm{H}_n(X)) \subseteq \mathrm{Supp}_R(X)$, and by assumption there exists a prime ideal $\mathfrak{p} \in \mathrm{Supp}_S(P)$ such that $\varphi^*(\mathfrak{p}) = \mathfrak{m}$. The local homomorphism $\varphi_{\mathfrak{p}}\colon R_{\mathfrak{m}} \to S_{\mathfrak{p}}$ and the complexes $X_{\mathfrak{m}}$ and $P_{\mathfrak{p}}$ satisfy the hypotheses of Foxby and Iyengar (2003, (3.1)), providing the second equality in the following sequence wherein the inequalities are straightforward

$$\inf\big(X \otimes_R^{\mathbf{L}} P\big) \leq \inf\big(\big(X \otimes_R^{\mathbf{L}} P\big)_{\mathfrak{p}}\big) = \inf\big(X_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}}^{\mathbf{L}} P_{\mathfrak{p}}\big) = \inf(X_{\mathfrak{m}}) + \inf(P_{\mathfrak{p}}) \leq n + \sup(P)$$

and the first equality follows from the isomorphism $\big(X \otimes_R^{\mathbf{L}} P\big)_{\mathfrak{p}} \simeq X_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}}^{\mathbf{L}} P_{\mathfrak{p}}$.

The second inequality is verified similarly. The third inequality is an immediate consequence of the first two, and statements (a), (b), and (c) follow directly. □

The proof of the next result is nearly identical to that of Iyengar and Sather-Wagstaff (2004, (2.10)), using $X = \mathrm{cone}(\alpha)$ in Theorem 4.2. Example 4.6 shows that the extra hypotheses are necessary for the nontrivial implication.

Corollary 4.3. *Let* P *be a homologically finite* S*-complex with* $\mathrm{fd}_R(P) < \infty$ *and* $\text{m-Spec}(R) \subseteq \varphi^*(\mathrm{Supp}_S(P))$. *If* α *is a morphism of homologically degree-wise finite* R*-complexes, then* α *is an isomorphism if and only if* $\alpha \otimes_R^{\mathbf{L}} P$ *is so.*

Here is a partial converse for Proposition 3.12. The first inequality can be strict: Use the guaranteed equality with R local and $\inf(P) < \mathrm{pd}_R(P)$. Example 4.6 shows that the second inequality may be strict and the first one may not.

Theorem 4.4. *Let* C, P, X *be homologically finite* R*-complexes with* C *semidualizing,* $\mathrm{pd}_R(P)$ *finite, and* $\text{m-Spec}(R)$ *contained in* $\mathrm{Supp}_R(P)$. *There are inequalities*

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R(X) + \inf(P) &\leq \mathrm{G}_C\text{-}\dim_R(X \otimes_R^{\mathbf{L}} P) \\ &\leq \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{pd}_R(P). \end{aligned}$$

In particular, the complexes X *and* $X \otimes_R^{\mathbf{L}} P$ *are* C*-reflexive simultaneously. If* R *is local or* $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(P, R)) = 0$, *then the second inequality is an equality.*

Proof. First we verify that X and $X \otimes_R^{\mathbf{L}} P$ are C-reflexive simultaneously. Theorem 4.2(b) and the isomorphism 1.7(b) imply that the complexes $\mathbf{R}\mathrm{Hom}_R(X, C)$

and $\mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R P, C)$ are homologically bounded simultaneously. The following commutative diagram from Christensen (2001, (3.17))

$$\begin{array}{ccc} X \otimes^{\mathbf{L}}_R P & \xrightarrow{\delta^C_{X\otimes^{\mathbf{L}}_R P}} & \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R\big(X \otimes^{\mathbf{L}}_R P, C\big), C\big) \\ {\scriptstyle \delta^C_X\otimes^{\mathbf{L}}_R P}\big\downarrow & & \big\uparrow {\scriptstyle 1.6(\mathrm{d})}\,\simeq \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C) \otimes^{\mathbf{L}}_R P & \xrightarrow[1.6(\mathrm{g})]{\simeq} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(P, \mathbf{R}\mathrm{Hom}_R(X, C)), C) \end{array}$$

shows that $\delta^C_{X\otimes^{\mathbf{L}}_R P}$ and $\delta^C_X \otimes^{\mathbf{L}}_R P$ are isomorphisms simultaneously. Corollary 4.3 then implies that $\delta^C_{X\otimes^{\mathbf{L}}_R P}$ and δ^C_X are isomorphisms simultaneously.

For the (in)equalities, we assume that X and $X \otimes^{\mathbf{L}}_R P$ are C-reflexive. The first inequality is verified in the next sequence where (1) is by definition and 1.7(b)

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R P\big) &\overset{(1)}{=} \inf(C) - \inf\big(\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big) \\ &\overset{(2)}{\geq} \inf(C) - \sup(\mathbf{R}\mathrm{Hom}_R(P, R)) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \\ &\overset{(3)}{\geq} \mathrm{G}_C\text{-}\dim_R(X) - \mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(P, R)) \\ &\overset{(4)}{=} \mathrm{G}_C\text{-}\dim_R(X) + \inf(P) \end{aligned}$$

(2) is by Theorem 4.2, (3) is standard, and (4) is by 1.4. The second inequality is in Proposition 3.12. When $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(P, R)) = 0$, there is an equality

$$\inf\big(\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big) = \inf(\mathbf{R}\mathrm{Hom}_R(P, R)) + \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$

by Theorem 4.2(c); the same equality holds by Nakayama's Lemma when R is local. Thus, under either of these hypotheses, the displayed sequence in the proof of Proposition 3.12 gives the desired equality. □

The next result contains Theorem II(a) from the introduction. Example 4.6 shows that the converse of the first implication can fail. To see that the inequalities can be strict, consult Example 5.11. For equality, use $C = R$.

Theorem 4.5. *Assume that* $\mathrm{fd}(\varphi)$ *is finite and* C *is a homologically degree-wise finite* R*-complex. If* C *is* R*-semidualizing, then* $C \otimes^{\mathbf{L}}_R S$ *is* S*-semidualizing with*

$$\inf\big(C \otimes^{\mathbf{L}}_R S\big) \geq \inf(C) \quad \textit{and} \quad \mathrm{amp}\big(C \otimes^{\mathbf{L}}_R S\big) \leq \mathrm{amp}(C).$$

Conversely, if $C \otimes^{\mathbf{L}}_R S$ *is* S*-semidualizing and* $\mathrm{Im}(\varphi^*)$ *contains* $\text{m-Spec}(R)$*, then* C *is* R*-semidualizing and* $\inf\big(C \otimes^{\mathbf{L}}_R S\big) = \inf(C)$*.*

Proof. The first implication and the inequalities are in Christensen (2001, (1.3.4), (5.1)). Assume that $C \otimes^{\mathbf{L}}_R S$ is S-semidualizing and $\text{m-Spec}(R) \subseteq \mathrm{Im}(\varphi^*)$. The equality is in Theorem 4.2(c), and Theorem 4.2(b) implies that C is homologically finite over R. The following commutative diagram shows that $\chi_C^R \otimes_R^{\mathbf{L}} S$ is an isomorphism

$$\begin{array}{ccc} S & \xrightarrow[\simeq]{\chi^S_{C\otimes^{\mathbf{L}}_R S}} & \mathbf{R}\mathrm{Hom}_S\big(C \otimes_R^{\mathbf{L}} S, C \otimes_R^{\mathbf{L}} S\big) \\ {\scriptstyle 1.6(\mathrm{a})}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle 1.9(\mathrm{a})}\big\downarrow{\scriptstyle \simeq} \\ R \otimes_R^{\mathbf{L}} S & \xrightarrow{\chi^R_C \otimes^{\mathbf{L}}_R S} & \mathbf{R}\mathrm{Hom}_R(C, C) \otimes_R^{\mathbf{L}} S \end{array}$$

and Corollary 4.3 implies that χ_C^R is an isomorphism as well. □

Example 4.6. We show: (1) the implication in Theorem 4.2(c) can fail when $\mathrm{amp}(P) > 0$; (2) the nontrivial implications in Theorem 4.2(a) and Corollary 4.3 can fail in the absence of the hypothesis on $\mathrm{Supp}_S(P)$; and (3) the first inequality in Proposition 3.12 and the second inequality in Theorem 4.4 can be strict, while equality can occur in the first inequality in Theorem 4.4.

Let $R = k[Y]$. Setting $P^1 = R/(Y-1) \oplus \Sigma R \oplus \Sigma^2 R/Y$ and $X^1 = R/(Y)$, one has $X^1 \otimes_R^{\mathbf{L}} P^1 \simeq \Sigma R/(Y)$, and so one verifies (1) from the next computations:

$$\inf(X^1) = \sup(X^1) = \mathrm{amp}(X^1) = 0$$
$$\inf(P^1) = 0 \qquad \sup(P^1) = \mathrm{amp}(P^1) = 2 \qquad \mathrm{fd}_R(P^1) = 3$$
$$\inf(X^1 \otimes_R^{\mathbf{L}} P^1) = \sup(X^1 \otimes_R^{\mathbf{L}} P^1) = 1 \qquad \mathrm{amp}(X^1 \otimes_R^{\mathbf{L}} P^1) = 0.$$

For (2) let $\alpha\colon R \to R/(Y) \oplus R$ be the natural map and $P^2 = R/(Y-1)$. It is straightforward to check that $\alpha \otimes_R^{\mathbf{L}} P^2$ is an isomorphism, even though α is not. Furthermore, with $X^2 = \mathrm{cone}(\alpha)$ one has $X^2 \otimes_R^{\mathbf{L}} P^2 \simeq 0$ while $X^2 \not\simeq 0$. With $X^3 = R \oplus \big(\bigoplus_{i\in\mathbb{Z}} R/(Y)\big)$, the complex $X^3 \otimes_R^{\mathbf{L}} P^2 \simeq P^2$ is homologically bounded, even though X^3 is not.

For (3), if $P^3 = R \oplus R/(Y-1)$, then $X^1 \otimes_R^{\mathbf{L}} P^3 \simeq R/(Y)$ and so

$$\mathrm{G}_C\text{-}\mathrm{dim}_R(X^1) + \inf(P^3) = \mathrm{G}_C\text{-}\mathrm{dim}_R\big(X^1 \otimes_R^{\mathbf{L}} P^3\big) < \mathrm{G}_C\text{-}\mathrm{dim}_R(X^1) + \mathrm{pd}_R(P^3).$$

Set $S = R/(Y)$ with $\varphi\colon R \to S$ the natural surjection. The module P^3 is not R-semidualizing, even though $P^3 \otimes_R^{\mathbf{L}} S \simeq S$ is S-semidualizing.

Next we refine the ascent property (Christensen, 2001, (5.10)). When φ is local, Theorem 4.8 shows that this inequality can be strict (if $\mathrm{amp}(C) > 0$) or not (if $\mathrm{amp}(C) = 0$). Example 5.14 shows that the converse to the final statement need not hold.

Proposition 4.7. *Assume that* $\mathrm{fd}(\varphi)$ *is finite, and let* C, X *be homologically finite* R-*complexes such that* C *is* R-*semidualizing. There is an inequality*

$$\mathrm{G}_{C\otimes_R^{\mathbf{L}} S}\text{-}\mathrm{dim}_S\big(X \otimes_R^{\mathbf{L}} S\big) \leq \mathrm{amp}(C) + \mathrm{G}_C\text{-}\mathrm{dim}_R(X).$$

In particular, if X *is* C-*reflexive, then* $X \otimes_R^{\mathbf{L}} S$ *is* $C \otimes_R^{\mathbf{L}} S$-*reflexive.*

Proof. The last statement is in Christensen (2001, (5.10)), so assume that $\mathrm{G}_C\text{-}\dim_R(X)$ and $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S(X\otimes^{\mathbf{L}}_R S)$ are finite. In the following sequence

$$\begin{aligned}\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S\bigl(X\otimes^{\mathbf{L}}_R S\bigr) &= \inf\bigl(C\otimes^{\mathbf{L}}_R S\bigr)-\inf\bigl(\mathbf{R}\mathrm{Hom}_S\bigl(X\otimes^{\mathbf{L}}_R S, C\otimes^{\mathbf{L}}_R S\bigr)\bigr)\\ &\le \sup(C)-\inf(\mathbf{R}\mathrm{Hom}_R(X,C))\\ &= \mathrm{amp}(C)+\inf(C)-\inf(\mathbf{R}\mathrm{Hom}_R(X,C))\\ &= \mathrm{amp}(C)+\mathrm{G}_C\text{-}\dim_R(X)\end{aligned}$$

the equalities are routine, and the inequality follows from 1.9(a) and 4.1. □

The following descent result is Theorem II(b) from the introduction.

Theorem 4.8. *Let C, X be homologically degree-wise finite R-complexes with C semidualizing. If $\mathrm{fd}(\varphi)$ is finite and $\mathrm{Im}(\varphi^*)$ contains $\text{m-Spec}(R)$, then*

$$\mathrm{G}_C\text{-}\dim_R(X)=\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S\bigl(X\otimes^{\mathbf{L}}_R S\bigr).$$

In particular, $X\otimes^{\mathbf{L}}_R S$ is $C\otimes^{\mathbf{L}}_R S$-reflexive if and only if X is C-reflexive.

Proof. One implication is in Proposition 4.7, so assume that $X\otimes^{\mathbf{L}}_R S$ is $C\otimes^{\mathbf{L}}_R S$-reflexive. Theorem 4.2(b) and 1.9(a) imply that X and $\mathbf{R}\mathrm{Hom}_R(X,C)$ are homologically bounded. With Corollary 4.3 the commutative diagram from Christensen (2001, (5.10))

$$\begin{array}{ccc} X\otimes^{\mathbf{L}}_R S & \xrightarrow[\simeq]{\delta^{C\otimes^{\mathbf{L}}_R S}_{X\otimes^{\mathbf{L}}_R S}} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(X\otimes^{\mathbf{L}}_R S, C\otimes^{\mathbf{L}}_R S), C\otimes^{\mathbf{L}}_R S)\\ & & \uparrow{\scriptstyle 1.9(\mathrm{a})}\ \simeq\\ {=}\downarrow & & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X,C)\otimes^{\mathbf{L}}_R S, C\otimes^{\mathbf{L}}_R S)\\ & & \uparrow{\scriptstyle 1.9(\mathrm{a})}\ \simeq\\ X\otimes^{\mathbf{L}}_R S & \xrightarrow{\delta^C_X\otimes^{\mathbf{L}}_R S} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,C),C)\otimes^{\mathbf{L}}_R S\end{array}$$

shows that δ^C_X is an isomorphism, and so X is C-reflexive.

Assuming that $\mathrm{G}_C\text{-}\dim_R(X)$ and $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S(X\otimes^{\mathbf{L}}_R S)$ are finite, one has

$$\begin{aligned}\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S\bigl(X\otimes^{\mathbf{L}}_R S\bigr) &= \inf\bigl(C\otimes^{\mathbf{L}}_R S\bigr)-\inf\bigl(\mathbf{R}\mathrm{Hom}_S\bigl(X\otimes^{\mathbf{L}}_R S, C\otimes^{\mathbf{L}}_R S\bigr)\bigr)\\ &= \inf(C)-\inf(\mathbf{R}\mathrm{Hom}_R(X,C))\\ &= \mathrm{G}_C\text{-}\dim_R(X)\end{aligned}$$

where the second equality is from Theorem 4.2(c) and 1.9(a). □

Here is Theorem II(c) from the introduction. It uses the functor $\mathrm{Pic}(-)$; see 1.11. The conclusion fails outright if C, C' are not semidualizing by

Example 4.10. Note that the injectivity of $\mathrm{Pic}(\varphi)$ in the hypotheses is not automatic, even when φ is faithfully flat (Fossum, 1973, (11.8)), unless φ is local or surjective; see Proposition 4.11. In fact, the inclusion $\mathrm{Pic}(R) \subseteq \mathfrak{S}(R)$ shows that this condition is necessary.

Theorem 4.9. *Assume that* $\mathrm{fd}(\varphi)$ *is finite,* $\mathrm{Im}(\varphi^*) \supseteq \text{m-Spec}(R)$, *and* $\mathrm{Pic}(\varphi)$ *is injective. If* C, C' *are R-semidualizing and* $C \otimes^{\mathbf{L}}_R S \simeq C' \otimes^{\mathbf{L}}_R S$, *then* $C \simeq C'$.

Proof. By Theorem 4.8 the isomorphism $C \otimes^{\mathbf{L}}_R S \simeq C' \otimes^{\mathbf{L}}_R S$ implies that C is C'-reflexive and vice versa. It follows from Lemma 3.2 that, for each $\mathfrak{m} \in \text{m-Spec}(R)$, there is an isomorphism $\mathbf{R}\mathrm{Hom}_R(C', C)_{\mathfrak{m}} \sim R_{\mathfrak{m}}$. The isomorphisms

$$R \otimes^{\mathbf{L}}_R S \simeq S \simeq \mathbf{R}\mathrm{Hom}_S\big(C' \otimes^{\mathbf{L}}_R S, C \otimes^{\mathbf{L}}_R S\big) \simeq \mathbf{R}\mathrm{Hom}_R(C', C) \otimes^{\mathbf{L}}_R S \qquad (\dagger)$$

along with Theorem 4.2, explain the following inequalities:

$$0 = \mathrm{amp}\big(\mathbf{R}\mathrm{Hom}_R(C', C) \otimes^{\mathbf{L}}_R S\big) \geq \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(C', C)) \geq 0.$$

Thus, $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(C', C)) = 0$ and $\mathbf{R}\mathrm{Hom}_R(C', C)_{\mathfrak{m}} \simeq \Sigma^i R_{\mathfrak{m}}$ for each $\mathfrak{m} \in \text{m-Spec}(R)$, where $i = \inf(\mathbf{R}\mathrm{Hom}_R(C', C))$. In other words, $\mathbf{R}\mathrm{Hom}_R(C', C) \simeq \Sigma^i L$ where $[L] \in \mathrm{Pic}(R)$. The isomorphisms $(\dagger)$ imply $S \simeq \Sigma^i L \otimes^{\mathbf{L}}_R S \simeq \Sigma^i L \otimes_R S$ and so $i = 0$. Applying $(\dagger)$ again yields $\mathrm{Pic}(\varphi)([L]) = [S] = \mathrm{Pic}(\varphi)([R])$ so the injectivity of $\mathrm{Pic}(\varphi)$ implies $L \cong R$. Hence, $\mathbf{R}\mathrm{Hom}_R(C', C) \simeq R$ and thus

$$C' \simeq R \otimes^{\mathbf{L}}_R C' \simeq \mathbf{R}\mathrm{Hom}_R(C', C) \otimes^{\mathbf{L}}_R C' \simeq C$$

where the last isomorphism is from Lemma 3.1(b). □

Example 4.10. The conclusions of Lemma 1.10 and Theorem 4.9 fail if φ is not flat and if the complexes are not semidualizing. Set $R = k[\![Y, Z]\!]$ and $S = R/(Y, Z)$ with $\varphi\colon R \to S$ the surjection. The complexes $C = R/(Y)$ and $C' = R/(Z)$ satisfy $C \otimes^{\mathbf{L}}_R S \simeq S \oplus \Sigma S \simeq C' \otimes^{\mathbf{L}}_R S$ and $C \not\simeq C'$.

Proposition 4.11. *If* φ *is surjective with* $\mathrm{fd}(\varphi)$ *finite and* $\text{m-Spec}(R)$ *is contained in* $\mathrm{Im}(\varphi^*)$, *then* $\mathrm{Pic}(\varphi)$ *is injective.*

Proof. Set $I = \mathrm{Ker}(\varphi)$ so that $S \cong R/I$, and note that our hypothesis on φ implies that the Jacobson radical of R contains I. Let L be a finitely generated rank 1 projective R-module such that $S \cong L \otimes_R S \cong L/IL$. Fix an element $x \in L$ whose residue in L/IL is a generator and let $\alpha\colon R \to L$ be given by $1 \mapsto x$. By construction, the induced map $\alpha \otimes_R S\colon S \to L \otimes_R S$ is bijective. Since L is a projective R-module, this says that the morphism $\alpha \otimes^{\mathbf{L}}_R S\colon S \to L \otimes^{\mathbf{L}}_R S$ is an isomorphism. By Corollary 4.3 it follows that α is also an isomorphism. □

5. FINITE RING HOMOMORPHISMS OF FINITE FLAT DIMENSION: COBASE CHANGE

Here we study the relation between the semidualizing and reflexivity properties and the functor $\mathbf{R}\mathrm{Hom}_R(S, -)$ where $\varphi\colon R \to S$ is a module-finite ring homomorphism of finite flat dimension. We begin with results that follow directly from 4.1–4.3 using 1.4 and 1.7(b); their limitations are shown by the same examples.

5.1. If X, P are R-complexes such that $\mathrm{H}(P) \neq 0$ is finite and $\mathrm{pd}_R(P) < \infty$, then

$$\begin{aligned}\inf(\mathbf{R}\mathrm{Hom}_R(P,X)) &\geq \inf(X) - \mathrm{pd}_R(P)\\ \sup(\mathbf{R}\mathrm{Hom}_R(P,X)) &\leq \sup(X) - \inf(P)\\ \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(P,X)) &\leq \mathrm{amp}(X) + \mathrm{pd}_R(P) - \inf(P).\end{aligned}$$

Corollary 5.2. *Let P be a homologically finite R-complex with $\mathrm{pd}_R(P)$ finite and such that $\mathrm{Supp}_R(P)$ contains $\text{m-Spec}(R)$. For each homologically degree-wise finite R-complex X, there are inequalities*

$$\begin{aligned}\inf(\mathbf{R}\mathrm{Hom}_R(P,X)) &\leq \inf(X) + \sup(\mathbf{R}\mathrm{Hom}_R(P,R))\\ \sup(\mathbf{R}\mathrm{Hom}_R(P,X)) &\geq \sup(X) + \inf(\mathbf{R}\mathrm{Hom}_R(P,R))\\ \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(P,X)) &\geq \mathrm{amp}(X) - \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(P,R)).\end{aligned}$$

In particular:

(a) $X \simeq 0$ *if and only if* $\mathbf{R}\mathrm{Hom}_R(P,X) \simeq 0$;
(b) X *is homologically bounded if and only if* $\mathbf{R}\mathrm{Hom}_R(P,X)$ *is so*;
(c) *If* $\mathrm{amp}((\mathbf{R}\mathrm{Hom}_R(P,R)) = 0$, *then the first inequality is an equality.*

Corollary 5.3. *Let P be a homologically finite R-complex with $\mathrm{pd}_R(P)$ finite and such that $\mathrm{Supp}_R(P)$ contains $\text{m-Spec}(R)$. If α is a morphism of homologically degree-wise finite R-complexes, then α is an isomorphism if and only if the induced morphism $\mathbf{R}\mathrm{Hom}_R(P,\alpha)$ is an isomorphism.*

Here is a partial converse for Proposition 3.12. For strictness in the first inequality, use the guaranteed equality with R local and $\mathrm{amp}(P) > 0$. Example 4.6 shows other limitations.

Theorem 5.4. *Let C, P, X be homologically finite R-complexes with C semidualizing, $\mathrm{pd}_R(P)$ finite, and $\text{m-Spec}(R)$ contained in $\mathrm{Supp}_R(P)$. There are inequalities*

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(X) - \sup(P) &\leq \mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(P,X))\\ &\leq \mathrm{G}_C\text{-}\dim_R(X) - \inf(P).\end{aligned}$$

In particular, the complexes X and $\mathbf{R}\mathrm{Hom}_R(P,X)$ are C-reflexive simultaneously. If R is local or $\mathrm{amp}(P) = 0$, then the second inequality is an equality.

Proof. Set $(-)^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(-,C)$. First we verify that X and $\mathbf{R}\mathrm{Hom}_R(P,X)$ are C-reflexive simultaneously. Theorem 4.2(b) and the Hom-evaluation isomorphism

$$\mathbf{R}\mathrm{Hom}_R(P,X)^{\dagger_C} \simeq P \otimes^{\mathbf{L}}_R X^{\dagger_C} \qquad (\dagger)$$

show that the complexes $\mathbf{R}\mathrm{Hom}_R(P, X)^{\dagger_C}$ and $X^{\dagger_C}$ are homologically bounded simultaneously. The following commutative diagram from Christensen (2001, (3.17))

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(P, X) & \xrightarrow{\delta^C_{\mathbf{R}\mathrm{Hom}_R(P,X)}} & \mathbf{R}\mathrm{Hom}_R(P, X)^{\dagger_C\dagger_C} \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(P,\delta^C_X)}\big\downarrow & & \simeq\big\downarrow{\scriptstyle 1.6(\mathrm{g})} \\
\mathbf{R}\mathrm{Hom}_R(P, X^{\dagger_C\dagger_C}) & \xrightarrow[1.6(\mathrm{d})]{\simeq} & (P \otimes^{\mathbf{L}}_R X^{\dagger_C})^{\dagger_C}
\end{array}$$

implies that $\delta^C_{\mathbf{R}\mathrm{Hom}_R(P,X)}$ and $\mathbf{R}\mathrm{Hom}_R(P, \delta^K_X)$ are isomorphisms simultaneously. Corollary 5.3 shows that the same is true for $\delta^C_{\mathbf{R}\mathrm{Hom}_R(P,X)}$ and δ^C_X.

Assume that X and $\mathbf{R}\mathrm{Hom}_R(P, X)$ are C-reflexive. The second inequality is in Proposition 3.12. The first inequality is verified in the following sequence

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(P, X)) &\overset{(1)}{=} \inf(C) - \inf\big(P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big) \\
&\overset{(2)}{\geq} \inf(C) - \sup(P) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C)) \\
&= \mathrm{G}_C\text{-}\dim_R(X) - \sup(P)
\end{aligned}$$

where (1) is by isomorphism (†), and (2) is by Theorem 4.2. If $\mathrm{amp}(P) = 0$, then

$$\inf\big(P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big) = \inf(\mathbf{R}\mathrm{Hom}_R(P, R)) + \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$

by Theorem 4.2(c); the same equality holds by Nakayama's Lemma if R is local. Thus, under either of these hypotheses, the displayed sequence in the proof of Proposition 3.12 gives the desired equality. □

Example 4.6 shows how the converse of the first implication of the next result can fail. If R is local and $\mathrm{amp}(C) = 0$, then the second and third inequalities are strict if and only if $\mathrm{pd}_R(S) > 0$. We do not know if the first inequality can be strict.

Theorem 5.5. *Assume that φ is module-finite with* $\mathrm{fd}(\varphi)$ *finite and C is a homologically degree-wise finite R-complex. If C is R-semidualizing, then* $\mathbf{R}\mathrm{Hom}_R(S, C)$ *is S-semidualizing and*

$$\inf(C) - \mathrm{pd}_R(S) \leq \inf(\mathbf{R}\mathrm{Hom}_R(S, C)) \leq \sup(C)$$

with equality on the left if R is local or $\mathrm{amp}(C) = 0$. *Conversely, if* $\mathbf{R}\mathrm{Hom}_R(S, C)$ *is S-semidualizing and* $\text{m-Spec}(R) \subseteq \mathrm{Im}(\varphi^*)$, *then C is R-semidualizing and*

$$\inf(\mathbf{R}\mathrm{Hom}_R(S, C)) \leq \inf(C) + \sup(\mathbf{R}\mathrm{Hom}_R(S, R))$$

with equality if $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, R)) = 0$.

Proof. First, assume that C is R-semidualizing. Mimic the proof of Christensen (2001, (6.1)) to show that $\mathbf{R}\mathrm{Hom}_R(S, C)$ is S-semidualizing. The first inequality and conditional equality follow immediately from Proposition 3.11. The second inequality is a consequence of 5.1 since $\inf(\mathbf{R}\mathrm{Hom}_R(S, C)) \leq \sup(\mathbf{R}\mathrm{Hom}_R(S, C))$.

Next, assume that $\mathbf{R}\mathrm{Hom}_R(S, C)$ is S-semidualizing and m-Spec$(R) \subseteq \mathrm{Im}(\varphi^*)$. Corollary 5.2(b) implies that C is homologically finite. The commutative diagram

$$\begin{array}{ccc} S & \xrightarrow[\simeq]{\chi^S_{\mathbf{R}\mathrm{Hom}_R(S,C)}} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S, C), \mathbf{R}\mathrm{Hom}_R(S, C)) \\ \simeq\downarrow & & \simeq\downarrow 1.9(b) \\ S \otimes^{\mathbf{L}}_R R & \xrightarrow{S\otimes^{\mathbf{L}}_R \chi^R_C} & S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, C) \end{array}$$

shows that $S \otimes^{\mathbf{L}}_R \chi^R_C$ is an isomorphism and Corollary 4.3 implies the same for χ^R_C. The last (in)equality is in Corollary 5.2. $\square$

Part (a) of the next result says, if $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, R)) = 0 = \mathrm{amp}(C)$, then $\mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, C)) = 0$.

Proposition 5.6. *Let C be a semidualizing R-module, and assume that φ is surjective and Cohen–Macaulay of grade d.*

(a) $\mathrm{Ext}^d_R(S, C)$ *is S-semidualizing and* $\mathrm{Ext}^i_R(S, C) = 0$ *for each $i \neq d$.*
(b) *If φ is Gorenstein, then the S-module* $\mathrm{Ext}^d_R(S, R)$ *is locally free of rank* 1.

Proof. (a) Let $\mathfrak{q} \subset S$ be prime and set $\mathfrak{p} = \varphi^*(\mathfrak{q})$ and $I = \mathrm{Ker}(\varphi)$. The S-complex $\mathbf{R}\mathrm{Hom}_R(S, C)$ is semidualizing by Theorem 5.5, so it suffices to show $\mathrm{Ext}^j_R(S, C)_\mathfrak{q} = 0$ for $j \neq d$. There is an $R_\mathfrak{p}$-sequence $\mathbf{y} \in I_\mathfrak{p}$ of length $d = \mathrm{grade}_{R_\mathfrak{p}}(S_\mathfrak{q})$. Since $C_\mathfrak{p}$ is $R_\mathfrak{p}$-semidualizing, $\mathbf{y}$ is also $C_\mathfrak{p}$-regular, and thus $\mathrm{Ext}^j_R(S, C)_\mathfrak{q} = \mathrm{Ext}^j_{R_\mathfrak{p}}(S_\mathfrak{q}, C_\mathfrak{p}) = 0$ for $j < d$. Also, $d = \mathrm{pd}_{R_\mathfrak{p}}(S_\mathfrak{q})$ implies $\mathrm{Ext}^j_R(S, C)_\mathfrak{p} = 0$ for $j > d$.

Part (b) follows from (a) and the definition of a Gorenstein homomorphism. $\square$

When φ is local, Theorem 5.8 shows that the next inequality can be strict (if $\mathrm{pd}_R(S) > 0$) or not (if $\mathrm{amp}(C) = 0 = \mathrm{pd}_R(S)$). Example 5.14 shows that the converse to the final statement need not hold.

Theorem 5.7. *Assume that φ is module-finite with* $\mathrm{fd}(\varphi) < \infty$, *and let C, X be homologically finite R-complexes with C semidualizing. There is an inequality*

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(\mathbf{R}\mathrm{Hom}_R(S, X)) \leq \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{amp}(C).$$

In particular, if X is C-reflexive, then $\mathbf{R}\mathrm{Hom}_R(S, X)$ *is* $\mathbf{R}\mathrm{Hom}_R(S, C)$*-reflexive.*

Proof. Set $(-)(\varphi) = \mathbf{R}\mathrm{Hom}_R(S, -)$. It suffices to verify the final statement. Indeed, if $\mathrm{G}_{C(\varphi)}\text{-}\dim_S(X(\varphi))$ and $\mathrm{G}_C\text{-}\dim_R(X)$ are both finite, then Theorem 5.5 and 1.9(b) explain (2) below

$$\begin{aligned} \mathrm{G}_{C(\varphi)}\text{-}\dim_S(X(\varphi)) &\overset{(1)}{=} \inf(C(\varphi)) - \inf(\mathbf{R}\mathrm{Hom}_S(X(\varphi), C(\varphi))) \\ &\overset{(2)}{\leq} \sup(C) - \inf\big(S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C)\big) \end{aligned}$$

$$\overset{(3)}{\leq} \sup(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$
$$\overset{(4)}{=} \mathrm{amp}(C) + \mathrm{G}_C\text{-}\dim_R(X)$$

while (1) and (4) are by definition, and (3) follows from 4.1.

Assume now that X is C-reflexive. The complex $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded below, so 1.9(b) and 4.1 imply the same for $\mathbf{R}\mathrm{Hom}_S(X(\varphi), C(\varphi))$. The commutative diagram shows that the biduality morphism $\delta_{X(\varphi)}^{C(\varphi)}$ is an isomorphism

$$\begin{array}{ccc}
X(\varphi) & \xrightarrow{\delta_{X(\varphi)}^{C(\varphi)}} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(X(\varphi), C(\varphi)), C(\varphi)) \\
 & & \Big\downarrow{\scriptstyle 1.9(\mathrm{b})}\ \simeq \\
\Big\| & & \mathbf{R}\mathrm{Hom}_S(S \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(X, C), \mathbf{R}\mathrm{Hom}_R(S, C)) \\
 & & \Big\uparrow{\scriptstyle 1.6(\mathrm{d})}\ \simeq \\
\mathbf{R}\mathrm{Hom}_R(S, X) & \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}_R(S, \delta_X^C)} & \mathbf{R}\mathrm{Hom}_R(S, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C))
\end{array}$$

and it follows that $X(\varphi)$ is $C(\varphi)$-reflexive. □

When φ local the first inequality in our next result can be strict (if $\mathrm{pd}_R(S) > 0$) or not (if $\mathrm{pd}_R(S) = 0$). We do not know if the second inequality can be strict.

Theorem 5.8. *Let C, X be homologically degree-wise finite over R with C semidualizing. If φ is module-finite with* $\mathrm{fd}(\varphi) < \infty$ *and* $\text{m-Spec}(R) \subseteq \mathrm{Im}(\varphi^*)$, *then*

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(\mathbf{R}\mathrm{Hom}_R(S, X)) \leq \mathrm{G}_C\text{-}\dim_R(X)$$
$$\leq \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(\mathbf{R}\mathrm{Hom}_R(S, X)) + \mathrm{pd}_R(S).$$

Thus, $\mathbf{R}\mathrm{Hom}_R(S, X)$ *is* $\mathbf{R}\mathrm{Hom}_R(S, C)$*-reflexive if and only if X is C-reflexive. If either R is local or* $\mathrm{amp}(C) = 0$, *then the second inequality is an equality.*

Proof. Set $(-)(\varphi) = \mathbf{R}\mathrm{Hom}_R(S, -)$. First, we assume that $X(\varphi)$ is $C(\varphi)$-reflexive and prove that X is C-reflexive; the converse is in Theorem 5.7. The complexes $X(\varphi)$ and $S \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(X, C)$ are homologically finite by 1.9(b). Theorem 4.2(b) and Corollary 5.2(b) imply the same for $\mathbf{R}\mathrm{Hom}_R(X, C)$ and X. In the commutative diagram from the proof of Theorem 5.7, the morphism $\delta_{X(\varphi)}^{C(\varphi)}$ is an isomorphism, hence so are $\mathbf{R}\mathrm{Hom}_R(S, \delta_X^C)$ and δ_X^C by Corollary 5.3.

Now assume $\mathrm{G}_C\text{-}\dim_R(X), \mathrm{G}_{C(\varphi)}\text{-}\dim_S(X(\varphi)) < \infty$. The first desired inequality follows from the numbered sequence in the proof of Theorem 5.7 because $\inf(C(\varphi)) \leq \inf(C)$ by Theorem 5.5. The second inequality is in the next sequence:

$$\mathrm{G}_C\text{-}\dim_R(X) \overset{(1)}{=} \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$
$$\overset{(2)}{=} \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_S(X(\varphi), C(\varphi)))$$

$$\stackrel{(3)}{\leq} \mathrm{pd}_R(S) + \inf(C(\varphi)) - \inf(\mathbf{R}\mathrm{Hom}_S(X(\varphi), C(\varphi)))$$
$$\stackrel{(4)}{=} \mathrm{G}_{C(\varphi)}\text{-}\dim_S(X(\varphi)) + \mathrm{pd}_R(S).$$

(1) and (4) are by definition, (2) is from 1.9(b) and Theorem 4.2(c), and (3) is in Theorem 5.5. If $\mathrm{amp}(C) = 0$ or R is local, (3) is an equality by Proposition 3.11. □

Here is a version of Theorem 4.9 for $\mathbf{R}\mathrm{Hom}_R(S, -)$; its proof is almost identical, using Theorem 5.8 in place of Theorem 4.8, and the isomorphism 1.9(b).

Theorem 5.9. *Assume that φ is module-finite with $\mathrm{fd}(\varphi) < \infty$, $\mathrm{Im}(\varphi^*)$ contains $\text{m-Spec}(R)$, and $\mathrm{Pic}(\varphi)$ is injective. If C, C' are semidualizing R-complexes such that $\mathbf{R}\mathrm{Hom}_R(S, C) \simeq \mathbf{R}\mathrm{Hom}_R(S, C')$, then $C \simeq C'$.*

Each inequality in the next result can be strict: for the first and third, let φ be local and $\mathrm{pd}_R(S) > 0$, and use the equality; for the second, see Example 5.11. To see that each one can be an equality, take $\mathrm{amp}(C) = 0 = \mathrm{pd}_R(S)$.

Proposition 5.10. *Let C be a semidualizing R-complex and X a homologically finite S-complex. If φ is module-finite and $\mathrm{fd}(\varphi) < \infty$, then there are inequalities*

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(X) - \mathrm{amp}(C) \leq \mathrm{G}_C\text{-}\dim_R(X)$$
$$\leq \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(X) + \mathrm{pd}_R(S)$$

with equality on the right if R is local or $\mathrm{amp}(C) = 0$. In particular, the complex X is simultaneously C-reflexive and $\mathbf{R}\mathrm{Hom}_R(S, C)$-reflexive. If $\mathrm{Im}(\varphi^*)$ contains $\text{m-Spec}(R)$, then $\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S(X) \leq \mathrm{G}_C\text{-}\dim_R(X)$.

Proof. Set $(-)(\varphi) = \mathbf{R}\mathrm{Hom}_R(S, -)$. Simultaneous reflexivity is proved in Christensen (2001, (6.5)), so assume that X is C-reflexive and $C(\varphi)$-reflexive. In the next sequence:

$$\mathrm{G}_{C(\varphi)}\text{-}\dim_S(X) \stackrel{(1)}{=} \inf(C(\varphi)) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$
$$\stackrel{(2)}{\leq} \sup(C) - \inf(\mathbf{R}\mathrm{Hom}_R(X, C))$$
$$\stackrel{(3)}{=} \mathrm{amp}(C) + \mathrm{G}_C\text{-}\dim_R(X).$$

(1) is by adjunction, (2) is by Theorem 5.5, and (3) is by definition. This is the first inequality. For the second inequality, start with adjunction in (4)

$$\mathrm{G}_C\text{-}\dim_R(X) \stackrel{(4)}{=} \inf(C) - \inf(\mathbf{R}\mathrm{Hom}_S(X, C(\varphi)))$$
$$\stackrel{(5)}{\leq} \mathrm{pd}_R(S) + \inf(C(\varphi)) - \inf(\mathbf{R}\mathrm{Hom}_S(X, C(\varphi)))$$
$$\stackrel{(6)}{=} \mathrm{pd}_R(S) + \mathrm{G}_{C(\varphi)}\text{-}\dim_S(X)$$

while Theorem 5.5 yields (5), and (6) is by definition. If R is local or $\operatorname{amp}(C) = 0$, then (5) is an equality by Theorem 5.5 and thus so is the second inequality.

If $\text{m-Spec}(R) \subseteq \operatorname{Im}(\varphi^*)$, then Corollary 5.2 gives $\inf(C(\varphi)) \leq \inf(C)$. Using this in (2) above gives the third inequality. □

Example 5.11. Certain inequalities in Propositions 2.9, 3.11, and 5.10 and in Theorems 4.5 and 5.5 can be strict, even when $\operatorname{Spec}(R)$ and $\operatorname{Spec}(S)$ are connected. Let k be a field and set

$$A = k[X, Y, Z]/(Y^2, YZ) \qquad U = A \backslash ((X, Y)A \cup (Y, Z)A) \qquad R = U^{-1}A.$$

The ring R has two maximal ideals and one nonmaximal prime ideal

$$\mathfrak{m} = (X, Y)R \qquad \mathfrak{n} = (Y, Z)R \qquad \mathfrak{p} = (Y)R$$

and $\operatorname{Spec}(R)$ is connected as $\mathfrak{p} \subseteq \mathfrak{m} \cap \mathfrak{n}$. The minimal injective resolution of the normalized dualizing complex for R has the form

$$D = 0 \to E(R/\mathfrak{p}) \to E(R/\mathfrak{m}) \oplus E(R/\mathfrak{n}) \to 0.$$

Since $R_\mathfrak{n}$ is not Cohen–Macaulay, one has $\mathrm{H}_1(D)_\mathfrak{n} \neq 0 \neq \mathrm{H}_0(D)_\mathfrak{n}$ and hence $\inf(D) = 0$. Also, $E(R/\mathfrak{m})_\mathfrak{p} = 0 = E(R/\mathfrak{n})_\mathfrak{p}$ implies $\mathrm{H}_1(D)_\mathfrak{p} \cong E(R/\mathfrak{p})_\mathfrak{p} \neq 0$. Since $R_\mathfrak{m}$ is Gorenstein, one has $D_\mathfrak{m} \sim R_\mathfrak{m}$; and since $0 \neq \mathrm{H}_1(D)_\mathfrak{p} = (\mathrm{H}_1(D)_\mathfrak{m})_{\mathfrak{p}_\mathfrak{m}}$ it follows that $\mathrm{H}_1(D)_\mathfrak{m} \neq 0$ and thus $D_\mathfrak{m} \simeq \Sigma R_\mathfrak{m}$.

With $S = R/(X)R \cong k[Y, Z]_{(Y)}/(Y^2)$ and $\varphi\colon R \to S$ the natural surjection, one has $\operatorname{pd}_R(S) = 1$ and $\operatorname{Supp}_R(S) = \{\mathfrak{m}\}$. The map $\varphi_{\mathfrak{m}S}\colon R_\mathfrak{m} \to S$ is local Gorenstein of grade 1, giving the third isomorphism below; the other computations are routine:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(S, D) &\simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(S, D_\mathfrak{m}) \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(S, \Sigma R_\mathfrak{m}) \simeq S \\
D \otimes^{\mathbf{L}}_R S &\simeq D_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} S \simeq \Sigma R_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} S \simeq \Sigma S \\
\mathrm{G}_D\text{-}\dim_R(S) &= 0 < 1 = \sup\{\mathrm{G}_{D_\mathfrak{m}}\text{-}\dim_{R_\mathfrak{m}}(S_\mathfrak{m}), \mathrm{G}_{D_\mathfrak{n}}\text{-}\dim_{R_\mathfrak{n}}(S_\mathfrak{n})\} \\
\mathrm{G}_D\text{-}\dim_R(S) &= 0 < 1 = \operatorname{pd}_R(S) \\
\inf(D) &= 0 < 1 = \inf\big(D \otimes^{\mathbf{L}}_R S\big) \\
\operatorname{amp}\big(D \otimes^{\mathbf{L}}_R S\big) &= 0 < 1 = \operatorname{amp}(D) \\
\inf(D) - \operatorname{pd}_R(S) &= -1 < 0 = \inf(\mathbf{R}\mathrm{Hom}_R(S, D)) \\
\mathrm{G}_D\text{-}\dim_R(S) &= 0 < 1 = \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,D)}\text{-}\dim_S(S) + \operatorname{pd}_R(S)
\end{aligned}$$

The next result follows from Propositions 3.12 and 5.10. If φ is local, the inequality can be strict (if $\operatorname{pd}(S) > 0$) or not (if $\operatorname{pd}(S) = 0 = \operatorname{amp}(C)$); see Theorem 5.13.

Theorem 5.12. *Let C, X be homologically finite R-complexes with C semidualizing. When φ is module-finite with $\operatorname{fd}(\varphi) < \infty$ there is an inequality*

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) \leq \mathrm{G}_C\text{-}\dim_R(X) + \operatorname{amp}(C) + \operatorname{pd}_R(S).$$

In particular, if X is C-reflexive, then $X \otimes^{\mathbf{L}}_R S$ is $\mathbf{R}\mathrm{Hom}_R(S, C)$-reflexive.

Here is Theorem III from the introduction. When φ is local, the equality guarantees that the inequalities are strict if and only if $\mathrm{pd}_R(S) > 0$.

Theorem 5.13. *Let C, X be homologically finite R-complexes with C semidualizing. If φ is module-finite with $\mathrm{fd}(\varphi) < \infty$ and $\text{m-Spec}(R) \subseteq \mathrm{Im}(\varphi^*)$, then*

$$\mathrm{G}_C\text{-}\dim_R(X) - \mathrm{pd}_R(S) \leq \mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) \leq \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{pd}_R(S).$$

Thus, $X \otimes^{\mathbf{L}}_R S$ is $\mathbf{R}\mathrm{Hom}_R(S, C)$-reflexive if and only if X is C-reflexive. If R is local or $\mathrm{amp}(C) = 0 = \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S, R))$, then

$$\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) = \mathrm{G}_C\text{-}\dim_R(X).$$

Proof. Set $(-)(\varphi) = \mathbf{R}\mathrm{Hom}_R(S, -)$. In the sequence

$$\begin{aligned}\mathrm{G}_{C(\varphi)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) &\overset{(1)}{\leq} \mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)\\ &\overset{(2)}{\leq} \mathrm{G}_C\text{-}\dim_R(X) + \mathrm{pd}_R(S)\\ &\overset{(3)}{\leq} \mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big) + \mathrm{pd}_R(S)\\ &\overset{(4)}{\leq} \mathrm{G}_{C(\varphi)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) + 2\mathrm{pd}_R(S)\end{aligned}$$

(1) and (4) are in Proposition 5.10, and (2) and (3) are in Theorem 4.4. When one of the extra conditions holds, there is a similar sequence

$$\mathrm{G}_{C(\varphi)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) = \mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big) - \mathrm{pd}_R(S) = \mathrm{G}_C\text{-}\dim_R(X)$$

by Theorem 4.4 and Proposition 5.10. □

Example 5.14. Without the hypothesis on m-Spec(R) in Theorems 4.4, 4.8, 5.4, 5.8, and 5.13, the nontrivial implications fail, even when Spec(R) is connected; one can have $\mathrm{G}_C\text{-}\dim_R(X) = \infty$ even though each of the following is finite: $\mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(S, X))$, $\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(S, X))$, $\mathrm{G}_C\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$, $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$, $\mathrm{G}_{\mathbf{R}\mathrm{Hom}_R(S,C)}\text{-}\dim_R\big(X \otimes^{\mathbf{L}}_R S\big)$.

Let $(R_0, \mathfrak{m}_0)$ be a non-Gorenstein local ring and set $R = R_0[Y]$ with $\mathfrak{m} = (\mathfrak{m}_0, Y)R$ and $X = R/\mathfrak{m} \oplus R$ and $S = R/(Y-1)$ with the natural surjection $R \to S$. Then X is not R-reflexive since if it were then $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ would be $R_\mathfrak{m}$-reflexive implying that $R_\mathfrak{m}$ is Gorenstein. However, $X \otimes^{\mathbf{L}}_R S \simeq S$ has finite projective dimension over S and over R, so it is reflexive with respect to each complex that is R-semidualizing or S-semidualizing; similarly for $\mathbf{R}\mathrm{Hom}_R(S, X) \simeq \Sigma^{-1}S$. Finally, Spec$(R)$ is connected as the existence of nontrivial idempotents in R would give rise to such elements in R_0; see, e.g., Atiyah and Macdonald (1969, Exer. 1.22).

6. FACTORIZABLE LOCAL HOMOMORPHISMS OF FINITE FLAT DIMENSION: COBASE CHANGE

Motivated by Avramov and Foxby (1997) and Iyengar and Sather-Wagstaff (2004) we extend results of Section 5 to special nonfinite cases.

Proposition 6.1. *Let $\dot{\varphi}$: $R \to R'$ and φ': $R' \to S$ be homomorphisms of finite flat dimension with φ' module-finite and X a homologically degree-wise finite R-complex.*

(a) *If the R-complex X is homologically bounded (respectively, semidualizing), then the S-complex $\mathbf{R}\mathrm{Hom}_{R'}(S, X \otimes^{\mathbf{L}}_R R')$ is so as well.*
(b) *Assume that $\dot{\varphi}$ is faithfully flat and $\mathrm{Im}((\varphi')^*)$ contains $\text{m-Spec}(R')$. If the S-complex $\mathbf{R}\mathrm{Hom}_{R'}(S, X \otimes^{\mathbf{L}}_R R')$ is homologically bounded (respectively, semidualizing), then the R-complex X is so as well.*

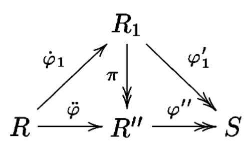
Proof. (a) If X is homologically bounded, then so is $\mathbf{R}\mathrm{Hom}_{R'}(S, X \otimes^{\mathbf{L}}_R R')$ by 4.1 and 5.1. Theorems 4.5 and 5.5 yield the other implication.

(b) When $\dot{\varphi}$ is flat, the isomorphism $\mathrm{H}_i(X \otimes^{\mathbf{L}}_R R') \cong \mathrm{H}_i(X) \otimes_R R'$ implies that the R'-complex $X \otimes^{\mathbf{L}}_R R'$ is homologically degree-wise finite. If $\mathbf{R}\mathrm{Hom}_{R'}(S, X \otimes^{\mathbf{L}}_R R')$ is homologically bounded, then so is $X \otimes^{\mathbf{L}}_R R'$ by Corollary 5.2(b), and so is X. Theorems 4.5 and 5.5 provide the remaining implication. □

For the rest of this article, we focus on local homomorphisms that factor nicely.

6.2. When φ is local, a *regular* (respectively, *Gorenstein*) *factorization* of φ is a pair of local homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ such that $\varphi = \varphi'\dot{\varphi}$, φ' is surjective, and $\dot{\varphi}$ is flat with regular (respectively, Gorenstein) closed fibre. In either case, the homomorphisms φ and φ' have finite flat dimension simultaneously by Foxby and Iyengar (2003, (3.2)). When the ring R' is complete, the regular factorzation is a *Cohen factorization*. It is straightforward to construct a regular factorization when φ is essentially of finite type. Also, if S is complete, then φ admits a Cohen factorization (Avramov et al., 1994, (1.1)).

Lemma 6.3. *Assume that φ is module-finite and local and that it admits a Gorenstein factorization $R \xrightarrow{\dot{\varphi}_1} R_1 \xrightarrow{\varphi'_1} S$ with R, R_1, S complete. Then there exists a commutative diagram of local homomorphisms*



where π is surjective with kernel generated by an R_1-sequence and the bottom row is a Gorenstein factorization of φ such that $\ddot{\varphi}$ is module-finite.

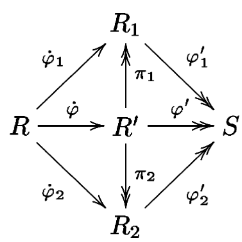
Proof. Since φ is module finite, the closed fibre $S/\mathfrak{m}S \cong R_1/(\mathrm{Ker}(\varphi'_1), \mathfrak{m})$ is Artinian and the extension of residue fields $k \to l$ is finite. In particular, the ideal $(\mathrm{Ker}(\varphi_1'), \mathfrak{m})R_1/\mathfrak{m}R_1$ is primary to the maximal ideal of $R_1/\mathfrak{m}R_1$. Let $\mathbf{y} = y_1, \ldots, y_d \in \mathrm{Ker}(\varphi_1')$ be a system of parameters for $R_1/\mathfrak{m}R_1$, that is, a maximal $R_1/\mathfrak{m}R_1$-sequence. Set $R'' = R_1/(\mathbf{y})$ with natural surjection $\pi\colon R_1 \to R''$, and let the maps $\ddot{\varphi}\colon R \to R''$ and $\varphi''\colon R'' \to S$ be induced by $\dot{\varphi}_1$ and φ_1', respectively.

One has $\varphi''\ddot{\varphi} = \varphi_1'\dot{\varphi}_1 = \varphi$, and φ'' is surjective because φ_1' is so. The closed fibre of $\ddot{\varphi}$ is $R''/\mathfrak{m}R'' \cong (R_1/\mathfrak{m}R_1)/(\mathbf{y})$ which is Gorenstein because $R_1/\mathfrak{m}R_1$ is so. The sequence $\mathbf{y}$ is R_1-regular, and the map $\ddot{\varphi}$ is flat; see, e.g., Matsumura (1989, Corollary to (22.5)). Finally, the equality in the next sequence is straightforward

$$\mathrm{length}_R(R''/\mathfrak{m}R'') = \mathrm{length}_{R''/\mathfrak{m}R''}(R''/\mathfrak{m}R'') \cdot \mathrm{rank}_k(l) < \infty$$

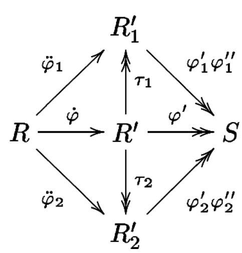
and the inequality is by construction. So, φ'' is module-finite by Matsumura (1989, (8.4)). $\square$

Proposition 6.4. *Assume that φ is local and admits Gorenstein factorizations $R \xrightarrow{\dot{\varphi}_1} R_1 \xrightarrow{\varphi_1'} S$ and $R \xrightarrow{\dot{\varphi}_2} R_2 \xrightarrow{\varphi_2'} S$ with each R_i complete. There exists a commutative diagram of local ring homomorphisms*



where $\varphi'\dot{\varphi}$ is a Cohen factorization of φ and each π_i is surjective and Gorenstein.

Proof. Taking Cohen factorizations $R \xrightarrow{\ddot{\varphi}_i} R_i' \xrightarrow{\varphi_i''} R_i$ of $\dot{\varphi}_i$, it is evident that the diagrams $R \xrightarrow{\ddot{\varphi}_i} R_i' \xrightarrow{\varphi_i'\varphi_i''} S$ are Cohen factorizations of φ. Since $\dot{\varphi}_i$ is flat with Gorenstein closed fibre, the surjection φ_i'' is Gorenstein by Avramov and Foxby (1992, (2.4)) and Avramov et al. (1994, (3.2)). The "comparison theorem" for Cohen factorizations (Avramov et al., 1994, (1.2)), provides a commutative diagram of local ring homomorphisms



where $\varphi'\dot{\varphi}$ is a Cohen factorization of φ and each τ_i is surjective with kernel generated by a regular sequence. Each τ_i is Gorenstein by Avramov and Foxby

(1992, (4.3)), and hence so is each $\pi_i = \varphi_i''\tau_i$. Thus, these maps yield a diagram with the desired properties. □

Theorem 6.5. *Let X be a homologically finite R-complex. Assume that φ is local with $\operatorname{fd}(\varphi)$ finite and that φ admits Gorenstein factorizations $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} S$ and $R \xrightarrow{\dot\varphi_2} R_2 \xrightarrow{\varphi_2'} S$. Set $d = \operatorname{depth}(\varphi)$ and $d_i = \operatorname{depth}(\dot\varphi_i)$ for $i = 1, 2$.*

(a) *The S-complexes $\Sigma^{d_1}\mathbf{R}\mathrm{Hom}_{R_1}\big(S, X \otimes^{\mathbf{L}}_R R_1\big)$ and $\Sigma^{d_2}\mathbf{R}\mathrm{Hom}_{R_2}\big(S, X \otimes^{\mathbf{L}}_R R_2\big)$ are isomorphic.*
(b) *When φ is Gorenstein at $\mathfrak{n}$, the S-complexes $\Sigma^{d_i}\mathbf{R}\mathrm{Hom}_{R_i}\big(S, X \otimes^{\mathbf{L}}_R R_i\big)$ and $\Sigma^{d} X \otimes^{\mathbf{L}}_R S$ are isomorphic.*
(c) *When φ is module-finite, the S-complexes $\Sigma^{d_i}\mathbf{R}\mathrm{Hom}_{R_i}\big(S, X \otimes^{\mathbf{L}}_R R_i\big)$ and $\mathbf{R}\mathrm{Hom}_R(S, X)$ are isomorphic.*

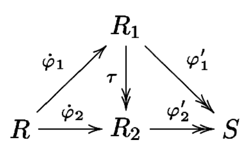
Proof. First, we show that, if φ is module-finite and Gorenstein at $\mathfrak{n}$, then the S-complexes $\Sigma^d X \otimes^{\mathbf{L}}_R S$ and $\mathbf{R}\mathrm{Hom}_R(S, X)$ are isomorphic. To this end, note that $\operatorname{grade}_R(S) = -d$ and so Proposition 5.6(b) implies $\Sigma^d S \simeq \mathbf{R}\mathrm{Hom}_R(S, R)$ since S is local. This provides the first of the following isomorphisms:

$$\Sigma^d S \otimes^{\mathbf{L}}_R X \simeq \mathbf{R}\mathrm{Hom}_R(S, R) \otimes^{\mathbf{L}}_R X \simeq \mathbf{R}\mathrm{Hom}_R(S, X)$$

where the other is from 1.7(b). This establishes the desired isomorphism.

The completed diagrams $\widehat{R} \xrightarrow{\widehat{\dot\varphi_i}} \widehat{R_i} \xrightarrow{\widehat{\varphi_i'}} \widehat{S}$ are Gorenstein factorizations of $\hat\varphi\colon \widehat{R} \to \widehat{S}$. Using Lemma 1.10, one can replace the given factorizations with the completed ones to assume that the local rings R, R_1, R_2, S are complete.

By considering the upper and lower halves of the diagram provided by Proposition 6.4, we assume that there is a commutative diagram of local homomorphisms



where τ is surjective and Gorenstein. By definition then, one has $d_2 = d_1 + \operatorname{depth}(\tau)$.

(a) The above diagram gives a sequence of isomorphisms

$$\begin{aligned}
\Sigma^{d_2}\mathbf{R}\mathrm{Hom}_{R_2}\big(S, X \otimes^{\mathbf{L}}_R R_2\big) &\overset{(1)}{\simeq} \Sigma^{d_2}\mathbf{R}\mathrm{Hom}_{R_2}\big(S, \big(X \otimes^{\mathbf{L}}_R R_1\big) \otimes^{\mathbf{L}}_{R_1} R_2\big) \\
&\overset{(2)}{\simeq} \Sigma^{d_2}\mathbf{R}\mathrm{Hom}_{R_2}\big(S, \Sigma^{-\operatorname{depth}(\tau)}\mathbf{R}\mathrm{Hom}_{R_1}\big(R_2, X \otimes^{\mathbf{L}}_R R_1\big)\big) \\
&\overset{(3)}{\simeq} \Sigma^{d_1}\mathbf{R}\mathrm{Hom}_{R_2}\big(S, \mathbf{R}\mathrm{Hom}_{R_1}\big(R_2, X \otimes^{\mathbf{L}}_R R_1\big)\big) \\
&\overset{(4)}{\simeq} \Sigma^{d_1}\mathbf{R}\mathrm{Hom}_{R_1}\big(S, X \otimes^{\mathbf{L}}_R R_1\big)
\end{aligned}$$

where (1) is by associativity, (2) follows from the the first paragraph since τ is Gorenstein and surjective, (3) follows from the final observation of the previous paragraph, and (4) is adjunction.

(b) When φ is Gorenstein, the same is true of each φ_i' by Avramov et al. (1994, (3.2)) and Avramov and Foxby (1992, (2.4)). Since each φ_i' is also surjective, the first paragraph gives the first isomorphism in the next sequence where the second isomorphism is associativity and cancellation:

$$\Sigma^{d_i}\mathbf{R}\mathrm{Hom}_{R_i}\big(S, X \otimes^{\mathbf{L}}_R R_i\big) \simeq \Sigma^{d_i+\mathrm{depth}(\varphi_i')}\big(X \otimes^{\mathbf{L}}_R R_i\big) \otimes^{\mathbf{L}}_{R_i} S \simeq \Sigma^d X \otimes^{\mathbf{L}}_R S.$$

(c) When φ is module-finite, the diagram provided by Lemma 6.3 yields a sequence of isomorphisms where $d'' = \mathrm{depth}(\ddot{\varphi})$:

$$\begin{aligned}\Sigma^{d_1}\mathbf{R}\mathrm{Hom}_{R_1}\big(S, X \otimes^{\mathbf{L}}_R R_1\big) &\overset{(5)}{\simeq} \Sigma^{d''}\mathbf{R}\mathrm{Hom}_{R''}\big(S, X \otimes^{\mathbf{L}}_R R''\big)\\ &\overset{(6)}{\simeq} \Sigma^{d''}\mathbf{R}\mathrm{Hom}_{R''}\big(S, \Sigma^{-d''}\mathbf{R}\mathrm{Hom}_R(R'', X)\big)\\ &\overset{(7)}{\simeq} \mathbf{R}\mathrm{Hom}_R(S, X).\end{aligned}$$

(5) is by part (a), (6) follow from the first paragraph, and (7) is by adjunction. □

We employ the following handy notation for the remainder of this section.

6.6. Assume that φ is local with $\mathrm{fd}(\varphi)$ finite and admits a Gorenstein factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ with $d = \mathrm{depth}(\dot{\varphi})$. For a homologically finite R-complex X, set

$$X(\varphi) = \Sigma^d\mathbf{R}\mathrm{Hom}_{R'}\big(S, X \otimes^{\mathbf{L}}_R R'\big).$$

Theorem 6.5 shows that this is independent of the choice of Gorenstein factorization and that $X(\varphi) \simeq \mathbf{R}\mathrm{Hom}_R(S, X)$ when φ is module-finite.

Remark 6.7. With φ as in 6.6, the complex $R(\varphi)$ is normalized dualizing for φ. If D is a (normalized) dualizing complex for R, then the complex $D(\varphi)$ is (normalized) dualizing for S; see Proposition 6.10.

Next is an alternate description of $X(\varphi)$ that follows directly from 1.7(b). In it, we tensor over S in order to stress that complexes are isomorphic over S and not just over R. A similar remark applies to Proposition 6.9.

Proposition 6.8. *If φ is as in* 6.6 *and X is a homologically finite R-complex, then there is an isomorphism $X(\varphi) \simeq \big(X \otimes^{\mathbf{L}}_R S\big) \otimes^{\mathbf{L}}_S R(\varphi)$.*

The next isomorphisms follows from parts (b) and (c) of 1.9.

Proposition 6.9. *If φ is as in* 6.6 *then there are isomorphisms*

$$\mathbf{R}\mathrm{Hom}_S(X(\varphi), Y(\varphi)) \simeq \mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_R S$$

$$\mathbf{R}\mathrm{Hom}_S\big(X \otimes^{\mathbf{L}}_R S, Y(\varphi)\big) \simeq \big(\mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_R S\big) \otimes^{\mathbf{L}}_S R(\varphi)$$

for all homologically finite R-complexes X, Y.

Proposition 6.10. *Assume that φ is local with $\mathrm{fd}(\varphi)$ finite and let C be a semidualizing R-complex. The Poincaré and Bass series of $C \otimes^{\mathbf{L}}_R S$ are*

$$P^S_{C\otimes^{\mathbf{L}}_R S}(t) = P^R_C(t) \qquad I_S^{C\otimes^{\mathbf{L}}_R S}(t) = I^C_R(t) I_\varphi(t).$$

If φ has a Gorenstein factorization, then the Poincaré and Bass series of $C(\varphi)$ are

$$P^S_{C(\varphi)}(t) = P^R_C(t) I_\varphi(t) \qquad I_S^{C(\varphi)}(t) = I^C_R(t).$$

Proof. The first Poincaré series is from Avramov and Foxby (1997, (1.5.3)), and the Bass series follows

$$I_S^{C\otimes^{\mathbf{L}}_R S}(t) \overset{(1)}{=} I^S_S(t)/P^S_{C\otimes^{\mathbf{L}}_R S}(t) \overset{(2)}{=} I^R_R(t) I_\varphi(t)/P^R_C(t) \overset{(3)}{=} I^C_R(t) I_\varphi(t)$$

where (1) and (3) are by 1.5 and (2) is from 1.8. If φ admits a Gorenstein factorization, then the second Poincaré series follows from Proposition 6.8 with Avramov and Foxby (1997, (1.5.3)) and Christensen (2001, (1.7.6)), and the second Bass series is computed like the first one. □

Here we record the analog of Theorem 5.8 for our new setting.

Corollary 6.11. *If φ is as in 6.6 and C, X are homologically finite R-complexes with C semidualizing, then $\mathrm{G}_{C(\varphi)}\text{-}\mathrm{dim}_S(X(\varphi)) = \mathrm{G}_C\text{-}\mathrm{dim}_R(X) + \mathrm{depth}(\varphi)$. In particular, $X(\varphi)$ is $C(\varphi)$-reflexive if and only if X is C-reflexive.*

Proof. Let $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ be a Gorenstein factorization of φ and set $d = \mathrm{depth}(\dot\varphi)$. Equalities (1) and (5) in the following sequence are by definition:

$$\begin{aligned}
\mathrm{G}_{C(\varphi)}\text{-}\mathrm{dim}_S(X(\varphi)) &\overset{(1)}{=} \mathrm{G}_{\Sigma^d(C\otimes^{\mathbf{L}}_R R')(\varphi')}\text{-}\mathrm{dim}_S\big(\Sigma^d\big(X \otimes^{\mathbf{L}}_R R'\big)(\varphi')\big)\\
&\overset{(2)}{=} \mathrm{G}_{(C\otimes^{\mathbf{L}}_R R')(\varphi')}\text{-}\mathrm{dim}_S\big(\big(X \otimes^{\mathbf{L}}_R R'\big)(\varphi')\big) + d\\
&\overset{(3)}{=} \mathrm{G}_{C\otimes^{\mathbf{L}}_R R'}\text{-}\mathrm{dim}_{R'}\big(X \otimes^{\mathbf{L}}_R R'\big) - \mathrm{pd}_{R'}(S) + d\\
&\overset{(4)}{=} \mathrm{G}_C\text{-}\mathrm{dim}_R(X) + \mathrm{depth}(\varphi') + d\\
&\overset{(5)}{=} \mathrm{G}_C\text{-}\mathrm{dim}_R(X) + \mathrm{depth}(\varphi)
\end{aligned}$$

while (2) is by Christensen (2001, (3.12)), (3) is Theorem 5.8, and (4) is from Theorem 4.8 and the Auslander–Buchsbaum formula. □

Theorems 4.9 and 5.9 provide the proof of the next result.

Corollary 6.12. *Let φ be as in* 6.6. *When C, C' are semidualizing R-complexes, one has $C(\varphi) \simeq C'(\varphi)$ if and only if $C \simeq C'$.*

Replace Proposition 5.10 with Theorem 5.13 in the proof of Corollary 6.11 to prove the next result.

Corollary 6.13. *If φ is as in* 6.6 *and C, X are homologically finite R-complexes with C semidualizing, then* $\mathrm{G}_{C(\varphi)}\text{-}\dim_S\big(X \otimes^{\mathbf{L}}_R S\big) = \mathrm{G}_C\text{-}\dim_R(X)$. *In particular, $X \otimes^{\mathbf{L}}_R S$ is $C(\varphi)$-reflexive if and only if X is C-reflexive.*

Remark 6.14. With the reflexivity relations of Theorem 4.8 and Corollaries 6.11 and 6.13 in mind, we wish to characterize the finiteness of $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S(X(\varphi))$. If $\mathrm{G}_C\text{-}\dim_R(X)$ is finite and φ is Gorenstein at $\mathfrak{n}$, then $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S(X(\varphi))$ is finite by Theorems 4.8 and 6.5(b). We wonder if the converse holds. Here is one instance of this: If $\mathrm{G}_{C\otimes^{\mathbf{L}}_R S}\text{-}\dim_S(C(\varphi))$ is finite, then $C \otimes^{\mathbf{L}}_R S$ and $C(\varphi)$ are shift isomorphic by Lemma 3.2, and Frankild and Sather-Wagstaff (Preprint, (3.7(c))) implies that φ is Gorenstein at $\mathfrak{n}$.

Proposition 6.15. *Let φ be local with* $\mathrm{fd}(\varphi)$ *finite and C, C' semidualizing R-complexes such that C' is C-reflexive. There are the coefficientwise equalities*

$$P^S_{\mathbf{R}\mathrm{Hom}_S(C'\otimes^{\mathbf{L}}_R S,\, C\otimes^{\mathbf{L}}_R S)}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(C',C)}(t)$$
$$I_S^{\mathbf{R}\mathrm{Hom}_S(C'\otimes^{\mathbf{L}}_R S,\, C\otimes^{\mathbf{L}}_R S)}(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(C',C)}(t) I_\varphi(t).$$

If φ has a Gorenstein factorization, then there are equalities

$$P^S_{\mathbf{R}\mathrm{Hom}_S(C'(\varphi),C(\varphi))}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(C',C)}(t)$$
$$I_S^{\mathbf{R}\mathrm{Hom}_S(C'(\varphi),C(\varphi))}(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(C',C)}(t) I_\varphi(t)$$
$$P^S_{\mathbf{R}\mathrm{Hom}_S(C'\otimes^{\mathbf{L}}_R S,\, C(\varphi))}(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(C',C)}(t) I_\varphi(t)$$
$$I_S^{\mathbf{R}\mathrm{Hom}_S(C'\otimes^{\mathbf{L}}_R S,\, C(\varphi))}(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(C',C)}(t).$$

Proof. For the first Poincaré series, use Christensen (2001, (1.7.6)) with 1.9(a). When φ admits a Gorenstein factorization, the other Poincaré series come from Proposition 6.9 with Avramov and Foxby (1997, (1.5.3)) and Christensen (2001, (1.7.6)). The Bass series follow as in Proposition 6.10. □

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