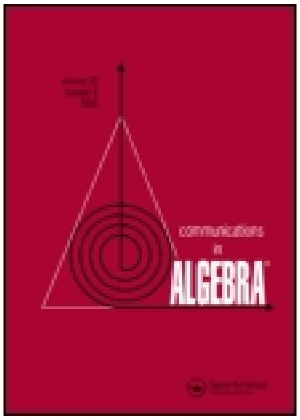

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DECOMPOSITIONS OF MONOMIAL IDEALS IN REAL SEMIGROUP RINGS

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Irreducible decompositions of monomial ideals in polynomial rings over a field are well-understood. In this article, we investigate decompositions in the set of monomial ideals in the semigroup ring $A[\mathbb{R}^d_{\geq 0}]$ where A is an arbitrary commutative ring with identity. We classify the irreducible elements of this set, which we call m-irreducible, and we classify the elements that admit decompositions into finite intersections of m-irreducible ideals.

Key Words: Irreducible decompositions; Monomial ideals; Semigroup rings.

2010 Mathematics Subject Classification: 13C05; 13F20.

1. INTRODUCTION

Throughout this article, let A be a commutative ring with identity.

When A is a field, the polynomial ring $P = A[X_1, \dots, X_d]$ is noetherian, so every ideal in this ring has an irreducible decomposition. For *monomial ideals*, that is, the ideals of P generated by sets of monomials, these decompositions are well understood: the non-zero irreducible monomial ideals are precisely the ideals $(X_{i_1}^{e_1}, \dots, X_{i_n}^{e_n})P$ generated by "pure powers" of some of the variables, and every monomial ideal in this setting decomposes as a finite intersection of irreducible monomial ideals. Furthermore, there are good algorithms for computing these decompositions, both by hand [6, 11, 12, 16, 24] and by computer [2, 7, 8].[1] Note that much of this discussion extends to the case where P is replaced by a numerical semigroup ring, that is, a ring of the form $A[S]$ where S is a sub-semigroup of $\mathbb{Z}^d$.

When A is not noetherian, some of the conclusions from the previous paragraph fail because P fails to be noetherian. However, P does behave somewhat "noetherianly" with respect to monomial ideals. For instance, all monomial ideals

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[1]One of the most interesting aspects of this theory is found in its interactions with combinatorics, including applications to graphs and simplicial complexes; see, e.g., [3–5, 9, 10, 15, 17–19, 21, 22, 25, 27]. Foundational material on the subject can be found in the following texts [1, 13, 14, 20, 23, 26, 28].

in P are finitely generated, by finite sets of monomials. The ideals that are indecomposable with respect to intersections of monomial ideals, which we call *m-irreducible*, are the ideals $(X_{i_1}^{e_1}, \ldots, X_{i_n}^{e_n})P$. Each monomial ideal of P admits an *m-irreducible decomposition*, a decomposition into a finite intersection of m-irreducible monomial ideals, and many of the algorithms carry over to this setting; see, e.g., [23].

In this article, we step even further from the noetherian setting by considering monomial ideals in the semigroup ring $R = A[\mathbb{R}^d_{\geq 0}]$ where A is an arbitrary commutative ring with identity and $\mathbb{R}^d_{\geq 0} = \{(r_1, \ldots, r_d) \in \mathbb{R}^d \mid r_1, \ldots, r_d \geq 0\}$. This ring can be thought of as the set $A[X_1^{\mathbb{R}_{\geq 0}}, \ldots, X_d^{\mathbb{R}_{\geq 0}}]$ of all polynomials in variables $X_1, \ldots, X_d$ with coefficients in A where the exponents are non-negative *real* numbers. For instance, in this ring, many of the monomial ideals are not finitely generated and many do not admit finite m-irreducible decompositions; see, e.g., Fact 3.3(c) and Example 4.13. On the other hand, every ideal admits a (possibly infinite) m-irreducible decomposition, by Proposition 4.14.

Our goal is to completely characterize the monomial ideals in R that admit m-irreducible decompositions. This is accomplished in two steps. First, we characterize the m-irreducible ideals. This is accomplished in Theorem 3.9, which we paraphrase in the following result.

Theorem 1.1. *A monomial ideal in the ring* $A[\mathbb{R}^d_{\geq 0}] = A[X_1^{\mathbb{R}_{\geq 0}}, \ldots, X_d^{\mathbb{R}_{\geq 0}}]$ *is m-irreducible if and only if it is generated by a set of pure powers of the variables* $X_1, \ldots, X_d$.

Our characterization of the monomial ideals in R that admit m-irreducible decompositions is more technical. However, our intuition is straightforward, and reflects the connection between the noetherian property and existence of decompositions: a monomial ideal in R admits an m-irreducible decomposition if and only if it is almost finitely generated. To make sense of this, we need to explain what we mean by "almost finitely generated." We build up the general definition in steps.

First, we consider the case of monomial ideals that are "almost principal." In one variable (i.e., the case $d = 1$) there are exactly two kinds of nonzero monomial ideals: given a real number $a \geq 0$ set

$$I_{a,0} = (X_1^r \mid r \geq a)R = (X_1^a)R$$
$$I_{a,1} = (X_1^r \mid r > a)R.$$

These ideals are completely determined by the sets of their exponents, corresponding exactly to open and closed rays in $\mathbb{R}_{\geq 0}$. We think of these ideals as being almost generated by X_1^a. This includes the ideal that *is* generated by X_1^a as a special case.

In two variables (i.e., the case $d = 2$) there is more variation. First, not every ideal is almost principal; in fact, we can find ideals here that are not almost finitely generated here. Second, the almost principal ideals come in four flavors in this setting: given real numbers $a, b \geq 0$ set

$$I_{(a,b),(0,0)} = (X_1^r X_2^s \mid r \geq a \text{ and } s \geq b)R = (X_1^a X_2^b)R$$
$$I_{(a,b),(1,0)} = (X_1^r X_2^s \mid r > a \text{ and } s \geq b)R$$

$$I_{(a,b),(0,1)} = (X_1^r X_2^s \,|, r \geq a \text{ and } s > b)R$$
$$I_{(a,b),(1,1)} = (X_1^r X_2^s \mid r > a \text{ and } s > b)R.$$

These ideals are completely determined by the sets of their exponent vectors, corresponding to combinations of open and closed rays on each axis of $\mathbb{R}^2$. We think of these ideals as being almost generated by $X_1^a X_2^b$. In general (i.e., for arbitrary $d \geq 1$) there are 2^d different flavors of almost principal monomial ideals, corresponding to the different choices of open and closed rays for the exponents of each variable; see Notation 4.2.

In general, a monomial ideal is "almost finitely generated" if it is a finite sum of almost principal monomial ideals. This definition is motivated by the fact that a finitely generated monomial ideal is a sum of principal monomial ideals. For instance, each m-irreducible monomial ideal is almost finitely generated. In these terms, our characterization of the decomposable monomial ideals, stated next, is quite straightforward; see Theorem 4.12:

Theorem 1.2. *A monomial ideal in R admits an m-irreducible decomposition if and only if it is almost finitely generated.*

As to the organization of this article, Section 2 consists of definitions and background results, and Sections 3 and 4 are primarily concerned with the proofs of Theorems 1.1 and 1.2, respectively.

2. BACKGROUND AND PRELIMINARY RESULTS

In this section, we lay the foundation for the proofs of our main results. We begin by establishing some notation for use throughout the article.

Definition 2.1. Set

$$\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0\} \qquad \mathbb{R}_{\infty\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Let d be a non-negative integer, and set

$$\mathbb{R}^d_{\geq 0} = (\mathbb{R}_{\geq 0})^d = \left\{(r_1, \ldots, r_d) \in \mathbb{R}^d \mid r_1, \ldots, r_d \geq 0\right\},$$

which is an additive semigroup, and

$$\mathbb{R}^d_{\infty\geq 0} = (\mathbb{R}_{\infty\geq 0})^d = \left\{(r_1, \ldots, r_d) \mid r_1, \ldots, r_d \in \mathbb{R}_{\infty\geq 0}\right\}$$

We consider the semigroup ring

$$R = A[\mathbb{R}^d_{\geq 0}],$$

which we think of as the set of all polynomials in variables $X_1, \ldots, X_d$ with coefficients in A where the exponents are non-negative real numbers. Moreover, for $i = 1, \ldots, d$, we set

$$X_i^\infty = 0.$$

When the number of variables is small ($d \leq 2$), we will use variables X, Y in place of X_1, X_2. A *monomial* in R is an element of the form

$$\underline{X}^{\underline{r}} = X_1^{r_1} \dots X_d^{r_d} \in R,$$

where $\underline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d_{\geq 0}$ is the *exponent vector* of the monomial f. Multiplication of monomials in this ring is defined analogously to its classical counterpart. For all $\underline{q}, \underline{r} \in \mathbb{R}^d_{\geq 0}$, and all $s \in \mathbb{R}$, we write

$$\underline{X}^{\underline{q}}\underline{X}^{\underline{r}} = \underline{X}^{\underline{q}+\underline{r}} \qquad (\underline{X}^{\underline{q}})^s = \underline{X}^{(s\underline{q})}.$$

An arbitrary element of R is a linear combination of monomials

$$f = \sum_{\underline{r} \in \mathbb{R}^d_{\geq 0}}^{\text{finite}} a_{\underline{r}} \underline{X}^{\underline{r}}$$

with coefficients $a_{\underline{r}} \in A$. A *monomial ideal* of R is an ideal $I = (S)R$ generated by a set S of monomials in R. Given a subset $G \subseteq R$, the *monomial set* of G is

$$[\![G]\!] = \{\text{monomials of } R \text{ in } G\} \subseteq G.$$

We next list some basic properties of monomial ideals.

Fact 2.2. Let I and J be monomial ideals of R.

(a) For any subset $G \subseteq R$, we have $[\![G]\!] = G \cap [\![R]\!]$, by definition.
(b) The monomial ideal I is generated by its monomial set: $I = ([\![I]\!])R$.
(c) Parts (a) and (b) combine to show that $I \subseteq J$ if and only if $[\![I]\!] \subseteq [\![J]\!]$, and hence $I = J$ if and only if $[\![I]\!] = [\![J]\!]$.

The next facts follow from the $\mathbb{R}^d_{\geq 0}$-graded structure on R, that is, the isomorphisms $R \cong \oplus_{\mathbb{R}^d_{\geq 0}} A\underline{X}^{\underline{r}} \cong \oplus_{f \in [\![R]\!]} Af$.

Fact 2.3. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a set of monomial ideals of R.

(a) Given monomials $f = \underline{X}^{\underline{r}}$ and $g = \underline{X}^{\underline{s}}$ in R, we have $f|g$ if and only if $g \in (f)R$ if and only if $s_i \geq r_i$ for all i. When these conditions are satisfied, we have $g = fh$ where $h = \underline{X}^{\underline{s}-\underline{r}}$.
(b) Given a monomial $f \in [\![R]\!]$ and a subset $S \subseteq [\![R]\!]$, we have $f \in (S)R$ if and only if $f \in (s)R$ for some $s \in S$.
(c) The sum $\sum_{\lambda \in \Lambda} I_\lambda$ is a monomial ideal such that $[\![\sum_{\lambda \in \Lambda} I_\lambda]\!] = \bigcup_{\lambda \in \Lambda} [\![I_\lambda]\!]$.
(d) The intersection $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a monomial ideal such that $[\![\bigcap_{\lambda \in \Lambda} I_\lambda]\!] = \bigcap_{\lambda \in \Lambda} [\![I_\lambda]\!]$.

Given a set $\{S_\lambda\}_{\lambda \in \Lambda}$ of subsets of R, the equality $\sum_{\lambda \in \Lambda}(S_\lambda)R = (\bigcup_{\lambda \in \Lambda} S_\lambda)R$ is standard. In general, one does not have such a nice description for the intersection $\bigcap_{\lambda \in \Lambda}(S_\lambda)R$. However, for monomial ideals in our ring R, the next result provides such a description. In a sense, it says that the monomial ideals of R behave like ideals in a unique factorization domain. First, we need a definition.

Definition 2.4. Let $\underline{X}^{\underline{r_1}}, \ldots, \underline{X}^{\underline{r_k}} \in [\![R]\!]$ with $\underline{r_i} = (r_{i,1}, \ldots, r_{i,d}) \in \mathbb{R}^d_{\geq 0}$. We define the *least common multiple* of these monomials as

$$\operatorname*{lcm}_{1\leq i\leq k}(\underline{X}^{\underline{r_k}}) = \underline{X}^{\underline{p}},$$

where $\underline{p}$ is defined componentwise by $p_j = \max_{1\leq i\leq k}\{r_{i,j}\}$.

Lemma 2.5. *Given subsets $S_1, \ldots, S_k \subseteq [\![R]\!]$, we have*

$$\bigcap_{i=1}^{k}(S_i)R = \left(\operatorname*{lcm}_{1\leq i\leq k}(f_i) \mid f_i \in S_i, for\ i = 1, \ldots, k\right) R.$$

Proof. Let $L = (\operatorname{lcm}_{1\leq i\leq k}(f_i) \mid f_i \in S_i \text{ for } i = 1, \ldots, k)R$, which is a monomial ideal of R by definition. Fact 2.3(d) implies that $\bigcap_{i=1}^k (S_i)R$ is also a monomial ideal with $[\![\bigcap_{i=1}^k (S_i)R]\!] = \bigcap_{i=1}^k [\![(S_i)R]\!]$. Thus, to show that $\bigcap_{i=1}^k (S_i)R = L$, we need only show that $\bigcap_{i=1}^k [\![(S_i)R]\!] = [\![L]\!]$.

For the containment $\bigcap_{i=1}^k [\![(S_i)R]\!] \subseteq [\![L]\!]$, let $\underline{X}^{\underline{t}} \in \bigcap_{i=1}^k [\![(S_i)R]\!]$. Fact 2.3(b) implies that $\underline{X}^{\underline{t}}$ is a multiple of one of the generators $\underline{X}^{\underline{r_i}} \in S_i$ for $i = 1, \ldots, k$. Hence we have $t_j \geq r_{i,j}$ for all i and j by Fact 2.3(a). It follows that $t_j \geq \max_{1\leq i\leq d}\{r_{i,j}\} = p_j$, hence $\underline{X}^{\underline{t}} \in (\underline{X}^{\underline{p}})R \subseteq L$.

For the reverse containment $\bigcap_{i=1}^k [\![(S_i)R]\!] \supseteq [\![L]\!]$, let $\underline{X}^{\underline{t}} \in [\![L]\!]$. Fact 2.3(b) implies that $\underline{X}^{\underline{t}}$ is a multiple of one of the monomial generators of L, so there exist $\underline{X}^{\underline{r_i}} \in S_i$ for $i = 1, \ldots, k$ such that $\underline{X}^{\underline{t}}$ is a multiple of $\operatorname{lcm}_{1\leq i\leq k}(\underline{X}^{\underline{r_i}})$. Since $\operatorname{lcm}_{1\leq i\leq k}(\underline{X}^{\underline{r_i}})$ is a multiple of $\underline{X}^{\underline{r_i}}$ for each i, we then have $\underline{X}^{\underline{t}} \in \bigcap_{i=1}^k (\underline{X}^{\underline{r_i}})R \subseteq \bigcap_{i=1}^k (S_i)R$, as desired. $\square$

Lemma 2.6. *Let $G \subseteq [\![R]\!]$, and set $I = (G)R$. Let $\underline{X}^{\underline{b}} \in [\![I]\!]$ be given, and for $j = 1, \ldots, d$ set $I_j = (G \cup \{X_j^{b_j}\})R$. Then we have $I = \bigcap_{j=1}^d I_j$.*

Proof. The containment $G \subseteq G \cup \{X_j^{b_j}\}$ implies that $I = (G)R \subseteq (G \cup \{X_j^{b_j}\})R = I_j$, so we have $I \subseteq \bigcap_{j=1}^d I_j$. For the reverse containment, Facts 2.2(c) and 2.3(d) imply that it suffices to show that $[\![I]\!] \supseteq \bigcap_{j=1}^d [\![I_j]\!]$.

Let $\alpha \in \bigcap_{j=1}^d [\![I_j]\!]$, and suppose that $\alpha \notin I$. For $j = 1, \ldots, d$, we have $\alpha \in I_j = (G \cup \{X_j^{b_j}\})R$. The condition $\alpha \notin I = (G)R$ implies that α is not a multiple of any element of G, so Fact 2.3(c) implies that $\alpha \in (X_j^{b_j})R$ for $j = 1, \ldots, d$. In other words, we have $\alpha \in \bigcap_{j=1}^d (X_j^{b_j})R = (\underline{X}^{\underline{b}})R \subseteq I$ by Lemma 2.5. This contradiction establishes the lemma. $\square$

The last result of this section describes the interaction between sums and intersections of monomial ideals in R.

Lemma 2.7. *For $t = 1, \ldots, l$, let $\{K_{t,i_t}\}_{i_t=1}^{m_t}$ be a collection of monomial ideals. Then the following equalities hold*:

(a) $$\bigcap_{t=1}^{l}\sum_{i_t=1}^{m_t} K_{t,i_t} = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_l=1}^{m_l}\bigcap_{t=1}^{l} K_{t,i_t};$$

(b) $$\sum_{t=1}^{l}\bigcap_{i_t=1}^{m_t} K_{t,i_t} = \bigcap_{i_1=1}^{m_1}\bigcap_{i_2=1}^{m_2}\cdots\bigcap_{i_l=1}^{m_l}\sum_{t=1}^{l} K_{t,i_t}.$$

Proof. (a) Note that the left- and right-hand sides of Equation (a) are monomial ideals of R by Fact 2.3(c)–(d). Because of Fact 2.2(c), Equation (a) follows from the next sequence of equalities:

$$\begin{aligned}
\left[\!\!\left[\bigcap_{t=1}^{l}\sum_{i_t=1}^{m_t} K_{t,i_t}\right]\!\!\right] &= \bigcap_{t=1}^{l}\left[\!\!\left[\sum_{i_t=1}^{m_t} K_{t,i_t}\right]\!\!\right] \\
&= \bigcap_{t=1}^{l}\left[\bigcup_{i_t=1}^{m_t} [\![K_{t,i_t}]\!]\right] \\
&= \bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_l=1}^{m_l}\left[\bigcap_{t=1}^{l} [\![K_{t,i_t}]\!]\right] \\
&= \bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_l=1}^{m_l}\left[\left[\!\!\left[\bigcap_{t=1}^{l} K_{t,i_t}\right]\!\!\right]\right] \\
&= \left[\!\!\left[\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_l=1}^{m_l}\bigcap_{t=1}^{l} K_{t,i_t}\right]\!\!\right].
\end{aligned}$$

Here, the third equality is from the distributive law for unions and intersections, while the remaining steps are from Fact 2.3(c)–(c).

The verification of Eq. (b) is similar, so we omit it. $\square$

3. M-IRREDUCIBLE IDEALS

The following notation is extremely convenient for our proofs. To motivate the notation, note that when ε is 1, we are thinking of ε as an arbitrarily small positive real number.

Notation 3.1. Let $\varepsilon \in \mathbb{Z}_2$. Given $r, \alpha \in \mathbb{R}$, we define

$$r \geq_\varepsilon \alpha \qquad \text{provided that} \begin{cases} r \geq \alpha & \text{if } \varepsilon = 0 \\ r > \alpha & \text{if } \varepsilon = 1. \end{cases}$$

Given $s \in \mathbb{R}_{\infty\geq 0}$, we define

$$s \geq_\varepsilon \infty \qquad \text{provided that } s = \infty.$$

Employing this new notation, we define a monomial ideal $J_{i,\alpha,\varepsilon}$ that is generated by pure powers of the single variable X_i. Recall our convention that $X_i^\infty = 0$.

Notation 3.2. Given $\alpha \in \mathbb{R}_{\infty\geq 0}$ and $\varepsilon \in \mathbb{Z}_2$, we set

$$J_{i,\alpha,\varepsilon} = (\{X_i^r \mid r \geq_\varepsilon \alpha\})\, R.$$

We use the term "pure power" to describe a monomial of the form X_i^r.

Given $\alpha \in \mathbb{R}_{\geq 0}$, we use $\varepsilon \in \mathbb{Z}_2$ to distinguish between two important cases. Essentially, they represent the difference between the closed interval $[\alpha, \infty)$ in the case $\varepsilon = 0$ and the open interval (α, ∞) in the case $\varepsilon = 1$. The important difference is the existence of a minimal element in the first case, but not in the second case. The case $\alpha = \infty$ may seem strange, but it is quite useful.

Fact 3.3. Let $\alpha \in \mathbb{R}_{\infty \geq 0}$ and $\varepsilon \in \mathbb{Z}_2$.

(a) $J_{i,\infty,\varepsilon} = 0$.
(b) If $\varepsilon = 0$, then $J_{i,\alpha,\varepsilon} = (X_i^{\alpha})\, R$.
(c) If $\alpha < \infty$, then $J_{i,\alpha,\varepsilon}$ is finitely generated if and only if $\varepsilon = 0$.

The ideal $J_{i,\alpha,\varepsilon}$ is generated by pure powers of the single variable X_i. Next, we consider the class of ideals generated by pure powers of more than one variable. This notation is the first place where we see the utility of our convention $X_i^{\infty} = 0$, since it allows us to consider all the variables simultaneously instead of worrying about partial lists of the variables.

Notation 3.4. Given $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{\infty \geq 0}$, and $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{Z}_2^d$, we set

$$J_{\underline{\alpha},\underline{\varepsilon}} = \left(\left\{X_i^{r_i} \mid i = 1, \ldots, d \text{ and } r_i \geq_{\varepsilon_i} \alpha_i\right\}\right) R.$$

Example 3.5. In the case $d = 2$, using constants $a, b \in \mathbb{R}_{\geq 0}$ and $\varepsilon, \varepsilon' \in \mathbb{Z}_2$, we have eight different possibilities for $J_{\underline{\alpha},\underline{\varepsilon}}$:

$$\begin{array}{ll}
J_{(a,b),(0,0)} = (X^a, Y^b)R & J_{(a,b),(0,1)} = (X^a, Y^{b'} \mid b' > b)R \\
J_{(a,b),(1,0)} = (X^{a'}, Y^b \mid a' > a)R & J_{(a,b),(1,1)} = (X^{a'}, Y^{b'} \mid a' > a \text{ and } b' > b)R \\
J_{(\infty,b),(\varepsilon,0)} = (Y^b)R & J_{(\infty,b),(\varepsilon,1)} = (Y^{b'} \mid b' > b)R \\
J_{(a,\infty),(1,\varepsilon')} = (X^{a'} \mid a' > a)R & J_{(\infty,\infty),(\varepsilon,\varepsilon')} = 0.
\end{array}$$

The following connection between ideals of the form $J_{\underline{\alpha},\underline{\varepsilon}}$ and those of the form $J_{i,\alpha,\varepsilon}$ is immediate.

Fact 3.6. Given $\underline{\alpha} \in \mathbb{R}^d_{\infty \geq 0}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^d$, we have the following equality:

$$J_{\underline{\alpha},\underline{\varepsilon}} = \sum_{i=1}^{d} J_{i,\alpha_i,\varepsilon_i}.$$

The ideals defined next are the irreducible elements of the set of monomial ideals.

Definition 3.7. A monomial ideal $I \subseteq R$ is *m-irreducible* (short for *monomial-irreducible*) provided that for all monomial ideals J and K of R such that $I = J \cap K$, either $I = J$ or $I = K$.

A straightforward induction argument establishes the following property.

Fact 3.8. Let I be an m-irreducible monomial ideal of R. Given monomial ideals $I_1, \ldots, I_n$ of R, if $I = \cap_{j=1}^{n} I_j$, then $I = I_j$ for some j.

Our first main result, which we prove next, is the fact that the m-irreducible monomial ideals of R are exactly the $J_{\underline{\alpha},\underline{\varepsilon}}$; it contains Theorem 1.1 from the introduction. Notice that it includes the case $J_{\underline{\infty},\underline{\varepsilon}} = 0$, where $\underline{\infty} = (\infty, \ldots, \infty)$.

Theorem 3.9. *Let $I \subseteq R$ be a monomial ideal. Then the following are equivalent*:

(i) *I is generated by pure powers of a subset of the variables $X_1, \ldots, X_d$*;
(ii) *There exist $\underline{\alpha} \in \mathbb{R}^d_{\infty \geq 0}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^d$ such that $I = J_{\underline{\alpha},\underline{\varepsilon}}$; and*
(iii) *I is m-irreducible.*

Proof. (i) $\Longrightarrow$ (ii) Assume that I is generated by pure powers of a subset of the variables. It follows that there are (possibly empty) sets $S_1, \ldots, S_d \subseteq \mathbb{R}_{\geq 0}$ such that $I = \sum_{i=1}^{d} (\{X_i^{z_i} \mid z_i \in S_i\})R$. For $i = 1, \ldots, d$ set $\alpha_i = \inf S_i \in \mathbb{R}_{\infty \geq 0}$, and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{\infty \geq 0}$. (Here we assume that $\inf \emptyset = \infty$.) Furthermore, for $i = 1, \ldots, d$ set

$$\varepsilon_i = \begin{cases} 0 & \text{if } \alpha_i \in S_i \\ 1 & \text{if } \alpha_i \notin S_i, \end{cases}$$

and let $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{Z}_2^d$. It is straightforward to show that $I = J_{\underline{\alpha},\underline{\varepsilon}}$.

(ii) $\Longrightarrow$ (iii) Let $\underline{\alpha} \in \mathbb{R}^d_{\infty \geq 0}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^d$ be given, and suppose by way of contradiction that $J_{\underline{\alpha},\underline{\varepsilon}}$ is not m-irreducible. By definition, there exist monomial ideals J and K such that $J_{\underline{\alpha},\underline{\varepsilon}} = J \cap K$, with $J_{\underline{\alpha},\underline{\varepsilon}} \neq J$ and $J_{\underline{\alpha},\underline{\varepsilon}} \neq K$. It follows that $J_{\underline{\alpha},\underline{\varepsilon}} \subsetneq J$ and $J_{\underline{\alpha},\underline{\varepsilon}} \subsetneq K$, so Fact 2.2(c) provides monomials $\underline{X}^{\underline{q}} \in J \setminus J_{\underline{\alpha},\underline{\varepsilon}}$ and $\underline{X}^{\underline{r}} \in K \setminus J_{\underline{\alpha},\underline{\varepsilon}}$. If there exist q_i or r_i such that $r_i \geq_{\varepsilon_i} \alpha_i$ or $q_i \geq_{\varepsilon_i} \alpha_i$, then Fact 2.3(a) implies that $\underline{X}^{\underline{q}} \in (X_i^{\alpha_i}) \subseteq J_{\underline{\alpha},\underline{\varepsilon}}$, a contradition.

We conclude that for all i, we have $q_i < \alpha_i$ and $r_i < \alpha_i$, so $p_i := \max\{q_i, r_i\} < \alpha_i$. By Fact 2.3(a), this implies that $\underline{X}^{\underline{p}} = \operatorname{lcm}(\underline{X}^{\underline{q}}, \underline{X}^{\underline{r}})$ is not a multiple of any generator of $J_{\underline{\alpha},\underline{\varepsilon}}$, hence $\underline{X}^{\underline{p}} \notin J_{\underline{\alpha},\underline{\varepsilon}}$ by Fact 2.3(b). However, Lemma 2.5 implies that $\underline{X}^{\underline{p}} \in J \cap K = J_{\underline{\alpha},\underline{\varepsilon}}$, a contradiction.

(iii) $\Longrightarrow$ (i) Assume that I is m-irreducible, and let $G \subseteq [\![R]\!]$ be a generating set for I. Suppose by way of contradiction that I is not generated by pure powers of some of the variables $X_1, \ldots, X_d$. Then there is a monomial $\underline{X}^{\underline{b}} \in G$ that is not a multiple of any pure power $X_i^a \in G$.

For $j = 1, \ldots, d$, we consider the monomial ideals $I_j = (G \cup \{X_j^{b_j}\})R$.

Claim: $I \neq I_j$ for $j = 1, \ldots, d$. Suppose by way of contradiction that there exists an index k such that $I = I_k$, that is, such that $(G)R = (G \cup \{X_k^{b_k}\})R$. Fact 2.2(c) implies that $X_k^{b_k}$ is a multiple of some $g \in G$. By Fact 2.3(a), we conclude that $g = X_k^a$ for some $a \leq b_k$, so we have $\underline{X}^{\underline{b}} \in (X_k^{b_k})R \subseteq (X_k^a)R$. That is, the monomial $\underline{X}^{\underline{b}}$ is a multiple of the pure power $X_k^a = g \in G$, a contradiction. This establishes the claim.

Lemma 2.6 implies that $I = \bigcap_{j=1}^{d} I_j$. Thus, the claim conspires with Fact 3.8 to contradict the fact that I is m-irreducible. $\square$

We explicitly document a special case of Theorem 3.9 for use in the sequel.

Corollary 3.10. *Each ideal $J_{i,\alpha,\varepsilon}$ is m-irreducible.*

Proof. This is the special case $\underline{\alpha} = (\infty, \ldots, \infty, \alpha, \infty, \ldots, \infty)$ of Theorem 3.9. □

4. IDEALS ADMITTING FINITE M-IRREDUCIBLE DECOMPOSITIONS

We now turn our attention to the task of characterizing the monomial ideals of R that admit decompositions into finite intersections of m-irreducible ideals.

Definition 4.1. Let $I \subseteq R$ be a monomial ideal. An *m-irreducible decomposition* of I is a decomposition $I = \bigcap_{\lambda\in\Lambda} I_\lambda$ where each I_λ is an m-irreducible monomial ideal of R. If the index set Λ is finite, we say that $I = \bigcap_{\lambda\in\Lambda} I_\lambda$ is a *finite m-irreducible decomposition*.

Our second main result shows that the monomial ideals of R that admit finite m-irreducible decompositions are precisely the finite sums of ideals of the next form.

Notation 4.2. Let $\underline{\alpha} \in \mathbb{R}^d_{\infty\geq 0}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^d$ be given, and set

$$I_{\underline{\alpha},\underline{\varepsilon}} = \left(\left\{\underline{X}^{\underline{r}} \mid i = 1, \ldots, d \text{ and } r_i \geq_{\varepsilon_i} \alpha_i\right\}\right) R.$$

Example 4.3. With the zero-vector $\underline{0} = (0, \ldots, 0)$, we have $I_{\underline{\alpha},\underline{0}} = (\underline{X}^{\underline{\alpha}})R$. If $\alpha_i = \infty$ for any i, then $I_{\underline{\alpha},\underline{\varepsilon}} = 0$.

As a first step, we show next that each ideal of the form $I_{\underline{\alpha},\underline{\varepsilon}}$ has a finite m-irreducible decomposition.

Lemma 4.4. *Given $\underline{\alpha} \in \mathbb{R}^d_{\infty\geq 0}$ and $\underline{\varepsilon} \in \mathbb{Z}_2^d$, we have $I_{\underline{\alpha},\underline{\varepsilon}} = \bigcap_{i=1}^d J_{i,\alpha_i,\varepsilon_i}$.*

Proof. If $\alpha_i = \infty$ for some i, then we have $J_{i,\alpha_i,\varepsilon_i} = 0$, so $\bigcap_{i=1}^d J_{i,\alpha_i,\varepsilon_i} = 0$; thus the desired equality follows from Example 4.3 in this case. Assume now that $\alpha_i \neq \infty$ for all i. In the following computation, the second equality is from Lemma 2.5:

$$\begin{aligned}
\bigcap_{i=1}^d J_{i,\alpha_i,\varepsilon_i} &= \bigcap_{i=1}^d \left(\{X_i^{r_i} \mid r_i \geq_{\varepsilon_i} \alpha_i\}\right) R \\
&= \left(\operatorname*{lcm}_{1\leq i\leq d} \left\{X_i^{r_i}\right\} \mid r_i \geq_{\varepsilon_i} \alpha_i\right) R \\
&= \left(\left\{X_1^{r_1} X_2^{r_2} \cdots X_d^{r_d} \mid r_i \geq_{\varepsilon_i} \alpha_i\right\}\right) R \\
&= \left(\left\{\underline{X}^{\underline{r}} \mid r_i \geq_{\varepsilon_i} \alpha_i\right\}\right) R \\
&= I_{\underline{\alpha},\underline{\varepsilon}}.
\end{aligned}$$

The first, fourth, and fifth equalities are by definition, and the third equality is straightforward. □

Remark 4.5. It is worth noting that the decomposition from Lemma 4.4 may be redundant, in the sense that some of the ideals in the intersection may be removed without affecting the intersection: if $\alpha_i = 0$ and $\varepsilon_i = 0$, then $J_{i,\alpha_i,\varepsilon_i} = (X_i^0)R = R$.

Lemma 4.4 not only provides a decomposition for the ideal $I_{\underline{\alpha},\underline{\varepsilon}}$, but also gives a first indication of how the ideals $J_{i,\alpha_i,\varepsilon_i}$ behave under intersections. The next lemmas partially extend this. The essential point of the first lemma is the fact that a finite intersection of real intervals of the form $[\alpha_i, \infty)$ and (α_j, ∞) is a real interval of the form $[\alpha, \infty)$ or (α, ∞).

Lemma 4.6. *Let $i, b \in \mathbb{Z}$ be given such that $1 \leq i \leq d$ and $b \geq 0$. Given $\alpha_1, \ldots, \alpha_b \in \mathbb{R}_{\infty \geq 0}$ and $\varepsilon_1, \ldots, \varepsilon_b \in \mathbb{Z}_2$, there exist $\beta \in \mathbb{R}_{\infty \geq 0}$ and $\delta \in \mathbb{Z}_2$ such that $\bigcap_{t=1}^{b} J_{i,\alpha_t,\varepsilon_t} = J_{i,\beta,\delta}$. Specifically, we have*

$$\beta = \begin{cases} \max\{\alpha_1, \ldots, \alpha_b\} & \text{if } b \geq 1 \\ 0 & \text{if } b = 0. \end{cases}$$

Proof. If $b = 0$, then we have

$$\bigcap_{t=1}^{b} J_{i,\alpha_t,\varepsilon_t} = \bigcap_{t=1}^{0} J_{i,\alpha_t,\varepsilon_t} = R = J_{i,0,0}$$

as claimed. Thus, we assume for the remainder of the proof that $b \geq 1$. Since $\bigcap_{t=1}^{b} J_{i,\alpha_t,\varepsilon_t} \subseteq J_{i,\alpha_j,\varepsilon_j}$ for each j, it suffices to find an index j such that $\alpha_j = \max\{\alpha_1, \ldots, \alpha_b\}$ and $J_{i,\alpha_j,\varepsilon_j} \subseteq J_{i,\alpha_t,\varepsilon_t}$ for all t. If $\alpha_j = \infty$ for some j, then we have $J_{i,\alpha_j,\varepsilon_j} = 0$, and we are done. Thus, we assume for the remainder of the proof that $\alpha_j \neq \infty$ for all j.

Choose k such that $\alpha_k = \max\{\alpha_1, \ldots, \alpha_b\}$. If there is an index j such that $\alpha_j = \alpha_k$ and $\varepsilon_j = 1$, then we have $J_{i,\alpha_j,\varepsilon_j} \subseteq J_{i,\alpha_t,\varepsilon_t}$ for all t since $J_{i,\alpha_j,\varepsilon_j}$ is generated by monomials of the form X_i^{α} where $\alpha > \alpha_j \geq \alpha_t$.

So, we assume that for every index j such that $\alpha_j = \alpha_k$, we have $\varepsilon_j = 0$. In this case, we have $J_{i,\alpha_k,\varepsilon_k} \subseteq J_{i,\alpha_t,\varepsilon_t}$ for all t, as follows. If $\alpha_k = \alpha_t$, then $\varepsilon_t = 0$, so we have $J_{i,\alpha_k,\varepsilon_k} = (X_i^{\alpha_k})R = (X_i^{\alpha_t})R = J_{i,\alpha_t,\varepsilon_t}$. On the other hand, if $\alpha_k \neq \alpha_t$, then $\alpha_k > \alpha_t$, and hence $J_{i,\alpha_k,\varepsilon_k} = (X_i^{\alpha_k})R \subseteq J_{i,\alpha_t,\varepsilon_t}$. ☐

Lemma 4.7. *Let k be a positive integer. For $t = 1, \ldots, k$ let $i_t \in \{1, \ldots, d\}$ be given, and fix $\alpha_t \in \mathbb{R}_{\infty \geq 0}$ and $\varepsilon_t \in \mathbb{Z}_2$. Then the intersection $\bigcap_{t=1}^{k} J_{i_t,\alpha_t,\varepsilon_t}$ is a monomial ideal of the form $I_{\underline{\beta},\underline{\delta}}$ for some $\underline{\beta} \in \mathbb{R}_{\geq 0}^{d}$ and $\underline{\delta} \in \mathbb{Z}_2^{d}$.*

Proof. If $\alpha_{i_j} = \infty$ for some j, then we have $J_{i_j,\alpha_j,\varepsilon_j} = 0$, so $\bigcap_{t=1}^{k} J_{i_t,\alpha_t,\varepsilon_t} = 0$, and the desired equality follows from Example 4.3 in this case. Thus, we assume for the remainder of the proof that $\alpha_{i_j} \neq \infty$ for all j.

Reorder the i_t's if necessary to obtain the first equality in the next sequence, where $0 \leq b_1 \leq b_2 \leq \cdots \leq b_d \leq b_{d+1} = k + 1$:

$$\bigcap_{t=1}^{k} J_{i_t,\alpha_t,\varepsilon_t} = \bigcap_{i=1}^{d} \bigcap_{t=b_i}^{b_{i+1}-1} J_{i,\alpha_t,\varepsilon_t} = \bigcap_{i=1}^{d} J_{i,\beta_i,\delta_i} = I_{\underline{\beta},\underline{\delta}}.$$

The second step is from Lemma 4.6, and the third step is from Lemma 4.4. ☐

We demonstrate the algorithm from the proof of Lemma 4.7 in the next example.

Example 4.8. Let $d = 2$. We show how to write the ideal

$$I = J_{1,2,1} \cap J_{2,\frac{3}{2},0} \cap J_{1,\frac{5}{3},0} \cap J_{2,1,1} = \left[J_{1,2,1} \cap J_{1,\frac{5}{3},0}\right] \cap \left[J_{2,\frac{3}{2},0} \cap J_{2,1,1}\right]$$

in the form $I_{\underline{\beta},\underline{\delta}}$. Define $\underline{\beta} \in \mathbb{R}^2_{\geq 0}$ and $\underline{\delta} \in \mathbb{Z}_2^2$ by $\beta_1 = \max\{2, \frac{5}{3}\} = 2$, $\delta_1 = 1$, $\beta_2 = \max\{\frac{3}{2}, 1\} = \frac{3}{2}$, and $\delta_2 = 0$. Then we have $I = J_{1,\beta_1,\delta_1} \cap J_{2,\beta_2,\delta_2} = I_{\underline{\beta},\underline{\delta}}$.

Lemmas 4.6 and 4.7 show how to simplify an arbitrary intersection of ideals of the form $J_{i,\alpha_i,\varepsilon_i}$. The next lemmas are proved similarly and show how to simplify an arbitrary sum of these ideals.

Lemma 4.9. *Let $i, b \in \mathbb{Z}$ be given such that $1 \leq i \leq d$ and $b \geq 0$. Given $\alpha_1, \ldots, \alpha_b \in \mathbb{R}_{\infty\geq 0}$ and $\varepsilon_1, \ldots, \varepsilon_b \in \mathbb{Z}_2$, there exist $\beta \in \mathbb{R}_{\infty\geq 0}$ and $\delta \in \mathbb{Z}_2$ such that $\sum_{t=1}^{b} J_{i,\alpha_t,\varepsilon_t} = J_{i,\beta,\delta}$. Specifically, we have*

$$\beta = \begin{cases} \min\{\alpha_1, \ldots, \alpha_b\} & \text{if } b \geq 1 \\ \infty & \text{if } b = 0. \end{cases}$$

Lemma 4.10. *Let k be a positive integer. For $t = 1, \ldots, k$, let $i_t \in \{1, \ldots, d\}$ be given, and fix $\alpha_{i_t} \in \mathbb{R}_{\infty\geq 0}$ and $\varepsilon_t \in \mathbb{Z}_2$. Then $\sum_{t=1}^{k} J_{i_t,\alpha_t,\varepsilon_t}$ is a monomial ideal of the form $J_{\underline{\beta},\underline{\delta}}$ for some $\underline{\beta} \in \mathbb{R}^d_{\infty\geq 0}$ and $\underline{\delta} \in \mathbb{Z}_2^d$.*

The next example is included to shed some light on Lemma 4.10.

Example 4.11. Let $d = 2$. We show how to write the ideal

$$I = J_{1,\frac{9}{8},1} + J_{2,\frac{11}{2},0} + J_{1,\frac{14}{3},0} + J_{2,3,1} = \left[J_{1,\frac{9}{8},1} + J_{1,\frac{14}{3},0}\right] + \left[J_{2,\frac{11}{2},0} + J_{2,3,1}\right]$$

in the form $J_{\underline{\beta},\underline{\delta}}$. Define $\underline{\beta} \in \mathbb{R}^2_{\geq 0}$ and $\underline{\delta} \in \mathbb{Z}_2^2$ by $\beta_1 = \min\{\frac{9}{8}, \frac{14}{3}\} = \frac{9}{8}$, $\delta_1 = 0$, $\beta_2 = \min\{\frac{11}{2}, 3\} = 3$, and $\delta_2 = 0$. Then we have $I = J_{1,\beta_1,\delta_1} + J_{2,\beta_2,\delta_2} = J_{\underline{\beta},\underline{\delta}}$.

We are now in a position to prove the main result of this section, which contains Theorem 1.2 from the introduction.

Theorem 4.12. *A monomial ideal $I \subseteq R$ has a finite m-irreducible decomposition if and only if it can be expressed as a finite sum of ideals of the form $I_{\underline{\alpha},\underline{\varepsilon}}$.*

Proof. $\Longrightarrow$: Assume that I has a finite m-irreducible decomposition. Theorem 3.9 explains the first equality in the next sequence:

$$I = \bigcap_{t=1}^{k} J_{\underline{\alpha}_t,\underline{\varepsilon}_t}$$

$$= \bigcap_{t=1}^{k} \sum_{i_t=1}^{d} J_{i_t,\alpha_{t,i_t},\varepsilon_{t,i_t}}$$

$$= \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_k=1}^{d} \bigcap_{t=1}^{k} J_{i_t,\alpha_{t,i_t},\varepsilon_{t,i_t}}$$

$$= \sum_{i_1=1}^{d} \sum_{i_2}^{d} \cdots \sum_{i_k=1}^{d} I_{\underline{\beta}_i,\underline{\delta}_i}.$$

The remaining equalities are from Fact 3.6, Lemma 2.7(a), and Lemma 4.7. Thus, the ideal I is a sum of the desired form.

$\Longleftarrow$: Assume that I is a finite sum of ideals of the form $I_{\underline{\alpha},\underline{\varepsilon}}$:

$$I = \sum_{t=1}^{k} I_{\underline{\alpha}_t,\underline{\varepsilon}_t}$$

$$= \sum_{t=1}^{k} \bigcap_{i_t=1}^{d} J_{i_t,\alpha_{t,i_t},\varepsilon_{t,i_t}}$$

$$= \bigcap_{i_1=1}^{d} \bigcap_{i_2=1}^{d} \cdots \bigcap_{i_k=1}^{d} \sum_{t=1}^{k} J_{i_t,\alpha_{t,i_t},\varepsilon_{t,i_t}}$$

$$= \bigcap_{i_1=1}^{d} \bigcap_{i_2=1}^{d} \cdots \bigcap_{i_k=1}^{d} J_{\underline{\beta}_i,\underline{\delta}_i}.$$

The second, third, and fourth steps in this sequence are from Lemmas 4.4, 2.7(b), and 4.10, respectively. This expresses I as a finite intersection of ideals of the form $J_{\underline{\beta},\underline{\delta}}$, so Theorem 3.9 implies that I has a finite m-irreducible decomposition. $\square$

The next example exhibits a monomial ideal in R that does not admit a finite m-irreducible decomposition. The discussion of the case $d = 1$ in the introduction shows every monomial ideal in this case is m-irreducible. Thus, our example must have $d \geq 2$. The essential point for the example is that the graph of the line $y = 1 - x$ that defines the exponent vectors of the generators of I is not a "descending staircase" which is the form required for an ideal to be a finite sum of ideals of the form $I_{\underline{\alpha},\underline{\varepsilon}}$. Note that any ideal defined by a similar curve (e.g., any curve of the form $y = f(x)$ where $f(x)$ is a non-negative, strictly decreasing, continuous function on the non-negative interval $[a, b]$) will have the same property.

Example 4.13. Set $d = 2$. We show that the ideal $I = (\{X^r Y^{1-r} \mid 0 \leq r \leq 1\})R$ does not admit a finite m-irreducible decomposition. By Theorem 4.12, it suffices to show that I cannot be written as a finite sum of ideals of the form $I_{\underline{\alpha},\underline{\varepsilon}}$.

Suppose by way of contradiction that $I = \sum_{i=1}^{k} I_{\underline{\alpha}_i,\underline{\varepsilon}_i}$. If there are indices i and j such that $I_{\underline{\alpha}_i,\underline{\varepsilon}_i} \subseteq I_{\underline{\alpha}_j,\underline{\varepsilon}_j}$, then we may remove $I_{\underline{\alpha}_j,\underline{\varepsilon}_j}$ from the list of ideals without changing the sum. Repeat this process for each pair of indices i, j such that $I_{\underline{\alpha}_i,\underline{\varepsilon}_i} \subseteq I_{\underline{\alpha}_j,\underline{\varepsilon}_j}$ to reduce to the case where no such containments occur in the sum; since the list of ideals is finite, this process terminates in finitely many steps.

We claim that for each real number r such that $0 < r < 1$, there is an index i such that $I_{\underline{\alpha}_i,\underline{\varepsilon}_i} = (X^r, Y^{1-r})R = I_{(r,1-r),(0,0)}$. (This will imply that the index set $\{1, \ldots, k\}$ cannot be finite, a contradiction.) The monomial X^rY^{1-r} is in $[\![I]\!] = \bigcup_{i=1}^k [\![I_{\underline{\alpha}_i,\underline{\varepsilon}_i}]\!]$ by Fact 2.3(c). It follows that $X^rY^{1-r} \in [\![I_{\underline{\alpha}_i,\underline{\varepsilon}_i}]\!]$ for some i. Since $X^{r'}Y^{1-r}, X^rY^{r''} \notin I$ for all $r', r'' \in \mathbb{R}_{\geq 0}$ such that $r' < r$ and $r'' < 1-r$, it follows readily that $I_{\underline{\alpha}_i,\underline{\varepsilon}_i} = (X^r, Y^{1-r})R = I_{(r,1-r),(0,0)}$.

Our final result shows that every monomial ideal of R has a possibly infinite m-irreducible decomposition and can be written as a possibly infinite sum of ideals of the form $I_{\underline{\alpha},\underline{\varepsilon}}$. [2]

Proposition 4.14. *Let I be a monomial ideal with monomial generating set S. Then there are equalities*

$$I = \sum_{\underline{X}^{\underline{r}} \in S} I_{\underline{r},\underline{0}} = \bigcap_{\underline{X}^{\underline{r}} \notin I} J_{\underline{r},\underline{1}}$$

where $\underline{0} = (0, \ldots, 0)$ and $\underline{1} = (1, \ldots, 1)$.

Proof. The first equality is straightforward, using Example 4.3:

$$I = (S)R = \sum_{\underline{X}^{\underline{r}} \in S} (\underline{X}^{\underline{r}})R = \sum_{\underline{X}^{\underline{r}} \in S} I_{\underline{r},\underline{0}}.$$

Fact 2.3(d) implies that the ideal $\bigcap_{\underline{X}^{\underline{r}} \notin I} J_{\underline{r},\underline{1}}$ is a monomial ideal of R, so Fact 2.2(c) implies that we need only show that $[\![I]\!] = \bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$.

For the containment $[\![I]\!] \subseteq \bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$ let $\underline{X}^{\underline{\alpha}} \in [\![I]\!]$ and $\underline{X}^{\underline{r}} \notin I$; we need to show that $\underline{X}^{\underline{\alpha}} \in J_{\underline{r},\underline{1}}$, that is, that $\alpha_i > r_i$ for some i. Suppose by way of contradiction that $\alpha_i \leq r_i$ for all i. Then $\underline{X}^{\underline{r}} \in (\underline{X}^{\underline{\alpha}})R \subseteq I$, a contradiction.

For the reverse containment $[\![I]\!] \supseteq \bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$, we show that $[\![R]\!] \setminus [\![I]\!] \subseteq [\![R]\!] \setminus \bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$. Let $\underline{X}^{\underline{\beta}} \in [\![R]\!] \setminus [\![I]\!]$. It follows that $\underline{X}^{\underline{\beta}}$ is in the index set for the intersection $\bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$. Since $\beta_i \not> \beta_i$ for all i, we have $\underline{X}^{\underline{\beta}} \notin J_{\underline{\beta},\underline{1}}$, so

$$\underline{X}^{\underline{\beta}} \in [\![R]\!] \setminus \left[\!\!\left[J_{\underline{\beta},\underline{1}} \right]\!\!\right] \subseteq \bigcup_{\underline{X}^{\underline{r}} \notin I} \left([\![R]\!] \setminus [\![J_{\underline{r},\underline{1}}]\!] \right) = [\![R]\!] \setminus \bigcap_{\underline{X}^{\underline{r}} \notin I} [\![J_{\underline{r},\underline{1}}]\!]$$

as desired. □

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[2]Note that the case $S = \emptyset$ is covered by the convention that the empty sum of ideals is the zero ideal; the case $[\![R]\!] \setminus [\![I]\!] = \emptyset$ is similarly covered since the empty intersection contains the empty product which is the unit ideal.

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