Journal of Algebra and Its Applications
Vol. 12, No. 5 (2013) 1250223 (24 pages)

DOI: 10.1142/S0219498812502234

EDGE IDEALS OF WEIGHTED GRAPHS

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Received 10 July 2012
Accepted 6 October 2012
Published 8 March 2013

Communicated by J. Trlifaj

Dedicated to Warren Shreve on the eve of his retirement

We study weighted graphs and their "edge ideals" which are ideals in polynomial rings that are defined in terms of the graphs. We provide combinatorial descriptions of m-irreducible decompositions for the edge ideal of a weighted graph in terms of the combinatorics of "weighted vertex covers". We use these, for instance, to say when these ideals are m-unmixed. We explicitly describe which weighted cycles, suspensions, and trees are unmixed and which ones are Cohen–Macaulay, and we prove that all weighted complete graphs are Cohen–Macaulay.

Keywords: Edge ideals; weighted graphs; unmixed; Cohen–Macaulay

Mathematics Subject Classification: Primary: 05C22, 05E40, 13F55, 13H10; Secondary: 05C05, 05C38, 05C69, 13C05

0. Introduction

Convention. Throughout this paper, let A be a nonzero commutative ring, and let R denote a polynomial ring $R = A[X_1, \ldots, X_d]$. Let $G = (V, E)$ be a (finite simple) graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. An edge between vertices v_i and v_j is denoted v_iv_j.

In this section, assume that A is a field.

Algebra and combinatorics have a rich history of interaction. In short, one can study combinatorial objects (graphs, posets, simplicial complexes, etc.) through algebraic constructions. In the other direction, one can use these constructions to find interesting examples of ideals and rings, for instance, families of Cohen–Macaulay rings. This paper continues in this tradition.

A relatively new (but well-studied) construction takes the graph G and associates to it the "edge ideal" $I(G)$ in the polynomial ring R. Much work has been done to relate the combinatorial properties of G to the algebraic properties of $I(G)$, and vice versa. For instance, one can explicitly describe the irreducible decomposition of $I(G)$ in terms of the combinatorial structure of G. In particular, one can easily describe when $I(G)$ is unmixed. On the other hand, the Cohen–Macaulay property for $R/I(G)$ is more subtle. Much work in the literature is devoted to providing classes of graphs G such that $R/I(G)$ is Cohen–Macaulay (or not) for instance in [1, 2, 5].

In this paper, we introduce and study a version of this construction for weighted graphs; see Secs. 1 and 2 for background material on weighted graphs and monomial ideals. We study the irreducible decompositions of these ideals via "weighted vertex covers" and characterize when these ideals are unmixed in Sec. 3. We apply this, for instance, to the situation of weighted cycles (which are almost always mixed) and weighted complete graphs (which are always unmixed) in Sec. 4. We conclude with Sec. 5 which describes some situations where these weighted graphs are Cohen–Macaulay. For instance, we completely characterize the Cohen–Macaulay weighted cycles.

Theorem A. *Consider a weighted d-cycle C_ω^d.*

(a) *If C_ω^d is Cohen–Macaulay, then $d \in \{3, 5\}$.*
(b) *C_ω^3 is always Cohen–Macaulay.*
(c) *C_ω^5 is Cohen–Macaulay if and only if it can be written in the form*

$$\begin{tikzcd}
v_1 \arrow[r, dash, "a"] \arrow[d, dash, "a"'] & v_2 \arrow[r, dash, "b"] & v_3 \arrow[dl, dash, "c"] \\
v_5 \arrow[r, dash, "d"'] & v_4 &
\end{tikzcd}$$

such that $a \le b \ge c \le d \ge e$.

This result is proved at the end of Sec. 5. In Theorem 5.10 we also completely characterize the Cohen–Macaulay weighted trees. This result contains the following theorem.

Theorem B. *If the weighted tree T_ω is Cohen–Macaulay, then the underlying tree T is a suspension of a tree, hence T is Cohen–Macaulay. Conversely, if T is a Cohen–Macaulay tree, then there is a weight function ω such that T_ω is Cohen–Macaulay.*

Recall that every suspension of a tree is Cohen–Macaulay. The same is not true for every weighted tree T_ω whose underlying graph is a suspension of a tree: if T is a suspension of a tree, then the weights on the "whiskers" determine whether T_ω is Cohen–Macaulay. This is a consequence of Theorem 5.7 which characterizes the Cohen–Macaulay weighted suspension.

As one may expect, we computed a number of examples using Macaulay 2 [3] in the process of proving our results, though none of our proofs depends on these computations.

1. Weighted Graphs and Weighted Vertex Covers

In this section, we introduce weighted vertex covers for weighted graphs and describe some of their basic properties for use in subsequent sections. Recall that G is a graph with vertex set $V = \{v_1, \ldots, v_d\}$.

Definition 1.1. A *vertex cover* of G is a subset $V' \subseteq V$ such that for each edge v_iv_j in G either $v_i \in V'$ or $v_j \in V'$. A vertex cover is *minimal* if it does not properly contain another vertex cover of G.

Definition 1.2. A *weight function* on a graph G is a function $\omega\colon E \to \mathbb{N}$ that assigns a *weight* to each edge.[a] A *weighted graph* G_ω is a graph G equipped with a weight function ω. A weighted graph G_ω where each edge has the same weight is a *trivially weighted graph*.

Note 1.3. We represent weighted graphs graphically, as in the statement of Theorem A in the introduction, by labeling each edge with its weight.

Definition 1.4. Let G_ω be a weighted graph. A *weighted vertex cover* of G_ω is an ordered pair (V', δ') such that V' is a vertex cover of G and $\delta'\colon V' \to \mathbb{N}$ is a function such that for each edge $e = v_iv_j \in E$ we have

(1) $v_i \in V'$ and $\delta'(v_i) \leq \omega(e)$, or
(2) $v_j \in V'$ and $\delta'(v_j) \leq \omega(e)$.

The number $\delta'(v_i)$ is the *weight* of v_i. When condition (1) is satisfied, we write that the vertex v_i *covers* the edge e, and similarly for condition (2).

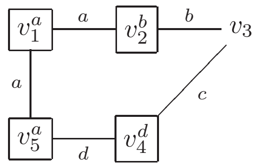
Notation 1.5. Given a weighted vertex cover (V', δ') of a weighted graph G_ω, we sometimes write $V' = \{v_{i_1}^{\delta'(v_{i_1})}, \ldots, v_{i_k}^{\delta'(v_{i_k})}\}$.

Notation 1.6. For $d \geq 3$, a *d-cycle* is the graph C^d with vertex set $V(C^d) = \{v_1, \ldots, v_d\}$ and edge set $E(C^d) = \{v_1v_2, v_2v_3, \ldots, v_dv_1\}$. We denote this graph as $C^d = v_1v_2 \ldots v_dv_1$.

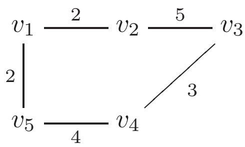
Note 1.7. As with weighted graphs, we represent weighted vertex covers graphically. For instance, the following sketch represents the weighted vertex cover

[a]We assume that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

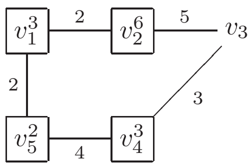
$\{v_1^a, v_2^b, v_4^d, v_5^a\}$ of the weighted 5-cycle from Theorem A in the introduction:

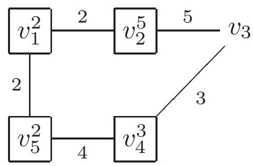


Example 1.8. Let C_ω^5 denote the following weighted 5-cycle:



Then the first sketch in the following display does not represent a weighted vertex cover of C_ω^5 because the edges v_1v_2 and v_2v_3 are not covered.





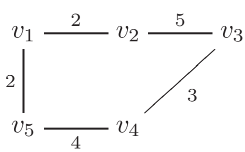
The second sketch in this display is a weighted vertex cover of C_ω^5.

We define an ordering of weighted vertex covers next.

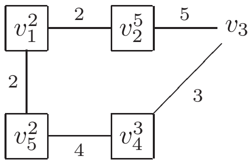
Definition 1.9. Let G_ω be a weighted graph. Given two weighted vertex covers (V', δ') and (V'', δ''), write $(V'', \delta'') \leq (V', \delta')$ if $V'' \subseteq V'$ and for all $v_i \in V''$ we have $\delta'(i) \leq \delta''(i)$. A weighted vertex cover (V', δ') is *minimal* if there does not exist another weighted vertex cover (V'', δ'') such that $(V'', \delta'') < (V', \delta')$. We define $|(V', \delta')| = |V'|$.

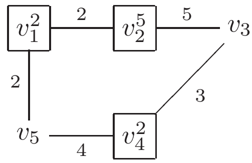
The graph G is said to be *unmixed* if all of the minimal vertex covers of G have the same cardinality. If G is not unmixed then G is *mixed*. Similarly, a weighted graph G_ω is *unmixed* if all of the minimal weighted vertex covers of G_ω have the same cardinality. If G_ω is not unmixed then G_ω is *mixed*.

Example 1.10. Let C_ω^5 denote the following weighted 5-cycle:

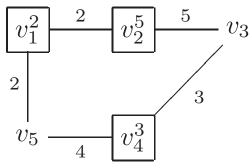


Then the first sketch in the following display is a weighted vertex cover of C_ω^5 that is not minimal, because the weighted vertex v_5^2 is not needed:





The second sketch in this display is also a non-minimal weighted vertex cover of C_ω^5 because the weight on v_4 can be increased to make the next weighted vertex cover which is minimal:



Note that this minimal weighted vertex cover can be obtained by removing the superfluous vertex from the first non-minimal weighted vertex cover.

The following results will be useful in the sections that follow. The first one says that, if the weight on a vertex in a weighted vertex cover can be increased without bound, then that vertex can be removed from the weighted vertex cover.

Lemma 1.11. *Let G_ω be a weighted graph, and assume that, for $j = 1, 2, \dots$ we have a weighted vertex cover $V_j = \{v_1^{a_1}, \dots, v_n^{a_n}, v_{n+1}^{b_j}\}$ of G_ω. If $b_1 < b_2 < \cdots$, then $V' = \{v_1^{a_1}, \dots, v_n^{a_n}\}$ is also a weighted vertex cover of G_ω.*

Proof. Let $e = v_i v_{n+1}$ be an edge in G_ω with weight $\omega(e)$. By assumption, there is an index j such that $b_j > \omega(e)$. Since V_j is a weighted vertex cover of G_ω, the edge e must be covered by v_i, that is, we must have $i \le n$ and $a_i \le \omega(e)$. Since this is so for each edge of the form $e = v_i v_{n+1}$, it follows that every edge of G_ω is covered by one of the weighted vertices $v_1^{a_1}, \dots, v_n^{a_n}$. In other words, V' is also a weighted vertex cover of G_ω, as desired. $\square$

Proposition 1.12. *Let G_ω be a weighted graph. Then for every weighted vertex cover (V', δ') of G_ω there is a minimal weighted vertex cover (V''', δ''') of G_ω such that $(V''', \delta''') \le (V', \delta')$.*

Proof. If (V', δ') is itself a minimal weighted vertex cover for G_ω, then we are done. If (V', δ') is not minimal, then either there is a $v_i \in V'$ that can be removed or for some $v_i \in V'$ the function $\delta'(v_i)$ can be increased, as in Example 1.10. In the first case, remove vertices from V' until the removal of one more vertex creates an ordered pair that is no longer a weighted vertex cover. Notice that this process terminates in finitely many steps because V' is finite. Let us denote this new weighted vertex cover as (V'', δ''). (If no vertices can be removed, then set $(V'', \delta'') = (V', \delta')$.)

Now, if (V'', δ'') is a minimal weighted vertex cover for G_ω, then we are done. If it is not minimal, then we can increase the weight of at least one of the vertices in V'. Increase the weight of each vertex (in sequence) such that any further increase would cause the ordered pair to not be a weighted vertex cover. This process also terminates in finitely many steps because the weight of each vertex of V'' cannot be increased without bound, by Lemma 1.11. We will denote this new ordered pair (V''', δ'''). Since no vertices can be removed from (V''', δ''') and the weight of each $v_i \in V'''$ cannot be increased, the pair (V''', δ''') is a minimal weighted vertex cover for G_ω such that $(V''', \delta''') \leq (V', \delta')$. □

Proposition 1.13. *Let G_ω be a weighted graph. Then every minimal vertex cover of the unweighted graph G occurs as a minimal weighted vertex cover of G_ω.*

Proof. Let V' be a minimal vertex cover for G. We need to show that (V', δ') is a minimal weighted vertex cover for G_ω for some δ'. For each $v_i \in V'$ define

$$\delta'(v_i) = \min\{\omega(e) \mid e = v_i v_j \in E \text{ for some } v_j\}.$$

We claim that (V', δ') is a weighted vertex cover for G_ω. Let $e = v_i v_j$ be an edge G. If $v_i \in V'$, then by definition of δ' we have $\delta'(v_i) \leq \omega(e)$; and if $v_j \in V'$, then $\delta'(v_j) \leq \omega(e)$. Hence (V', δ') is a weighted vertex cover.

Proposition 1.12 provides a minimal weighted vertex cover (V''', δ''') of G_ω such that $(V''', \delta''') \leq (V', \delta')$. Since V' is a minimal vertex cover, we cannot remove any vertices from V'. Since V''' is a vertex cover for G, the condition $V''' \subseteq V'$ implies that $V''' = V'$. Thus, V' occurs as the minimal weighted vertex cover (V''', δ'''). □

Proposition 1.14. *If G is mixed, then G_ω is mixed.*

Proof. Assume that G is mixed. Then there are minimal vertex covers V' and V'' for G such that $|V'| \neq |V''|$. By Proposition 1.13, we have functions $\delta' \colon V' \to \mathbb{N}$ and $\delta'' \colon V'' \to \mathbb{N}$ such that (V', δ') and (V'', δ'') are minimal weighted vertex covers for G_ω. Since $|(V', \delta')| \neq |(V'', \delta'')|$ we conclude that G_ω is mixed. □

2. Monomial Ideals

In this section, we include some background material on monomial ideals in the polynomial ring $R = A[X_1, \ldots, X_d]$.

Definition 2.1. A *monomial* in R is an element of the form $X_1^{a_1} \cdots X_d^{a_d}$ where the a_i are non-negative integers. A *monomial ideal* in R is an ideal generated by a (possibly empty) set of monomials of R. For each monomial ideal $I \subset R$, let $[\![I]\!]$ denote the set of all monomials contained in I.

Definition 2.2. For each subset $V' \subseteq V$, let $P(V') \subseteq R$ be the ideal "generated by the elements of V'":

$$P(V') = (X_i \mid v_i \in V')R.$$

For each subset $V' \subseteq V$ and for each function $\delta' \colon V' \to \mathbb{N}$, let $P(V', \delta') \subseteq R$ be the ideal "generated by the elements of (V', δ')":

$$P(V', \delta') = (X_i^{\delta'(v_i)} \mid v_i \in V')R.$$

We say that the ideals $P(V', \delta')$ are m-*irreducible*, to indicate that they are irreducible with respect to intersections of monomial ideals.

Note 2.3. The notation $V' = \{v_{i_1}^{\delta'(v_{i_1})}, \ldots, v_{i_k}^{\delta'(v_{i_k})}\}$ is handy because it essentially lists the generators of $P(V', \delta')$.

Example 2.4. The ideals $P(V', \delta')$ coming from the three weighted vertex covers in Example 1.10 are $(X_1^2, X_2^5, X_4^3, X_5^2)R$, $(X_1^2, X_2^5, X_4^2)R$, and $(X_1^2, X_2^5, X_4^3)R$. Notice that the ideal corresponding to the minimal weighted vertex cover is contained in the ideals corresponding to non-minimal weighted vertex covers.

Example 2.5. We have $P(\emptyset) = (\emptyset)R = 0$ and $P(\emptyset, \delta') = (\emptyset)R = 0$.

Note 2.6. A monomial ideal $I \subseteq R$ is of the form $P(V', \delta')$ if and only if it is generated by "pure powers" of the variables, that is, by monomials of the form $X_i^{e_i}$. When A is a field, the ideals $P(V')$ are precisely the prime monomial ideals, and the ideals $P(V', \delta')$ are precisely the irreducible monomial ideals.

Definition 2.7. Given an ordered pair (V', δ') the m-*height* of $P(V', \delta')$ is

$$\text{m-ht}(P(V', \delta')) = |V'|.$$

Given a monomial ideal $I \subset R$ such that $I = \bigcap_{i=1}^m P(V_i, \delta_i)$, the m-*height* of I is

$$\text{m-ht}(I) = \min_i \{\text{m-ht}(P(V_i, \delta_i))\}.$$

Note 2.8. Assume that A is a field. In this case, each ideal $P(V')$ is prime in R, and $\text{m-ht}(P(V', \delta')) = \text{m-ht}(P(V'))$ is the same as $\text{ht}(P(V', \delta')) = \text{ht}(P(V'))$. We use the notation m-ht in general to indicate that we are taking the height with respect to monomial prime ideals.

Definition 2.9. Assume that $I = \bigcap_{i=1}^m P(V_i, \delta_i)$ is an irredundant m-irreducible decomposition, that is, such that there are no containment relations between the ideals in the intersection. Then I is m-*unmixed* provided that $\text{m-ht}(P(V_i, \delta_i)) = \text{m-ht}(P(V_j, \delta_j))$ for all i, j, that is, if $\text{m-ht}(P(V_i, \delta_i)) = \text{m-ht}(I)$ for all i. We say that I is m-*mixed* if it is not m-unmixed.

Note 2.10. If A is a field, then a monomial ideal $I \subset R$ is m-unmixed if and only if it is unmixed.

3. Weighted Edge Ideals and Their Decompositions

In this section, we define the edge ideal of a weighted graph and establish some of its fundamental properties. Recall that G is a graph with vertex set V and edge set E, and $R = A[X_1, \dots, X_d]$.

Convention. In this section, G_ω is a weighted graph.

Definition 3.1. The *edge ideal* associated to G is the ideal $I(G) \subseteq R$ that is "generated by the edges of G":

$$I(G) = (X_i X_j \mid v_i v_j \in E)R.$$

The *weighted edge ideal* associated to G_ω is the ideal $I(G_\omega) \subseteq R$ that is "generated by the weighted edges of G":

$$I(G_\omega) = (X_i^{\omega(e)} X_j^{\omega(e)} \mid e = v_i v_j \in E)R.$$

Note 3.2. M-irreducible decompositions for the edge ideal $I(G)$ are $I(G) = \bigcap_{V'} P(V') = \bigcap_{\min V'} P(V')$ where the first intersection is taken over the set of all vertex covers of G, and the second intersection is taken over the set of all minimal vertex covers of G; see, e.g. [4, Theorem 5.3.9]. Furthermore, the second decomposition is irredundant. One of the points of this section is to provide analogous decompositions for $I(G_\omega)$. This is done in Theorem 3.5.

The following lemma is the first key to decomposing the edge ideal of G_ω.

Lemma 3.3. *Given subsets $V', V'' \subseteq V$ and functions $\delta' \colon V' \to \mathbb{N}, \delta'' \colon V'' \to \mathbb{N}$, we have $(V'', \delta'') \leq (V', \delta')$ if and only if $P(V'', \delta'') \subseteq P(V', \delta')$.*

Proof. Let us begin by assuming that $(V'', \delta'') \leq (V', \delta')$. Then we have $V'' \subseteq V'$ and $\delta'(v_i) \leq \delta''(v_i)$ for all $v_i \in V''$. To show that $P(V'', \delta'') \subseteq P(V', \delta')$ we need to show that each generator $X_i^{\delta''(v_i)}$ of $P(V'', \delta'')$ is in $P(V', \delta')$. By assumption, we have $v_i \in V'' \subseteq V'$ and $\delta''(v_i) \geq \delta'(v_i)$. Thus, the condition $X_i^{\delta'(v_i)} \in P(V', \delta')$ implies that $X_i^{\delta''(v_i)} = X_i^{\delta''(v_i) - \delta'(v_i)} X_i^{\delta'(v_i)} \in P(V', \delta')$. Hence $P(V'', \delta'') \subseteq P(V', \delta')$.

For the converse assume that $P(V'', \delta'') \subseteq P(V', \delta')$. Then $X_i^{\delta''(v_i)} \in P(V', \delta')$ for all $v_i \in V''$. Therefore, there is a generator $X_j^{\delta'(v_j)} \in P(v', \delta')$ such that $X_j^{\delta'(v_j)} | X_i^{\delta'(v_i)}$. Since $\delta'(v_j) \geq 1$, it follows that $i = j$ and $\delta'(v_j) \leq \delta''(v_i)$. Thus, $v_i = v_j \in V'$ and $\delta'(v_i) = \delta'(v_j) \leq \delta''(v_j)$. Since this is so for all $v_i \in V''$, we have $(V'', \delta'') \leq (V', \delta')$, by definition. $\square$

The next result is the second key to decomposing $I(G_\omega)$.

Lemma 3.4. *Given a subset $V' \subseteq V$ and a function $\delta' \colon V' \to \mathbb{N}$, one has $I(G_\omega) \subseteq P(V', \delta')$ if and only if (V', δ') is a weighted vertex cover of G_ω.*

Proof. Write $V' = \{v_{i_1}, \ldots, v_{i_k}\}$. Assume first that $I(G_\omega) \subseteq P(V', \delta')$. Then for all $e = v_i v_j \in E$, we have $X_i^{\omega(e)} X_j^{\omega(e)} \in I(G_\omega) \subseteq P(V', \delta') = (X_{i_1}^{\delta'(v_{i_1})}, \ldots, X_{i_k}^{\delta'(v_{i_k})})$. Thus $X_{i_\ell}^{\delta'(v_{i_\ell})} \mid X_i^{\omega(e)} X_j^{\omega(e)}$ for some $1 \leq \ell \leq k$. Since $\delta'(v_{i_\ell}) \geq 1$, we conclude that either $i_\ell = i$ and $\delta'(v_{i_\ell}) \leq \omega(e)$, or $i_\ell = j$ and $\delta'(v_{i_\ell}) \leq \omega(e)$. Thus, either $v_i = v_{i_\ell} \in V'$ and $\delta'(v_{i_\ell}) \leq \omega(e)$, or $v_j = v_{i_\ell} \in V'$ and $\delta'(v_{i_\ell}) \leq \omega(e)$. Since this is so for each edge in G, we conclude that (V', δ') is a weighted vertex cover of G_ω.

For the converse assume that (V', δ') is a weighted vertex cover of G_ω. We need to show that each generator of $I(G_\omega)$ is an element of $P(V', \delta')$. Let $X_i^{\omega(e)} X_j^{\omega(e)}$ be a generator of $I(G_\omega)$ corresponding to the edge $e = v_i v_j$ with weight $\omega(e)$ in G_ω. Since (V', δ') is a weighted vertex cover, we have two cases. The first case is when $v_i \in V'$ and $\delta'(v_i) \leq \omega(e)$; in this case, we have $X_i^{\delta'(i)} \mid X_i^{\omega(e)} X_j^{\omega(e)}$ and so $X_i^{\omega(e)} X_j^{\omega(e)} \in P(V', \delta')$. Similarly, if $v_j \in V'$ and $\delta'(v_j) \leq \omega(e)$, then $X_i^{\omega(e)} X_j^{\omega(e)} \in P(V', \delta')$. Thus $I(G_\omega) \subseteq P(V', \delta')$. $\square$

Here is our decomposition result for $I(G_\omega)$.

Theorem 3.5. *Let G_ω be a weighted graph with vertex set $V = \{v_1, \ldots, v_d\}$. Then*

$$I(G_\omega) = \bigcap_{(V',\delta')} P(V', \delta') = \bigcap_{\min(V',\delta')} P(V', \delta'),$$

where the first intersection is taken over all weighted vertex covers of G_ω and the second intersection is taken over all minimal weighted vertex covers of G_ω. Furthermore, the second decomposition is irredundant.

Proof. Every monomial ideal can be written as a finite (possibly empty) intersection of m-irreducible ideals, i.e. ideals of the form $P(V', \delta')$; see, e.g. [4, Theorems 4.1.4 and 4.3.1]. This implies that $I(G_\omega)$ is a finite intersection of ideals of the form $P(V', \delta')$, and Lemma 3.4 implies that the only (V', δ') that can occur in such a decomposition are weighted vertex covers for G_ω. Thus, we have $I(G_\omega) = \bigcap_{(V',\delta')} P(V', \delta')$.

Since every minimal weighted vertex cover is a weighted vertex cover we have

$$\bigcap_{(V',\delta')} P(V', \delta') \subseteq \bigcap_{\min(V',\delta')} P(V', \delta').$$

The reverse containment

$$\bigcap_{(V',\delta')} P(V', \delta') \supseteq \bigcap_{\min(V',\delta')} P(V', \delta')$$

follows from Proposition 1.12 and Lemma 3.3.

Finally, the intersection $\bigcap_{\min(V',\delta')} P(V', \delta')$ is irredundant by Lemma 3.3. $\square$

Theorem 3.5 proves the next result that connects unmixedness for graphs and edge ideals.

Corollary 3.6. *The graph G is unmixed if and only if the ideal $I(G)$ is* m*-unmixed. The weighted graph G_ω is unmixed if and only if the ideal $I(G_\omega)$ is* m*-unmixed.*

Remark 3.7. Corollary 3.6 shows that m-unmixedness of $I(G_\omega)$ is independent of the ring A since it only depends on the unmixedness of G_ω.

Example 3.8. We decompose $I(P^2_\omega)$ where P^2_ω is the following weighted 2-path:

$$v_1 \xrightarrow{\;a\;} v_2 \xrightarrow{\;b\;} v_3.$$

Assume by symmetry that $a \le b$. In this case, we have

$$\begin{aligned} I(P^2_\omega) &= (X_1^a X_2^a, X_2^b X_3^b)R \\ &= (X_1^a, X_2^b X_3^b)R \cap (X_2^a, X_2^b X_3^b)R \\ &= (X_1^a, X_2^b)R \cap (X_1^a, X_3^b)R \cap (X_2^a)R. \end{aligned}$$

If $a < b$, then this decomposition is irredundant. By Lemma 3.3, we conclude that there are exactly three minimal weighted vertex covers for P^2_ω, namely $\{v_1^a, v_2^b\}$, $\{v_1^a, v_3^b\}$ and $\{v_2^a\}$.

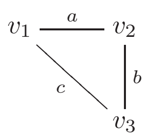
On the other hand, if $a = b$, then we have $(X_2^a)R = (X_2^b)R \subseteq (X_1^a, X_2^b)R$, and hence

$$I(P^2_\omega) = (X_1^a, X_3^b)R \cap (X_2^a)R.$$

We deduce that there are exactly two minimal weighted vertex covers for P^2_ω in this case.

In either case, we conclude that P^2_ω is mixed and $I(P^2_\omega)$ is m-mixed. See Sec. 5 for more information about weighted paths.

Example 3.9. We decompose $I(C^3_\omega)$ where C^3_ω is the following weighted 3-cycle:



Assume by symmetry that $a \le b \le c$. In this case, we decompose as in Example 3.8 to find

$$\begin{aligned} I(C^3_\omega) &= (X_1^a X_2^a, X_2^b X_3^b, X_1^c X_3^c)R \\ &= (X_1^a, X_2^b)R \cap (X_1^a, X_3^b)R \cap (X_1^c, X_2^a)R \cap (X_2^a, X_3^c)R. \end{aligned}$$

It follows that C^3_ω is unmixed and $I(C^3_\omega)$ is m-unmixed. It is worth noting that, when $a < b < c$, this decomposition is irredundant with two ideals of the form $(X_1^p, X_2^q)R$; this sort of behavior does not occur in the unweighted case. See Secs. 4 and 5 for more information about weighted cycles.

We end this section with a few results about associated primes and (un)mixedness.

Definition 3.10. Let I be a monomial ideal in R. The *monomial radical* of I is the monomial ideal $\text{m-rad}(I) = (S)R$ where

$$S = [\![R]\!] \cap \operatorname{rad}(I) = \{z \in [\![R]\!] \mid z^n \in I \text{ for some } n \geq 1\},$$

where $\operatorname{rad}(I)$ is the radical of I and $[\![R]\!]$ is from Definition 2.1.

Note 3.11. If A is a field (more generally, if A is reduced), then $\text{m-rad}(I) = \operatorname{rad}(I)$ for each monomial ideal $I \subseteq R$.

Lemma 3.12. *We have* $\text{m-rad}(I(G_\omega)) = I(G)$ *and* $\text{m-rad}(P(V', \delta')) = P(V')$ *for each ordered pair* (V', δ').

Proof. Given a monomial $f = X_{i_1}^{a_{i_1}} \cdots X_{i_n}^{a_{i_n}}$ where each $a_i \geq 1$, set $\operatorname{red}(f) = X_{i_1} \cdots X_{i_n}$. If I is generated by the set $S \subseteq [\![R]\!]$, then $\text{m-rad}(I)$ is generated by the set $\{\operatorname{red}(f) \mid f \in S\}$; see, e.g. [4, Proposition 3.5.5]. The desired conclusions now follow. $\square$

Proposition 3.13. *Assume that A is an integral domain.*

(a) *The minimal primes of $I(G_\omega)$ are the ideals $P(V')$ such that V' is a minimal vertex cover of G.*
(b) *The associated primes of $I(G_\omega)$ are the ideals $P(V')$ such that (V', δ') is a minimal weighted vertex cover of G_ω.*

Proof. (a) The minimal primes of $I(G_\omega)$ are the m-irreducible components of $\operatorname{rad}(I(G_\omega)) = \text{m-rad}(I(G_\omega)) = I(G_\omega)$ by Note 3.11 and Lemma 3.12. From Note 3.2 we know that $I(G) = \bigcap_{\min V'} P(V')$ is an irredundant irreducible decomposition where the intersection is taken over the set of all minimal vertex covers of G. It follows that the minimal primes of $I(G_\omega)$ are the ideals $P(V')$ such that V' is a minimal vertex cover of G, as claimed.

(b) The associated primes of $I(G_\omega)$ are the radicals of the m-irredundant irreducible components of $I(G_\omega)$. Theorem 3.5 implies that $I(G_\omega) = \bigcap_{\min(V', \delta')} P(V', \delta')$ is an irredundant m-irreducible decomposition where the intersection is take over the set of all minimal weighted vertex covers of G_ω. Hence, the associated primes of $I(G_\omega)$ are the ideals $\operatorname{rad}(P(V', \delta')) = P(V')$ where (V', δ') is a minimal weighted vertex cover of G_ω. $\square$

Proposition 3.14. *A trivially weighted graph G_ω is unmixed if and only if G is unmixed.*

Proof. The forward implication is from Proposition 1.14.

For the converse assume that G is unmixed. Since G_ω is trivially weighted, let the weight of the each edge in G_ω be a. Given a monomial ideal $I \subseteq R$, set $I^{[a]} = (\{f^a \mid f \in [\![I]\!]\})R$ where the notation $[\![I]\!]$ is from Definition 2.1. Since G_ω is trivially weighted, it is straightforward to show that $I(G_\omega) = I(G)^{[a]}$. Furthermore, given the m-irreducible decomposition $I(G) = \bigcap_{\min V'} P(V')$ from Note 3.2, we have

$$I(G_\omega) = I(G)^{[a]} = \bigcap_{\min V'} P(V')^{[a]} = \bigcap_{(V',\delta')} P(V', \delta'),$$

where $\delta'(v_i) := a$ for all $v_i \in V'$; see, e.g. [4, Proposition 7.1.3]. Since G is unmixed, each V' in this intersection has the same cardinality. It follows that each (V', δ') has the same cardinality. Therefore G_ω is unmixed. $\square$

4. Weighted Cycles and Weighted Complete Graphs

In this section, we characterize the weighted cycles and complete graphs that are unmixed.

Fact 4.1. C^n is unmixed if and only if $n \in \{3, 4, 5, 7\}$; see, e.g. [6, Exercise 6.2.15].

We treat the weighted cycles case-by-case. Here is a convenient summary of these results.

(a) If C^d_ω is unmixed, then $d \in \{3, 4, 5, 7\}$ by Proposition 1.14 and Fact 4.1.
(b) Every C^3_ω is unmixed by Example 3.9.
(c) C^4_ω is unmixed if and only if it is trivially weighted by Propositions 4.3 and 4.2.
(d) C^5_ω: see Theorem 4.4.
(e) C^7_ω is unmixed if and only if it is trivially weighted by Propositions 4.2 and 4.5.

Proposition 4.2. *For $n \in \{3, 4, 5, 7\}$, every trivially weighted n-cycle C^n_ω is unmixed.*

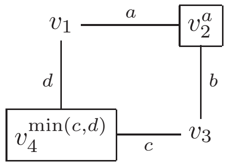
Proof. From Fact 4.1 we know that C^n is unmixed. Thus, Proposition 3.14 implies that C^n_ω is unmixed. $\square$

Proposition 4.3. *Every non-trivially weighted 4-cycle, C^4_ω is mixed.*

Proof. Let us consider a non-trivially weighted 4-cycle whose underlying unweighted graph is $C^4 = v_1v_2v_3v_4v_1$ and the weights of the edges are as follows:

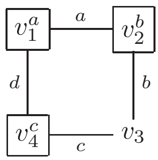
$$\begin{tikzcd} v_1 \arrow[r, dash, "a"] \arrow[d, dash, "d"'] & v_2 \arrow[d, dash, "b"] \\ v_4 \arrow[r, dash, "c"'] & v_3 \end{tikzcd}$$

By symmetry, assume that a is the smallest weight on any edge. Then since C^4 is not trivially weighted, at least one edge has weight strictly greater than a. By symmetry assume that $a < b$. Now we demonstrate two minimal vertex covers of different cardinalities. First, we consider $V' = \{v_2^a, v_4^{\min(c,d)}\}$.



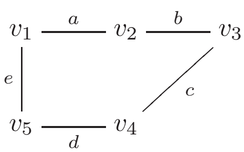
Notice that since $a < b$, the edges v_1v_2 and v_2v_3 are covered by v_2^a and since $\min(c,d) \le c, d$, the edges v_3v_4 and v_4v_1 are covered. The removal of either of these vertices would not result in a vertex cover. If we increase the weight on the vertex v_2, then the edge v_1v_2 will not be covered; and if we increase the weight on the vertex v_4, then the edge with the smaller weight (either v_3v_4 or v_4v_1) would not be covered. Thus, V' is a minimal weighted vertex cover with cardinality 2.

Now let $V'' = \{v_1^a, v_2^b, v_4^c\}$.



Notice that the vertex v_1^a covers the edges v_1v_2 and v_1v_4, the vertex v_2^b covers the edge v_2v_3 and the vertex v_4^c covers the edge v_3v_4. Furthermore, if we remove v_1^a from V'' or increase the weight, the edge v_1v_2 is not covered. If we remove the vertex v_2^b from V'' or increase the weight, the edge v_2v_3 is not covered. If we remove the vertex v_4^c from V'' or increase the weight then the edge v_3v_4 is not covered. Hence V'' is a minimal vertex cover with cardinality 3. Thus C_ω^4 is mixed. □

Theorem 4.4. *Let C_ω^5 be a weighted 5-cycle whose underlying unweighted graph is $C^5 = v_1v_2v_3v_4v_5v_1$. Then C_ω^5 is unmixed if and only if it is isomorphic to the weighted 5-cycle*



(4.1)

such that $e = a \le b \ge c \le d \ge e$.

Proof. Let us first assume that C_ω^5 is isomorphic to the weighted 5-cycle (4.1) and that $e = a \le b \ge c \le d \ge e$; we show that C_ω^5 is unmixed. In order to do this we decompose the edge ideal of C_ω^5 as in Example 3.8.

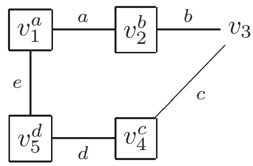
$$
\begin{aligned}
I(C_\omega^5) &= (X_1^a X_2^a, X_2^b X_3^b, X_3^c X_4^c, X_4^d X_5^d, X_5^e X_1^e) \\
&= (X_1^a, X_2^b, X_4^c) \cap (X_1^a, X_3^c, X_4^d) \cap (X_1^a, X_3^c, X_5^d) \bigcap (X_1^a, X_3^b, X_4^c) \\
&\quad \times \cap (X_2^a, X_3^c, X_5^e) \cap (X_2^a, X_4^c, X_5^e) \cap (X_2^a, X_4^c, X_1^e)
\end{aligned}
$$

Therefore C_ω^5 is unmixed when the weight on the edges are as specified.

For the converse we will assume that the weighted 5-cycle (4.1) is unmixed. By Proposition 1.13 and Fact 4.1, every minimal weighted vertex cover of C_ω^5 has cardinality 3. We proceed by steps to eliminate all possible cases of the comparability of the weights of the edges of C_ω^5 besides our hypothesized conclusion. In each step we derive contradictions by building minimal weighted vertex covers for that have cardinality greater than 3.

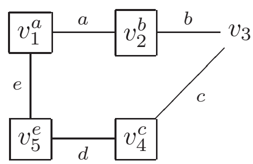
Step 1. Let us first suppose that $e < a < b$. We consider two cases.

Case 1: $e \geq d < c$. We claim $V' = \{v_1^a, v_2^b, v_4^d, v_5^e\}$ is a minimal weighted vertex cover.



It is routine to verify that all the weighted edges are covered by V'. We verify that V' is a minimal weighted vertex cover. Notice if we remove the weighted vertex v_1^a or increase the weight, then the edge v_1v_2 is not covered. If we remove the weighted vertex v_2^b or increase the weight, then the edge v_2v_3 is not covered. If we remove the weighted vertex v_4^c or increase the weight, then the edge v_3v_4 is not covered. If we remove the weighted vertex v_5^d or increase the weight, then the edge v_4v_5 is not covered. Thus V' is a minimal weighted vertex cover of cardinality 4, contradicting the unmixedness of C_ω^5.

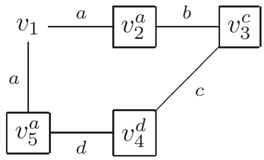
Case 2: Either $e \leq d$ or $e \geq d \geq c$. We claim that $V'' = \{v_1^a, v_2^b, v_4^c, v_5^e\}$ is a minimal weighted vertex cover.



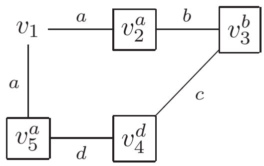
As in Case 1, it follows readily that V'' is a minimal weighted vertex cover of cardinality 4.

Step 2. Let us next suppose that no two adjacent edges have the same weight. Then since the cycle is of odd length we conclude by symmetry that C_ω^5 is isomorphic to a graph (4.1) such that $e < a < b$. Step 1 shows that this contradicts the unmixedness of C_ω^5. We conclude that there are two adjacent edges with the same weight.

Step 3. By symmetry, we assume that $e = a$. Now, suppose that $b < a$ and $d < a$. We again consider two cases. If $d > c \leq b$, then it is readily shown that $V' = \{v_2^a, v_3^c, v_4^d, v_5^a\}$ is a minimal weighted vertex cover.



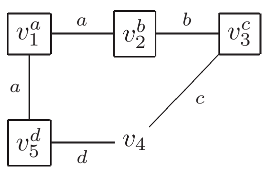
On the other hand, if $d \geq c \geq b$, $d \leq c \leq b$, or $d \leq c \geq b$, then $V'' = \{v_2^a, v_3^b, v_4^d, v_5^a\}$ is a minimal weighted vertex cover.



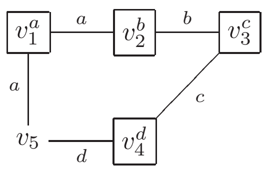
In each case we have exhibited a minimal weighted vertex cover of cardinality 4, which is a contradiction. Therefore, either $a \leq b$ or $a \leq d$.

Step 4. By symmetry, assume that $a \leq b$. Suppose that $b < c$. We consider six cases.

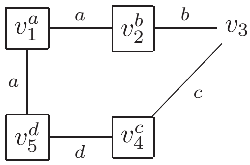
Case 1: $a < b < c$. In this case, $V' = \{v_1^a, v_2^b, v_3^c, v_5^d\}$ is a minimal weighted vertex cover.



Case 2: $a = b < c < d$. Here, $V'' = \{v_1^a, v_2^b, v_3^c, v_4^d\}$ is a minimal weighted vertex cover.



Case 3: $a = b < c > d > a$. In this case, $V''' = \{v_1^a, v_2^b, v_4^c, v_5^d\}$ is a minimal weighted vertex cover.



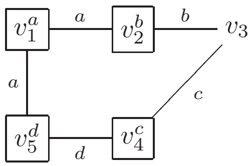
Case 4: $a = b < c \geq d \leq a$. This case fits our hypothesized conclusion.

Case 5: $a = b < c = d \leq a$. This case is not possible because it states that $a < c \leq a$.

Case 6: $a = b < c = d > a$. This case is covered by Step 3.

Step 5. Assume that $a \leq b \geq c$ and suppose $c > d$. We consider three cases.

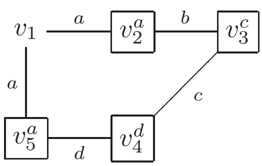
Case 1: $a < b \geq c > d$. Here, $V''' = \{v_1^a, v_2^b, v_4^c, v_5^d\}$ is a minimal weighted vertex cover.



Case 2: $a = b = c > d < a$ or $a = b = c > d \leq a$. This case fits our desired conclusion.

Case 3: $a = b \geq c > d \geq a$. This case is not possible because it states that $a \geq c > a$.

Step 6. Assume that $a \leq b \geq c \leq d$ and suppose that $a > d$. If $c < d$, then $V' = \{v_1^a, v_2^b, v_4^c, v_5^a\}$ is a minimal weighted vertex cover.

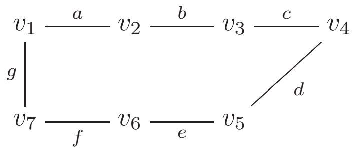


On the other hand, if $c = d$, then we have $c = d < e = a \leq b \geq c$ which fits our conclusion.

Thus, if C_ω^5 is unmixed, we have $e = a \leq b \geq c \leq d \geq e$, as claimed. $\square$

Proposition 4.5. *Every non-trivially weighted* 7*-cycle is mixed.*

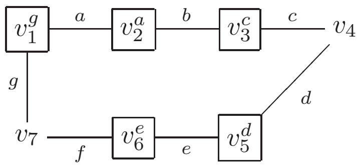
Proof. Let us consider a weighted 7-cycle whose underlying unweighted graph is $C^7 = v_1v_2v_3v_4v_5v_6v_7v_1$, weighted as follows:



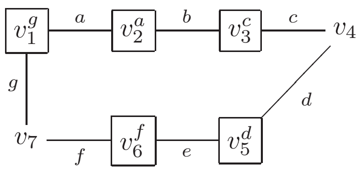
By symmetry, let us say that a is the smallest weight on any edge. Then since C^7 is not trivially weighted, at least one edge has weight strictly greater than a. By symmetry, assume that $a < g$.

Since the unweighted C^7 is unmixed and each minimal vertex cover has cardinality 4, Proposition 1.13 provides a minimal weighted vertex cover of C^7_ω with cardinality 4. Now we will demonstrate a weighted vertex cover V'' such that $|V''| = 5$. We consider two cases.

Case 1: $f \geq e \leq d$. In this case, $V'' = \{v_1^g, v_2^a, v_3^c, v_5^d, v_6^e\}$ is a minimal weighted vertex cover of C^7_ω.



Case 2: $f \leq e \geq d$, $f \geq e \geq d$ or $f \leq e \leq d$. In this case, $V'' = \{v_1^g, v_2^a, v_3^c, v_5^d, v_6^f\}$ is a minimal weighted vertex cover of C^7_ω.



Since we have demonstrated 2 minimal weighted vertex covers for C^7_ω of different cardinalities we conclude that our non-trivially weighted C^7_ω is mixed. $\square$

Proposition 4.6. *Every weighted complete graph K^n_ω is unmixed.*

Proof. It is easily verified that the smallest minimal vertex cover for K^n is of cardinality $n - 1$. Therefore by Proposition 1.13 the smallest minimal weighted vertex cover for K^n_ω also has cardinality $n - 1$. We show that there is not a minimal weighted vertex cover of cardinality n. Assume that (V', δ') is a minimal weighted vertex cover with cardinality n. By symmetry assume the vertex v_1 has the maximal weight of all the vertices of V'. Now consider the removal of v_1 from V'. Since $\delta'(v_1)$ was maximal and all other vertices of V are in (V', δ') then every edge adjacent to v_1

must be covered by the other vertex adjacent to that edge. Thus V' is not minimal and every minimal weighted vertex cover has cardinality $n-1$ which implies K_ω^n is unmixed. □

5. Cohen–Macaulay Weighted Graphs

In this section, we prove Theorems A and B from the introduction.

Convention. In this section, A is a field.

Definition 5.1. The weighted graph G_ω is *Cohen–Macaulay over A* if the ring $R/I(G_\omega)$ is Cohen–Macaulay. If G_ω is Cohen–Macaulay over every field, we simply say that it is *Cohen–Macaulay*.

The Cohen–Macaulay weighted complete graphs are easily identified.

Proposition 5.2. *Every weighted complete graph K_ω^n is Cohen–Macaulay.*

Proof. By Proposition 4.6 we know that K_ω^n is unmixed. Since A is a field, we also know that $\dim(K_\omega^n) = 1$ because the cardinality of the minimal vertex covers are $n-1$. Since unmixed in dimension 1 implies Cohen–Macaulay, we conclude that K_ω^n is Cohen–Macaulay. □

Next, we characterize the Cohen–Macaulay weighted suspensions and trees. One main point is the following lemma whose proof is essentially due to Herzog; see [5, Proposition 2.2; 6, Proposition 6.3.2].

Lemma 5.3. *Let $S = A[Y_1,\ldots,Y_n,Z_1,\ldots,Z_n]$ be a polynomial ring over A, and fix a subset $M \subseteq \{(i,j) \mid 1 \le i < j \le n\}$. Then the ideal*

$$I = (Y_i^{a_i}Z_i^{a_i}, Z_i^{b_{ij}}Z_j^{b_{ij}} \mid i = 1,\ldots,n \quad \text{and} \quad (i,j) \in M \quad \text{and} \quad a_i \ge b_{ij} \le a_j)S$$

is such that S/I is Cohen–Macaulay.

Proof. We polarize the ideal I to obtain

$$\begin{aligned} S' &= k[Y_{1,1},\ldots,Y_{1,a_1},\ldots,Y_{n,1}\ldots,Y_{n,a_n},Z_{1,1}\ldots,Z_{1,a_1},\ldots,Z_{n,1}\ldots,Z_{n,a_n}],\\ I' &= (Y_{1,1}\cdots Y_{1,a_1}Z_{1,1}\cdots Z_{1,a_1},\ldots,Y_{n,1}\cdots Y_{n,a_n}Z_{n,1}\cdots Z_{n,a_n},\\ &\qquad Z_{i,1}\cdots Z_{i,b_{ij}}Z_{j,1}\cdots Z_{j,b_{ij}})S'. \end{aligned}$$

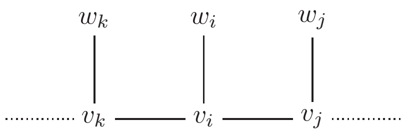
By general properties of polarization, the next sequence is S'-regular and S'/I'-regular:

$$\begin{gathered} Y_{1,1}-Y_{1,2},\ldots,Y_{1,1}-Y_{1,a_1},\ldots,Y_{n,1}-Y_{n,2},\ldots,Y_{n,1}-Y_{n,a_n},\\ Z_{1,1}-Z_{1,2},\ldots,Z_{1,1}-Z_{1,a_1},\ldots,Z_{n,1}-Z_{n,2},\ldots,Z_{n,1}-Z_{n,a_n}. \end{gathered}$$

Note that I' is a polarization of the ideal $J = (Z_i^{2a_i}, Z_i^{b_{i,j}}Z_j^{b_{i,j}})T$ where $T = A[Z_1,\ldots,Z_n]$. The ring T/J is Artinian, so it is Cohen–Macaulay. Since T/J is obtained from S'/I' by modding out by a homogeneous regular sequence, it follows that S'/I' is Cohen–Macaulay. Similarly, we conclude that S/I is Cohen–Macaulay, as desired. □

Definition 5.4. Recall that G has vertex set $V(G) = \{v_1, \dots, v_d\}$. A *suspension* of G is a graph H whose vertex set is $V(H) = V(G) \cup \{w_1, \dots, w_d\}$ and whose edge set is $E(H) = E(G) \cup \{v_1w_1, \dots, v_dw_d\}$. In other words, H is obtained from G by adding to G a new vertex w_i and edge (sometimes called a "whisker") v_iw_i for each vertex v_i of G.

Note 5.5. Let H be a suspension of G. Graphically, this says that T has the form



where the bottom "row" is the graph G. (Note that this sketch is deceptively oversimplified since the bottom row can be any graph.)

Definition 5.6. Let G_ω and H_λ be weighted graphs. Then H_λ is a *weighted suspension* of G_ω if the underlying graph H is a suspension of G and (with notation as in Definition 5.4) we have $\lambda(v_iv_j) = \omega(v_iv_j)$ for all $v_iv_j \in E(G)$.

Theorem 5.7. *Let H_λ be a weighted suspension of a weighted graph G_ω, with notation as in Definition* 5.4. *Then the following conditions are equivalent*:

(i) H_λ *is Cohen–Macaulay*,
(ii) H_λ *is unmixed*,
(iii) *for each* $v_iv_j \in E(G)$ *we have* $\lambda(v_iv_j) \le \lambda(v_iw_i)$ *and* $\lambda(v_iv_j) \le \lambda(w_jv_j)$.

Proof. (i) ⇒ (ii) This is standard.

(ii) ⇒ (iii) Assume that H_λ is unmixed. Since the underlying unweighted graph H is a suspension, it is Cohen–Macaulay by [5, Proposition 2.2]. In particular, it is unmixed. It is straightforward to show that the set $V' = E(G)$ is a minimal vertex cover for H, so each minimal vertex cover of H has cardinality d. Proposition 1.13 implies that the cardinality of each minimal weighted vertex cover of H_λ is also d. Suppose that there exists some i such that $a = \lambda(w_iv_i) < \lambda(v_iv_j) = b$ and $c = \lambda(w_jv_j)$. We derive a contradiction by constructing a minimal weighted vertex cover V'' such that $|V''| = d + 1$.

For each $k \neq i, j$ set

$$e_k = \min\{\lambda(w_kv_k), \lambda(v_kv_l) \mid v_kv_l \in E(H_\lambda)\}.$$

Let $V' = \{v_i^b, w_i^a, w_j^c, v_k^{e_k} \mid k \neq i, j\}$. It is routine to verify that this is indeed a weighted vertex cover of H_λ. Proposition 1.12 implies that there is a minimal weighted vertex cover (V'', δ'') of G_λ such that $(V'', \delta'') \le (V', \delta')$. Note that for

$k \neq i, j$, the vertex v_k cannot be removed from V' since this would leave the edge $w_k v_k$ uncovered. (However, it may be that the weight on v_k can be increased.) The vertex w_j cannot be removed from V', and its weight cannot be increased, because this would leave the edge $w_j v_j$ uncovered. If we remove v_i^b from V' or increase the weight, then the edge $v_i v_j$ is not covered. If we remove w_i^a or increase the weight, then the edge $w_i v_i$ is not covered. Thus V'' is a minimal weighted vertex cover such that $|V''| = |V'| = r + 1$, providing the desired contradiction.

(iii) $\Rightarrow$ (i) This follows from Lemma 5.3. $\square$

Remark 5.8. Remark 3.7 shows that the equivalence of conditions (ii) and (iii) of Theorem 5.7 do not need the assumption that A is a field. Similar comments hold for Theorem 5.10 and Corollary 5.11.

Note 5.9. Cohen–Macaulay unweighted trees have been explicitly characterized as follows: a tree T is Cohen–Macaulay if and only if either $|V(T_\omega)| \leq 2$, or T is a suspension of a tree; see, e.g. [6, Theorem 6.3.4 and Corollary 6.3.5]. We see next that a weighted tree is Cohen–Macaulay if and only if its underlying unweighted graph has this form, with some restrictions on the weights.

Theorem B from the introduction is a consequence of the next result.

Theorem 5.10. *Let H_λ be a weighted tree. Then the following conditions are equivalent:*

(i) *H_λ is Cohen–Macaulay,*
(ii) *H_λ is unmixed,*
(iii) *one of the following holds:*
   (1) *$|V(H_\lambda)| \leq 2$ or*
   (2) *H_λ is a weighted suspension of a weighted tree G_ω such that (with notation as in Definition 5.4) we have $\lambda(v_i v_j) \leq \lambda(v_i w_i)$ and $\lambda(v_i v_j) \leq \lambda(w_j v_j)$ for each $v_i v_j \in E(G)$.*

In particular, if H_λ is Cohen–Macaulay, then so is H.

Proof. (i) $\Rightarrow$ (ii) This is standard.

(ii) $\Rightarrow$ (iii) Assume that H_λ is unmixed and that $|V(H_\lambda)| > 2$; we need to show that condition (2) is satisfied. By Proposition 1.14 the underlying unweighted graph H is unmixed. Since we have $|V(H_\lambda)| > 2$, it follows from Note 5.9 that H is a suspension of a tree G. It follows readily that H_λ is a weighted suspension of a weighted tree G_ω. The condition $\lambda(v_i v_j) \leq \lambda(v_i w_i)$ and $\lambda(v_i v_j) \leq \lambda(w_j v_j)$ for each $v_i v_j \in E(G)$ follows from Theorem 5.7.

(iii) $\Rightarrow$ (i) This follows from Lemma 5.3 or Theorem 5.7. $\square$

For instance, Theorem 5.10 provides the following explicit characterization of Cohen–Macaulay weighted paths.

Corollary 5.11. *Let P_ω be a weighted path. Then the following conditions are equivalent*:

(i) *P_ω is Cohen–Macaulay,*
(ii) *P_ω is unmixed,*
(iii) *one of the following holds*: *P_ω is of length 1 or of length 3 of the following form*

$$x_1 \overset{a}{\text{———}} y_1 \overset{b}{\text{———}} y_2 \overset{c}{\text{———}} x_2$$

such that $b \leq a$ and $b \leq c$.

The following examples are useful for the proof of Proposition 5.17.

Example 5.12. Let P_ω^4 be a trivially weighted 4-path where each edge has weight a.

$$v_1 \overset{a}{\text{———}} v_2 \overset{a}{\text{———}} v_3 \overset{a}{\text{———}} v_4 \overset{a}{\text{———}} v_5.$$

We show that $R/I(P_\omega^4)$ has dimension 3, depth 2, and type 1.

As in Example 3.8, we decompose:

$$\begin{aligned} I(P_\omega^4) &= (X_1^aX_2^a, X_2^aX_3^a, X_3^aX_4^a, X_4^aX_5^a)R \\ &= (X_1^a, X_3^a, X_4^a)R \cap (X_1^a, X_3^a, X_5^a)R \cap (X_2^a, X_3^a, X_5^a)R \cap (X_2^a, X_4^a)R. \end{aligned}$$

It follows that $\dim(R/I(P_\omega^4)) = 3$.

Using the above decomposition, we conclude that the associated prime ideals of $I(P_\omega^4)$ are $(X_1, X_3, X_4)R$, $(X_1, X_3, X_5)R$, $(X_2, X_3, X_5)R$, and $(X_2, X_4)R$. In particular, the element $X_4 - X_5$ is $R/I(P_\omega^4)$-regular. We simplify the quotient

$$R/(I(P_\omega^4) + (X_4 - X_5)R) \cong R'/(X_1^aX_2^a, X_2^aX_3^a, X_3^aX_4^a, X_4^{2a})R',$$

where $R' = A[X_1, X_2, X_3, X_4]$. As before, we decompose:

$$\begin{aligned} &(X_1^aX_2^a, X_2^aX_3^a, X_3^aX_4^a, X_4^{2a})R' \\ &\quad = (X_1^a, X_3^a, X_4^{2a})R' \cap (X_2^a, X_3^a, X_4^{2a})R' \cap (X_2^a, X_4^a)R'. \end{aligned}$$

The associated primes of this ideal are $(X_1, X_3, X_4)R'$, $(X_2, X_3, X_4)R'$, and $(X_2, X_4)R'$. It follows that the element $X_1 - X_2$ is $R/(I(P_\omega^4) + (X_4 - X_5)R)$-regular, so we have $\operatorname{depth}(R/I(P_\omega^4)) \geq 2$, as claimed. We simplify the quotient

$$R/(I(P_\omega^4) + (X_4 - X_5, X_1 - X_2)R) \cong R''/(X_2^{2a}, X_2^aX_3^a, X_3^aX_4^a, X_4^{2a})R'', \tag{5.1}$$

where $R'' = A[X_2, X_3, X_4]$ and decompose:

$$(X_2^{2a}, X_2^aX_3^a, X_3^aX_4^a, X_4^{2a})R'' = (X_2^{2a}, X_3^a, X_4^{2a})R'' \cap (X_2^a, X_4^a)R''. \tag{5.2}$$

Since the maximal ideal $(X_2, X_3, X_4)R''$ is associated to $(X_2^{2a}, X_2^aX_3^a, X_3^aX_4^a, X_4^{2a})R''$, this shows that $\operatorname{depth}(R/I(P_\omega^4)) = 2$. Furthermore, this explains the non-vanishing in the next computation:

$$\begin{aligned}
0 &\neq \operatorname{Ext}^2_R(R/(X_1,\ldots,X_5)R, R/I(P^4_\omega)) \\
&\cong \operatorname{Hom}_{R''}(R''/(X_2,X_3,X_4)R'', R''/(X_2^{2a},X_2^aX_3^a,X_3^aX_4^a,X_4^{2a})R'') \\
&\cong ((X_2^{2a},X_2^aX_3^a,X_3^aX_4^a,X_4^{2a})R'' :_{R''} (X_2,X_3,X_4)R'') \\
&= ((X_2^{2a},X_3^a,X_4^{2a})R'' \cap (X_2^a,X_4^a)R'' :_{R''} (X_2,X_3,X_4)R'') \\
&\subseteq \big((X_2^{2a},X_3^a,X_4^{2a})R'' :_{R''} (X_2,X_3,X_4)R''\big) \\
&= (X_2^{2a-1}X_3^{a-1}X_4^{2a-1})R''.
\end{aligned}$$

The first isomorphism is standard from the fact that $X_4 - X_5, X_1 - X_2$ is R-regular and $R/I(P^4_\omega)$-regular with the isomorphism (5.1). The second isomorphism and the containment are routine. The first equality comes from the decomposition (5.2), and the second equality is from the fact that A is a field. It follows that $\operatorname{Ext}^2_R(R/(X_1,\ldots,X_5)R, R/I(P^4_\omega))$ is cyclic, so $R/I(P^4_\omega)$ has type 1, as claimed.

Example 5.13. Let P^5_ω be a trivially weighted 5-path where each edge has weight a.

$$v_1 \overset{a}{\text{———}} v_2 \overset{a}{\text{———}} v_3 \overset{a}{\text{———}} v_4 \overset{a}{\text{———}} v_5 \overset{a}{\text{———}} v_6.$$

As in Example 5.12, the quotient $R/I(P^4_\omega)$ has dimension 3, depth 2, and type 1.

Now we turn our attention to Cohen–Macaulayness of weighted cycles.

Proposition 5.14. *Every weighted* 3*-cycle* C^3_ω *is Cohen–Macaulay.*

Proof. From the decomposition of $I(C^3_\omega)$ in Example 3.9, we see that $I(C^3_\omega)$ is m-unmixed; since $R/I(C^3_\omega)$ has dimension 1, it is Cohen–Macaulay. □

Proposition 5.15. *No weighted* 4*-cycle is Cohen–Macaulay.*

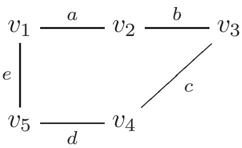
Proof. Let C^4_ω be a weighted 4-cycle. If C^4_ω is non-trivially weighted, then it is mixed by Proposition 4.3, hence it is not Cohen–Macaulay. Thus, we assume that C^4_ω is trivially weighted. Write the underlying unweighted graph of C^4_ω as $C^4 = v_1v_2v_3v_4v_1$, and let the weight of each edge of C^4_ω be a. Then $I(C^4_\omega) = (X_1^aX_2^a, X_2^aX_3^a, X_3^aX_4^a, X_4^aX_1^a)$. Decomposing $I(C^4_\omega)$ and computing associated primes as in Example 5.12, we see that $X_1 - X_2$ is a regular element for $R/I(C^4_\omega)$ such that

$$R' = R/(I(C^4_\omega) + (X_1 - X_2)R) \cong A[X_1,X_3,X_4]/(X_1^{2a}, X_1^aX_3^a, X_3^aX_4^a, X_4^aX_1^a).$$

Also, as in Example 5.12, the maximal ideal of R' is associated to R'. It follows that $R/I(C^4_\omega)$ has depth 1 and dimension 2, so C^4_ω is not Cohen–Macaulay. □

Theorem 5.16. *A weighted* 5*-cycle* C^5_ω *is Cohen–Macaulay if and only if it is unmixed.*

Proof. One implication is standard. For the converse, assume that C_ω^5 is unmixed. Theorem 4.4 implies that C_ω^5 is isomorphic to the weighted 5-cycle



such that $e = a \leq b \geq c \leq d \geq e$. Partially decomposing the edge ideal of C_ω^5 we obtain:

$$I(C_\omega^5) = (X_1^a X_2^a, X_2^b X_3^b, X_3^c X_4^c, X_4^d X_5^d, X_5^e X_1^e) = J \cap K$$

where $J = (X_1^a X_2^a, X_3^c, X_4^d X_5^d, X_5^e X_1^e)$ and $K = (X_1^a X_2^a, X_2^b X_3^b, X_4^c, X_5^e X_1^e)$. It is straightforward to show that these ideals fit into an exact sequence of the following form:

$$0 \to \frac{R}{I(C_\omega^5)} \to \frac{R}{J} \oplus \frac{R}{K} \to \frac{R}{(X_3^c, X_4^c, X_1^a X_2^a, X_5^e X_1^e)} \to 0.$$

The quotient $R/(X_3^c, X_4^c, X_1^a X_2^a, X_5^e X_1^e)$ has depth 1 and dimension 2, because it can be obtained from the ring $A[X_1, X_2, X_5]/(X_1^a X_2^a, X_5^e X_1^e)$ which has depth 1 and dimension 2.

Furthermore, Corollary 5.11 implies that $A[X_1, X_2, X_4, X_5]/(X_1^a X_2^a,\ X_4^d X_5^d, X_5^e X_1^e)$ and $A[X_1, X_2, X_3, X_5]/(X_1^a X_2^a, X_2^b X_3^b, X_5^e X_1^e)$ are Cohen–Macaulay of dimension 2. Hence, R/J and R/K are Cohen–Macaulay of depth 2. Thus by the Depth lemma, $R/I(C_\omega^5)$ has depth at least 2. Since it has dimension 2, it is Cohen–Macaulay. $\square$

Proposition 5.17. *No weighted 7-cycle is Cohen–Macaulay.*

Proof. Let C_ω^7 be a weighted 7-cycle. If C_ω^7 is non-trivially weighted, then it is mixed by Proposition 4.5, hence it is not Cohen–Macaulay. Thus, we assume that C_ω^7 is trivially weighted. Write the underlying unweighted graph of C_ω^7 as $C^7 = v_1v_2v_3v_4v_5v_6v_7v_1$, and let the weight of each edge of C_ω^7 be a. Partially decomposing the edge ideal of C_ω^7 we obtain:

$$I(C_\omega^7) = (X_1^a X_2^a, X_2^a X_3^a, X_3^a X_4^a, X_4^a X_5^a, X_5^a X_6^a, X_6^a X_7^a, X_7^a X_1^a) = J \cap K$$

where

$$J = (X_1^a, X_2^a X_3^a, X_3^a X_4^a, X_4^a X_5^a, X_5^a X_6^a, X_6^a X_7^a),$$
$$K = (X_2^a, X_3^a X_4^a, X_4^a X_5^a, X_5^a X_6^a, X_6^a X_7^a, X_7^a X_1^a).$$

It is routine to show that these ideals fit into an exact sequence of the following form:

$$0 \to \frac{R}{I(C_\omega^7)} \to \frac{R}{J} \oplus \frac{R}{K} \to \frac{R}{L} \to 0, \tag{5.3}$$

where

$$L = (X_1^a, X_2^a, X_3^a X_4^a, X_4^a X_5^a, X_5^a X_6^a, X_6^a X_7^a)R.$$

Example 5.12 implies that the ring $A[X_3, X_4, X_5, X_6, X_7]/(X_3^a X_4^a, X_4^a X_5^a, X_5^a X_6^a, X_6^a X_7^a)$ has depth 2 and type 1, and it follows that R/L also has depth 2 and type 1. Similarly, Example 5.13 implies that R/J and R/K both have depth 2 and type 1. The Depth lemma applied to the sequence (5.3) implies that $\operatorname{depth}(R/I(C_\omega^7)) \geq 2$. Furthermore, for the ideal $\mathfrak{m} = (X_1, \ldots, X_7)R$, part of the long exact sequence in $\operatorname{Ext}_R^2(R/\mathfrak{m}, -)$ associated to the sequence (5.3) has the form

$$\begin{aligned} 0 &\to \operatorname{Ext}_R^2(R/\mathfrak{m}, R/I(C_\omega^7)) \to \operatorname{Ext}_R^2(R/\mathfrak{m}, R/J) \oplus \operatorname{Ext}_R^2(R/\mathfrak{m}, R/K) \\ &\to \operatorname{Ext}_R^2(R/\mathfrak{m}, R/L). \end{aligned}$$

Using the type computations we have already made, this sequence has the form

$$0 \to \operatorname{Ext}_R^2(R/\mathfrak{m}, R/I(C_\omega^7)) \to k^2 \to k.$$

It follows that $\operatorname{Ext}_R^2(R/\mathfrak{m}, R/I(C_\omega^7)) \neq 0$, so $\operatorname{depth}(R/I(C_\omega^7)) = 2 < 3 = \dim(R/I(C_\omega^7))$. It follows that C_ω^7 is not Cohen–Macaulay, as claimed. □

Proof of Theorem A. (a) Assume that C_ω^n is Cohen–Macaulay. Then it is unmixed, so Proposition 1.14 implies that the unweighted cycle C^n is unmixed. From Fact 4.1, we conclude that $n \in \{3, 4, 5, 7\}$. Propositions 5.15 and 5.17 imply that $n \neq 4, 7$, so we have $n \in \{3, 5\}$.
(b) This is Proposition 5.14.
(c) Theorems 4.4 and 5.16. □

Acknowledgments

We thank the referee for her/his thoughtful comments.

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