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Torsion in kernels of induced maps on divisor class groups

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We investigate torsion elements in the kernel of the map on divisor class groups of excellent local normal domains A and A/I, for an ideal I of finite projective dimension. The motivation for this work is a result of Griffith–Weston which applies when I is principal.

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0. Introduction

In this section, let $A \to B$ be a homomorphism of noetherian normal integral domains. Under certain circumstances, one can show that such a map induces a group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ on divisor class groups given by $[\mathfrak{a}] \mapsto [(\mathfrak{a} \otimes_A B)^{BB}]$. (See Sec. 1 for definitions and background material.) For instance, if A is excellent and local, t is a prime element of A such that A/tA satisfies the regularity condition (R_1), and B is the integral closure of A/tA, then it is well known that the natural map $A \to B$ induces such a homomorphism. In this setting, the kernel of this induced map has been studied by Danilov, Griffith, Lipman, Miller, Spiroff, Weston, and others [5, 9, 10, 17, 19, 24, 25]. For instance, Griffith and Weston prove the following theorem.

Theorem A ([10, Theorem 1.2]). *Let $(A, \mathfrak{m})$ be an excellent, local, normal domain and let t be a principal prime in $\mathfrak{m}$ such that A/tA satisfies the condition (R_1). Let B denote the integral closure of A/tA and let $e > 1$ be an integer which represents a unit in A. Then the kernel of the homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ contains no element of order e.*

A careful analysis of their intricate argument reveals a detail that may warrant additional consideration. To be brief, the authors suppose that $[\mathfrak{a}]$ is an element of order e in $\mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(B))$, and let $a \in A$ be such that $\mathfrak{a}^{(e)} = aA$. They describe an A-algebra structure on the direct sum $R = \bigoplus_{i=0}^{e-1} \mathfrak{a}^{(i)}$ that makes R isomorphic to the integral closure of $A[T]/(T^e - a)$. They then show that the integral closure of $R \otimes_A B$ is étale over B and conclude that $e = 1$. In their proof of the last step, they seem to assume that the integral closure of $R \otimes_A B$ is local. While this is, in fact, true, more clarity is probably appropriate. We provide this, using a result of Hochster and Huneke [14, (3.9) Proposition c)], interpreting the integral closure of $R \otimes_A B$ as the S_2-ification of $R \otimes_A B$.

Using the point of view of S_2-ifications, we obtain the theorem below. Its proof is the content of Sec. 3. This result allows us to obtain the special case of Theorem A where A/tA is normal; see Corollary 4.1.

Theorem B. *Let $(A, \mathfrak{m})$ be an excellent, local, normal domain and I a prime ideal of A with finite projective dimension such that:*

(i) *$\bar{A} = A/I$ is normal;*
(ii) *I is a complete intersection on the punctured spectrum of A, i.e. for each prime ideal $\mathfrak{p} \neq \mathfrak{m}$, the localization $I_{\mathfrak{p}}$ is either equal to $A_{\mathfrak{p}}$ or generated by a regular sequence in $A_{\mathfrak{p}}$; and*
(iii) *$\mu(I) \leq \dim A - 2$, where $\mu(I)$ is the minimal number of generators of I.*

Then for any positive integer $e > 1$ which represents a unit in A, the kernel of the homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(\bar{A})$ contains no elements of order e.

A significant tool in the proof of Theorem B is the next result, the proof of which is the content of Sec. 2.

Theorem C. *Let $(A, \mathfrak{m})$ be a complete, local, normal domain such that $A/\mathfrak{m}$ is separably closed and let I be a prime ideal of A with finite projective dimension such that $\bar{A} = A/I$ is normal. Let $e > 1$ be an integer which represents a unit in A and assume that A contains a primitive eth root of unity. Let $[\mathfrak{a}]$ be an element in $\mathrm{Cl}(A)$ with order e. Set $R = A \oplus \mathfrak{a} \oplus \mathfrak{a}^{(2)} \oplus \cdots \oplus \mathfrak{a}^{(e-1)}$. If any of the equivalent conditions of Fact 1.14 hold for R/IR, then $[\mathfrak{a}] \notin \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(\bar{A}))$.*

Note that the A-algebra structure on R is described explicitly in paragraph 2.1, and that Lemma 2.4 shows that R/IR satisfies the hypotheses of Fact 1.14. Also, the proofs of Theorems B and C use ideas from [10, Theorem 1.2], but with some key changes. For completeness and clarification, we provide the details. Because of their length and technicality, the proofs are presented in a series of lemmas. We conclude the paper with a few corollaries and an example in Sec. 4.

Finally, we mention that a thorough discussion of the background material and list of terminology is given in Sec. 1 for the reader's convenience. We provide useful definitions and results on divisor class groups, S_2-ifications, and unramified/étale extensions. Undefined terms can be found in [18].

1. Background

Throughout this paper, we assume that all rings are commutative and noetherian.

Divisor class groups

We begin with our working definition of the divisor class group of a normal domain. It can be found in [17, §0] and is equivalent to the classical additive definition of the divisor class group appearing in [4, VII §1; 8, §6]. A discussion of this equivalence appears in [22, 2.10].

Definition 1.1. Let A be a normal domain and M a finitely generated A-module. The *dual* of M is $M^A = \mathrm{Hom}_A(M, A)$ and the *double dual* of M is $M^{AA} = (M^A)^A$. The natural *biduality map* $\sigma_M^A : M \to M^{AA}$ is the A-module homomorphism given by $\sigma_M^A(m)(g) = g(m)$ for all $m \in M$ and all $g \in M^A$. We say that M is *reflexive* if σ_M^A is an isomorphism.

Remark 1.2. As in our previous work [23], we use the notation M^A for the dual of an A-module M in order to avoid ambiguity when we work with two or more rings simultaneously. This replaces the notation M^* from the classical literature.

Definition 1.3. The *divisor class group* of a normal domain A, denoted $\mathrm{Cl}(A)$, is the group of isomorphism classes of reflexive A-modules of rank one, or equivalently, non-zero reflexive ideals of A. An element $[\mathfrak{a}] \in \mathrm{Cl}(A)$ is called a *divisor class*, and multiplication is defined by $[\mathfrak{a}] \cdot [\mathfrak{b}] = [(\mathfrak{a} \otimes_A \mathfrak{b})^{AA}]$. The identity element is $[A]$, and the inverse of $[\mathfrak{a}]$ is $[\mathfrak{a}^A]$.

As A is normal, it satisfies the Serre conditions (R_1) and (S_2). Thus, each reflexive ideal $\mathfrak{a}$ can be written uniquely as the primary decomposition $\mathfrak{a} = \bigcap_{j=1}^s \mathfrak{p}_j^{(e_j)}$, where the $\mathfrak{p}_j$ are the height one prime ideals containing $\mathfrak{a}$. For each positive integer d, we set $\mathfrak{a}^{(d)} = \bigcap_{j=1}^s \mathfrak{p}_j^{(e_j d)}$. In $\mathrm{Cl}(A)$, this definition implies that $d[\mathfrak{a}] = [\mathfrak{a}^{(d)}]$.

Fact 1.4. Let $A \to B$ be a homomorphism of finite flat dimension between normal domains. (For instance, if I is a prime ideal of A with finite projective dimension such that A and A/I are normal domains, then we can consider the natural surjection $A \to A/I$.) Then [22, Theorem A] provides a well-defined group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ given by $[\mathfrak{a}] \mapsto [(\mathfrak{a} \otimes_A B)^{BB}]$.

We complement this with the following.

Lemma 1.5. *Let A be a normal domain, and let I be a prime ideal of A such that $\bar{A} = A/I$ is excellent and satisfies (R_1). If B is the integral closure of $\bar{A}$, then the natural ring homomorphism $\varphi : A \to B$ induces a well-defined group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ given by $[\mathfrak{a}] \mapsto [(\mathfrak{a} \otimes_A B)^{BB}]$.*

Proof. Note that the fact that $\bar{A}$ is excellent implies that B is noetherian. Let Q be a height one prime ideal of B. Then $\bar{P} := Q \cap \bar{A} = P/I$ for some prime ideal P of A. By [23, Theorem 1.10], it suffices to show that A_P is a unique factorization domain, e.g. that it is regular. By [15, Theorems 4.8.6 and B.5.1], we know that $\operatorname{ht} \bar{P} = \operatorname{ht} Q$; this also uses the fact that $\bar{A}$ is excellent. Since $\bar{A}$ satisfies (R_1), it follows that $\bar{A}_{\bar{P}}$ is regular. Since the induced map $A_P \to \bar{A}_{\bar{P}}$ has finite projective dimension, we conclude from [3, Theorem 6.1(1)] that A_P is regular. □

S_2-ifications

We present here an exposition on S_2-ifications, taking much of our content from a paper by Hochster and Huneke [14]. This topic comes to the fore in the guise of the integral closure when one assumes that the domain in question satisfies the regularity condition (R_1), but is not necessarily normal. This allows one to consider a broader class of rings when studying divisor class groups.

Definition 1.6. Let $(A, \mathfrak{m})$ be a local ring, and let E be the injective hull of $A/\mathfrak{m}$ over A. A *canonical module* for A is a finitely generated A-module ω such that $\mathrm{Hom}_A(\omega, E) \cong \mathrm{H}^{\dim(A)}_{\mathfrak{m}}(A)$.

Fact 1.7. Let A be a local ring. If A is a homomorphic image of a Gorenstein local ring, e.g. A is complete, then it has a canonical module.

Definition 1.8. Let A be a local ring. Denote by $j(A)$ the largest ideal which is a submodule of A of dimension smaller than $\dim A$. Specifically,

$$j(A) = \{a \in A : \dim(A/\mathrm{ann}_A(a)) < \dim A\}.$$

Fact 1.9. The following items are from [14, (2.2f), (2.1)].

(a) If A is local with canonical module ω, then $j(A) = \mathrm{Ker}(A \to \mathrm{Hom}_A(\omega, \omega))$.
(b) A local ring A is equidimensional and (0) has no embedded primes if and only if $j(A) = (0)$. In particular, if A is a local domain, then $j(A) = 0$.

Definition 1.10 ([14, (2.3)]). Let A be a local ring.

(a) If $j(A) = 0$, then a subring S of the total ring of quotients of A is an *S_2-ification* of A if:
   - S is module finite over A;
   - S satisfies the Serre condition (S_2) over A; and
   - $\mathrm{Coker}(A \to S)$ has no prime ideal of A of height less than two in its support.

(b) When A is equidimensional but possibly $j(A) \neq 0$, then by an S_2-*ification* of A, we mean an S_2-ification of $A/j(A)$.

Fact 1.11 ([14, (2.2), (2.7)]). A local ring A has an S_2-*ification* if it has a canonical module ω. Specifically, $\mathrm{Hom}_A(\omega, \omega)$ is a commutative A-algebra that satisfies (S_2) both as a ring and an A-module, and it is an S_2-ification of A.

Lemma 1.12. *Assume that A is a local ring with a canonical module that satisfies (R_1), such that $j(A) = 0$. Then the S_2-ification S of A coincides with the integral closure of A in its total ring of quotients.*

Proof. Since the S_2-ification of A is unique up to isomorphism, by [14, (2.5)], it suffices to show that S is integrally closed in its total ring of fractions. It suffices to show that S satisfies (R_1) and (S_2) as a ring. Let P be a prime ideal of S, and set $\mathfrak{p} = P \cap A$. From [14, Proposition 3.5(a)], we know that $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,P$. Hence, the fact that S satisfies (S_2) as a ring follows from the fact that it satisfies (S_2) as an A-module, by [12, Corollaire (5.7.11)].

To verify that S satisfies (R_1), assume that $\mathrm{ht}\,P \leq 1$. Since A satisfies (R_1), the ring $A_\mathfrak{p}$ is regular. Since $\mathrm{Coker}(A \to S)$ has no prime ideal of A of height less than two in its support, we conclude that $A_\mathfrak{p} \cong S_\mathfrak{p}$, hence $A_\mathfrak{p} \cong S_P$ is regular. □

Fact 1.13. When $(A, \mathfrak{m})$ is an excellent local domain and t is an element of $\mathfrak{m}$ such that A/tA satisfies (R_1), then the integral closure of A/tA in its total ring of quotients is local. (See [13, §XIII], or [14, Proposition 3.9].) It follows from [20, Corollary 37.6] that t is a prime element.

The fact above, that the S_2-ification of A/tA is local, does not easily generalize to A/I for a non-principal ideal I of A, even under the assumptions that A/I is complete, local, and equidimensional. One would need to know, for example, that the canonical module of A/I is indecomposable; or more importantly, that any of the equivalent conditions below hold for the ring A/I.

Fact 1.14 ([14, Theorem 3.6]). For $(T, \mathfrak{n})$ a complete local equidimensional ring, the following conditions are equivalent:

(i) $H_\mathfrak{n}^{\dim T}(T)$ is indecomposable.
(ii) The canonical module of T is indecomposable.
(iii) The S_2-ification of $T/j(T)$ is local.
(iv) For every ideal J of height at least two, $\mathrm{Spec}(T) - V(J)$ is connected.
(v) Given any two distinct minimal primes $\mathfrak{p}, \mathfrak{q}$ of T, there is a sequence of minimal primes $\mathfrak{p} = \mathfrak{p}_0, \ldots, \mathfrak{p}_r = \mathfrak{q}$ such that for $0 \leq i < r$, $\mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) = 1$.

We apply this result in Sec. 4. (See Lemmas 3.5–3.7.)

Unramified and étale extensions, and isomorphisms in codimension one

For the remainder of this section, let R be a ring, and let S be an R-algebra.

Definition 1.15. For a prime ideal P of S and $\mathfrak{p} = P \cap R$, we say that P is *unramified over* R if $\mathfrak{p}S_P = PS_P$ and $S_P/\mathfrak{p}S_P$ is a separable field extension of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Definition 1.16. A prime ideal $\mathfrak{p}$ of R is *unramified in* S if (1) every prime lying over $\mathfrak{p}$ is unramified over R, and (2) there are only finitely many prime ideals of S lying over $\mathfrak{p}$. (Note that if $R \hookrightarrow S$ is module finite, then condition (2) is automatic.)

Definition 1.17. For S to be *unramified over* R means that (1) every prime ideal of S is unramified over R, and (2) for every $\mathfrak{p} \in \operatorname{Spec}(R)$ there are only finitely many prime ideals of S lying over $\mathfrak{p}$. (Note that if $R \hookrightarrow S$ is module finite, then condition (2) is automatic.)

Definition 1.18. We say that S is *unramified in codimension* i (or more accurately, in codimension less than or equal to i) over R if every prime ideal of S with height less than or equal to i is unramified over R.

Definition 1.19. For S to be *étale* over R means that S is unramified and flat over R.

Definition 1.20. We say that S is *étale in codimension one* (or more accurately, in codimension less than or equal to one) over R if, for every prime ideal P of S with height less than or equal to one and $\mathfrak{p} = P \cap R$, the ring S_P is étale over $R_{\mathfrak{p}}$.

The next fact is a version of "purity of branch locus" from Auslander and Buchsbaum [2, Corollary 3.7] that is very useful for the proof of Theorem B.

Fact 1.21. Assume that R is local and that $R \hookrightarrow S$ is a module finite extension of normal domains which is unramified in codimension one. If S is free as an R-module, then S is unramified over R.

Definition 1.22. Let M, N be finitely generated modules over a ring R. The map $\varphi : M \to N$ is said to be an *isomorphism in codimension one* if for each prime ideal $\mathfrak{p}$ in R of height less than or equal to one, the induced homomorphism $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism.

The next fact follows with a bit of work from a result of Auslander and Buchsbaum [2, Proposition 3.4]. It is a useful tool in establishing many of the ring isomorphisms in our main argument.

Fact 1.23. Let A be a normal domain. If M is a reflexive A-module, N is a torsion-free A-module, and $\psi : M \to N$ is an A-module homomorphism, then ψ is an isomorphism if and only if it is an isomorphism in codimension one.

2. Proof of Theorem C (and Additional Details)

2.1. Let $(A, \mathfrak{m})$ be a complete, local, normal domain such that $A/\mathfrak{m}$ is separably closed; let I be a prime ideal of A with finite projective dimension such that $\bar{A} = A/I$ is normal. Let $e > 1$ be an integer which represents a unit in A and assume that A contains a primitive eth root of unity. Let $\mathfrak{a}$ be a non-zero reflexive ideal of A such that $[(\mathfrak{a} \otimes_A \bar{A})^{\bar{A}\bar{A}}]$ is trivial in $\mathrm{Cl}(\bar{A})$, and suppose that the order of $[\mathfrak{a}]$ in $\mathrm{Cl}(A)$ is e. Thus, $\mathfrak{a}^{(e)} = aA$ for some $a \in \mathfrak{a}$. Set K to be the fraction field of A.

Since A is a normal ring, it is the intersection over all height one primes $\mathfrak{p}$ of the family of discrete valuation rings $\{A_\mathfrak{p}\}$, each of which is contained in K, with corresponding valuations $v_\mathfrak{p}$. There are height one primes $\mathfrak{p}_i$ of A and positive integers n_i such that $\mathfrak{a} = \mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_r^{(n_r)}$. Then $\mathfrak{a}^{(e)} = \mathfrak{p}_1^{(en_1)} \cap \cdots \cap \mathfrak{p}_r^{(en_r)} = aA$ and $v_{\mathfrak{p}_i}(a) = en_i$. By the Approximation Theorem [18, Theorem 12.6], there is an element $b \in A$ such that $v_{\mathfrak{p}_i}(b) = n_i$ for $i = 1, \ldots, r$ and $v_\mathfrak{q}(b) \geq 0$ for all other height one primes $\mathfrak{q}$ of A. Set $u = b^e/a \in K$. Then $v_{\mathfrak{p}_i}(u) = en_i - en_i = 0$ for $i = 1, \ldots, r$ and $v_\mathfrak{q}(u) = v_\mathfrak{q}(b^e) \geq 0$ for all other $\mathfrak{q}$.

Let $\sqrt[e]{u}$ be a fixed eth root of u in an algebraic closure of K. Then $\sqrt[e]{a} := b/\sqrt[e]{u}$ is an eth root of a. Since $au = b^e$, we have $K[\sqrt[e]{a}] = K[\sqrt[e]{u}]$. Moreover, the ring

$$R = A \oplus \left[\mathfrak{a} \cdot \frac{\sqrt[e]{u}}{b}\right] \oplus \left[\mathfrak{a}^{(2)} \cdot \frac{\sqrt[e]{u^2}}{b^2}\right] \oplus \cdots \oplus \left[\mathfrak{a}^{(e-1)} \cdot \frac{\sqrt[e]{u^{e-1}}}{b^{e-1}}\right] \tag{2.1.1}$$

is the integral closure of A in the field extension $K[\sqrt[e]{a}]$, as per [9, Theorem 2.4]. In particular, R is a domain. It is worth describing the ring structure on R: for $a_i \in \mathfrak{a}^{(i)}$ and $a_j \in \mathfrak{a}^{(j)}$, we have

$$\left(a_i \frac{\sqrt[e]{u^i}}{b^i}\right)\left(a_j \frac{\sqrt[e]{u^j}}{b^j}\right) = \begin{cases} a_i a_j \dfrac{\sqrt[e]{u^{i+j}}}{b^{i+j}} & \text{if } i+j < e, \\[2ex] a_i a_j \dfrac{\sqrt[e]{u^{i+j}}}{b^{i+j}} = \dfrac{a_i a_j}{a} \dfrac{\sqrt[e]{u^{i+j-e}}}{b^{i+j-e}} & \text{if } i+j \geq e. \end{cases}$$

Lemma 2.2. *Let $P \in \mathrm{Spec}(R)$ and set $\mathfrak{p} = P \cap A$. Then $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,P$. Furthermore, if $A_\mathfrak{p}$ is regular, then the extensions $A_\mathfrak{p} \to R_\mathfrak{p}$ and $A_\mathfrak{p} \to R_P$ are étale. In particular, the extension $A \hookrightarrow R$ is étale in codimension one.*

Proof. Since A is normal and R is a domain that is module finite over A, the theorems of going-up and going-down apply, and hence $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,P$.

Now assume that $A_\mathfrak{p}$ is regular. There are two cases to consider.

Suppose $a \notin \mathfrak{p}$. Then a is a unit in $A_\mathfrak{p}$. By [6, Theorem III.4.4, p. 113], the polynomial $T^e - a$ is separable in $A_\mathfrak{p}[T]$. Thus, $A_\mathfrak{p}[T]/(T^e - a)A_\mathfrak{p}[T]$ is a separable $A_\mathfrak{p}$-algebra, by definition, cf. [6, p. 109]. Since $a \notin \mathfrak{p}$, we have $\mathfrak{a}_\mathfrak{p} = A_\mathfrak{p}$, so $R_\mathfrak{p} \cong R \otimes_A A_\mathfrak{p} \cong A_\mathfrak{p} \oplus A_\mathfrak{p} \oplus \cdots \oplus A_\mathfrak{p}$ (e copies); i.e. $R_\mathfrak{p} \cong A_\mathfrak{p}[T]/(T^e - a)A_\mathfrak{p}[T]$. In other words, $R \otimes_A A_\mathfrak{p}$ is a separable $A_\mathfrak{p}$-algebra. Set $R_\mathfrak{p} = R \otimes_A A_\mathfrak{p}$. By definition

[6, p. 40], $R_{\mathfrak{p}}$ is a projective $(R_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} R_{\mathfrak{p}})$-module. In addition, it is the integral closure of $A_{\mathfrak{p}}$ in $K[\sqrt[e]{a}] = K[\sqrt[e]{u}]$. Thus, by [2, Proposition 2.2], $R_{\mathfrak{p}}$ is unramified over $A_{\mathfrak{p}}$.

On the other hand, if $a \in \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some i. Let S denote the complement in A of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, the height one primes in the decomposition of $\mathfrak{a}$ i.e. $S = A - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$. Note that u is a unit in A_S because $v_{\mathfrak{p}_i}(u) = 0$ for all $i = 1, \ldots, r$. As above, $T^e - u$ is separable in $A_S[T]$. Thus, $A_S[T]/(T^e - u)A_S[T]$ is a separable A_S-algebra, by definition, [6, p. 109]. Since $u \in S$, we have $R \otimes_A A_S \cong \bigoplus^e A_S$ and the proof proceeds as in the case above. In conclusion, for each i, $A_{\mathfrak{p}_i} \to R_{\mathfrak{p}_i}$ is unramified.

The fact that $A_{\mathfrak{p}}$ is regular implies that $R_{\mathfrak{p}}$ is free because $\mathfrak{a}A_{\mathfrak{p}} \cong A_{\mathfrak{p}}$. Thus, $R_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$. In particular, the extension $A_{\mathfrak{p}} \to R_{\mathfrak{p}}$ is étale. Moreover, it follows that R_P is also flat over $A_{\mathfrak{p}}$, and hence the extension $A_{\mathfrak{p}} \to R_P$ is étale. Finally, since $A_{\mathfrak{p}}$ is regular for any prime $\mathfrak{p} = P \cap A$, where $\operatorname{ht}_R P \leq 1$, it follows that $A \to R$ is étale in codimension one. $\square$

Lemma 2.3. *The ring R is a complete local normal domain.*

Proof. As noted above R is a domain. Since R is the integral closure of A in $K[\sqrt[e]{a}]$, it is normal. Moreover, R is complete since it is a module finite extension of the complete ring A. The fact that R is local follows from [20, (30.5)] since A is a complete, hence Henselian, local integral domain. $\square$

Lemma 2.4. *Set $\bar{R} = R \otimes_A \bar{A} = R/IR$. Let $\bar{P} \subset \bar{R}$ be a prime ideal and set $\bar{\mathfrak{p}} = \bar{P} \cap \bar{A}$. Then $\operatorname{ht} \bar{\mathfrak{p}} = \operatorname{ht} \bar{P}$. Furthermore, the ring $\bar{R}$ is equidimensional, complete, and local. In particular, Fact* 1.14 *applies with $T = \overline{R}$.*

Proof. By Lemma 2.3, the ring $\bar{R}$ is local and complete. Since A embeds in R as the degree zero direct summand, we see that $\bar{A}$ embeds in $\bar{R}$ as the degree zero direct summand. It follows that the induced map $\bar{A} \to \bar{R}$ is a module finite monomorphism. Hence, going-up holds, and as a result $\operatorname{ht} \bar{\mathfrak{p}} \geq \operatorname{ht} \bar{P}$.

We consider the following commutative diagram of local ring homomorphisms,

$$\begin{array}{ccc} A & \longrightarrow & R \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\pi} \\ \bar{A} & \longrightarrow & \bar{R} \end{array}$$

where the vertical maps are the natural surjections, and the horizontal maps are the inclusions. Set $P = \pi^{-1}(\bar{P})$ and $\mathfrak{p} = \tau^{-1}(\bar{\mathfrak{p}}) = P \cap A$.

Note that the map $A \to R$ satisfies going-down because A is integrally closed and R is an integral domain. Let $0 = \bar{\mathfrak{p}}_0 \subset \bar{\mathfrak{p}}_1 \subset \cdots \subset \bar{\mathfrak{p}}_h = \bar{\mathfrak{p}}$ be a chain of prime ideals of $\bar{A}$ such that $h = \operatorname{ht} \bar{\mathfrak{p}}$. It follows that $I = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h = \mathfrak{p}$ is a chain of prime ideals of A. By going-down, there are prime ideals $P_0 \subset P_1 \subset \cdots \subset P_h = P$

of R such that $P_i \cap A = \mathfrak{p}_i$ for each i. In particular, we have $I = \mathfrak{p}_0 = P_0 \cap A$ and so $IR \subseteq P_0$. It follows that the ideals $\bar{P}_i = P_i\bar{R} \subset \bar{R}$ are prime and form a chain $\bar{P}_0 \subset \bar{P}_1 \subset \cdots \subset \bar{P}_h = \bar{P}$ and thus $\operatorname{ht}\bar{\mathfrak{p}} = h \leq \operatorname{ht}\bar{P}$, as desired.

To show that $\bar{R}$ is equidimensional, assume that $\bar{P}$ is a minimal prime of $\bar{R}$. Then $\operatorname{ht}\bar{\mathfrak{p}} = \operatorname{ht}\bar{P} = 0$ and so $\bar{\mathfrak{p}} = 0$. Thus, the induced map $\bar{A} \to \bar{R}/\bar{P}$ is a module finite monomorphism. It follows that $\dim(\bar{R}/\bar{P}) = \dim\bar{A} = \dim\bar{R}$. □

Lemma 2.5. *For any prime ideal $\bar{P} \subset \bar{R}$ of height less than or equal to one and $\bar{\mathfrak{p}} = \bar{P} \cap \bar{A}$, the extensions $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{\mathfrak{p}}}$ and $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{P}}$ are étale. In particular, the extension $\bar{A} \to \bar{R}$ is étale in codimension one. Moreover, the ring $\bar{R}$ satisfies the Serre condition (R_1).*

Proof. Let $\bar{P} \subset \bar{R}$ be a prime of height less than or equal to one. Let P, $\bar{\mathfrak{p}}$ and $\mathfrak{p}$ be as in the proof of Lemma 2.4, hence $\operatorname{ht}\bar{\mathfrak{p}} = \operatorname{ht}\bar{P}$. Since $\bar{A}$ is a normal domain, it follows that $\bar{A}_{\bar{\mathfrak{p}}}$ is regular. The mapping $A_{\mathfrak{p}} \to \bar{A}_{\mathfrak{p}} = \bar{A}_{\bar{\mathfrak{p}}}$ has finite projective dimension since, by hypothesis, $A \to \bar{A}$ has finite projective dimension. It now follows from [3, Theorem 6.1(1)] that $A_{\mathfrak{p}}$ is regular. Next, by Lemma 2.2, the extensions $A_{\mathfrak{p}} \to R_{\mathfrak{p}}$ and $A_{\mathfrak{p}} \to R_P$ are étale. It follows by base-change that the extensions $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{\mathfrak{p}}}$ and $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{P}}$ are étale. Thus, the extension $\bar{A} \to \bar{R}$ is étale in codimension one. Moreover, the extension $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{P}}$ has a regular closed fiber (in fact, the closed fiber is a field because the extension is étale). Since the extension is also flat, as it is étale, and $\bar{A}_{\bar{\mathfrak{p}}}$ is regular, it follows that $\bar{R}_{\bar{P}}$ is regular. Therefore, $\bar{R}$ satisfies (R_1). □

Lemma 2.6. *For the biduality map $\sigma : \bar{R} \to \bar{R}^{\bar{A}\bar{A}}$, we have* $\operatorname{Ker}\sigma = \operatorname{nil}(\bar{R}) = j(\bar{R})$; *see Definition* 1.8.

Proof. Recall that $j(\bar{R}) = \{\bar{r} \in \bar{R} : \dim(\bar{R}/\operatorname{ann}_{\bar{R}}(\bar{r})) < \dim\bar{R}\}$ is an ideal of $\bar{R}$.

First of all, to see that $\operatorname{nil}(\bar{R}) \subseteq j(\bar{R})$, let $\bar{r} \in \operatorname{nil}(\bar{R})$. By the previous lemma, the extension $\bar{A}_{(0)\bar{A}} \to \bar{R}_{(0)\bar{A}}$ is étale. It is also module finite. Since $\bar{A}_{(0)\bar{A}}$ is a field, the ring $\bar{R}_{(0)\bar{A}}$ is a finite product of fields; hence, it is reduced. Thus, the image of $\bar{r}$ in $\bar{R}_{(0)\bar{A}}$ is zero. In other words, we have $(\bar{r}\bar{R})_{(0)\bar{A}} = \bar{r}\bar{R}_{(0)\bar{A}} = (0)$ and so $(0)\bar{A} \notin \operatorname{Supp}_{\bar{A}}(\bar{r}\bar{R})$. It follows that $\dim_{\bar{A}}(\bar{r}\bar{R}) < \dim(\bar{A})$. Now $\dim_{\bar{R}}(\bar{r}\bar{R}) = \dim_{\bar{A}}(\bar{r}\bar{R})$ and $\dim\bar{R} = \dim\bar{A}$, hence $\bar{r} \in j(\bar{R})$.

On the other hand, let $\bar{r} \in j(\bar{R})$. We need to show that $\bar{r} \in \bar{P}$ for each prime ideal $\bar{P} \subset \bar{R}$. If it is not, then $\bar{r} \notin \bar{P}_0$, for some minimal prime $\bar{P}_0$ of $\bar{R}$. The element $\bar{r}$ represents a unit in $\bar{R}_{\bar{P}_0}$. In particular, we have $0 \neq \bar{r}/1 \in \bar{R}_{\bar{P}_0}$ and so $\bar{P}_0 \in \operatorname{Supp}_{\bar{R}}(\bar{r}\bar{R})$. It follows that $\dim_{\bar{R}}(\bar{r}\bar{R}) = \dim\bar{R}$ since $\bar{R}$ is equidimensional. This contradicts the fact that $\bar{r} \in j(\bar{R})$. Therefore, $j(\bar{R}) \subseteq \operatorname{nil}(\bar{R})$.

We now show that $\operatorname{Ker}\sigma = j(\bar{R})$. The module $\bar{R}_{(0)\bar{A}}$ is free over $\bar{A}_{(0)\bar{A}}$, hence the localized map $\sigma_{(0)\bar{A}} : \bar{R}_{(0)\bar{A}} \to (\bar{R}^{\bar{A}\bar{A}})_{(0)\bar{A}}$ is an isomorphism. Thus, $(\operatorname{Ker}\sigma)_{(0)\bar{A}} \cong \operatorname{Ker}(\sigma_{(0)\bar{A}}) = (0)$, so $\dim_{\bar{A}}(\operatorname{Ker}\sigma) < \dim\bar{A}$. Now $\dim_{\bar{R}}(\operatorname{Ker}\sigma) = \dim_{\bar{A}}(\operatorname{Ker}\sigma)$ and $\dim\bar{R} = \dim\bar{A}$, hence $\operatorname{Ker}\sigma \subseteq j(\bar{R})$.

For the reverse containment, let $\bar{r} \in j(\bar{R})$. Then $\dim_{\bar{A}}(\bar{r}\bar{R}) < \dim \bar{A}$ as above, which means that $\bar{r}\bar{R}_{(0)\bar{A}} = (0)$. It follows that there exists a non-zero element $\bar{b} \in \bar{A}$ such that $\bar{b}\bar{r} = 0$, and so $\dim_{\bar{A}}(\bar{r}\bar{A}) < \dim \bar{A}$. For each $\psi \in \operatorname{Hom}_{\bar{A}}(\bar{R}, \bar{A})$, $\dim_{\bar{A}}(\psi(\bar{r}\bar{A})) \leq \dim_{\bar{A}}(\bar{r}\bar{A}) < \dim \bar{A}$, since $\bar{r}\bar{A} \twoheadrightarrow \psi(\bar{r}\bar{A})$ via ψ. However, $\psi(\bar{r}\bar{A})$ is an $\bar{A}$-submodule of $\bar{A}$; that is, an ideal of $\bar{A}$. If $\psi(\bar{r}\bar{A}) \neq (0)$, then $\psi(\bar{r}\bar{A}) \hookrightarrow \psi(\bar{r}\bar{A})_{(0)\bar{A}}$ because $\bar{A}$ is a domain; but this would mean that $\psi(\bar{r}\bar{A})_{(0)\bar{A}} \neq 0$, hence $\dim_{\bar{A}}(\psi(\bar{r}\bar{A})) = \dim \bar{A}$. This is a contradiction. Therefore, $\psi(\bar{r}\bar{A}) = (0)$. This says that $\psi(\bar{r}) = 0$ for all $\psi \in \operatorname{Hom}_{\bar{A}}(\bar{R}, \bar{A})$. By definition, $\sigma(\bar{r}) : \operatorname{Hom}_{\bar{A}}(\bar{R}, \bar{A}) \to \bar{A}$ is the map $\sigma(\bar{r})(\psi) = \psi(\bar{r}) = 0$. Thus, $\sigma(\bar{r}) = 0$, which means that $\bar{r} \in \operatorname{Ker}\sigma$, as desired. $\square$

Lemma 2.7. *The ring $\bar{R}/j(\bar{R})$ satisfies (R_1), so its integral closure $(\bar{R}/j(\bar{R}))'$ in its total ring of quotients is its S_2-ification.*

Proof. Take $\tilde{P} = \bar{P}/j(\bar{R})$ a height one prime of $\bar{R}/j(\bar{R})$. Then $\bar{P}$ is a height one prime of $\bar{R}$ since $j(\bar{R}) = \operatorname{nil}(\bar{R})$, and $\big(\bar{R}/j(\bar{R})\big)_{\tilde{P}} = \bar{R}_{\bar{P}}/(j(\bar{R}))_{\bar{P}}$. Since $\bar{R}$ is complete and local, it has a canonical module $\omega_{\bar{R}}$. Recalling Fact 1.9, $j(\bar{R}) = \operatorname{Ker}(\bar{R} \to \operatorname{Hom}_{\bar{R}}(\omega_{\bar{R}}, \omega_{\bar{R}}))$, hence $j(\bar{R})_{\bar{Q}} \cong j(\bar{R}_{\bar{Q}})$ for any prime ideal $\bar{Q}$ of $\bar{R}$. Note that this uses the fact that $\bar{R}$ is equidimensional; see Lemma 2.4 and [14, (2.2i)]. In particular, since $\bar{R}$ satisfies the (R_1) condition as per Lemma 2.5, we have $j(\bar{R}_{\bar{P}}) = (0)$ and hence $\big(\bar{R}/j(\bar{R})\big)_{\tilde{P}} \cong \bar{R}_{\bar{P}}$ is regular.

Thus, $\bar{R}/j(\bar{R})$ satisfies (R_1). Lemma 1.12 implies that the integral closure of $\bar{R}/j(\bar{R})$ in its total ring of quotients is the S_2-ification of $\bar{R}/j(\bar{R})$. $\square$

Lemma 2.8. *The $\bar{A}$-module $\bar{R}^{\bar{A}\bar{A}}$ is free of rank e and thus satisfies (S_2) over $\bar{A}$.*

Proof. Since $[\mathfrak{a}] \in \operatorname{Ker}(\operatorname{Cl}(A) \xrightarrow{\gamma} \operatorname{Cl}(\bar{A}))$, we have $(\mathfrak{a} \otimes_A \bar{A})^{\bar{A}\bar{A}} \cong (\bar{\alpha})$, for some $\bar{\alpha} \in \bar{A}$. Thus, using the fact that γ is a group homomorphism, $(\mathfrak{a}^{(t)} \otimes_A \bar{A})^{\bar{A}\bar{A}} = \gamma([\mathfrak{a}^{(t)}]) = \big(\gamma([\mathfrak{a}])\big)^t \cong (\bar{\alpha}^t)$, for all t; i.e. each component of $\bar{R}^{\bar{A}\bar{A}}$ is a free $\bar{A}$-module. Consequently, $\bar{R}^{\bar{A}\bar{A}}$ is free of rank e and satisfies the (S_2) condition, since $\bar{A}$ is (S_2). $\square$

Lemma 2.9. *For each prime $\tilde{P}$ in $(\bar{R}/j(\bar{R}))'$, we have*

$$\operatorname{ht}(\tilde{P} \cap \bar{A}) = \operatorname{ht}(\tilde{P} \cap (\bar{R}/j(\bar{R}))) = \operatorname{ht}\tilde{P}.$$

Moreover, $(\bar{R}/j(\bar{R}))'$ satisfies (S_2) as an $\bar{A}$-module and as a ring.

Proof. The equality $\operatorname{ht}(\tilde{P} \cap (\bar{R}/j(\bar{R}))) = \operatorname{ht}\tilde{P}$ is from [14, Proposition 3.5(a)]. Lemma 2.4 implies that $\operatorname{ht}(\tilde{P} \cap \bar{A}) = \operatorname{ht}\tilde{P}$. Since $(\bar{R}/j(\bar{R}))'$ satisfies (S_2) as an $\bar{R}/j(\bar{R})$-module by Lemma 2.7, it follows from [12, Corollaire (5.7.11)] that $(\bar{R}/j(\bar{R}))'$ satisfies (S_2) as an $\bar{A}$-module and as a ring. $\square$

Lemma 2.10. *There are $\bar{A}$-isomorphisms $(\bar{R}/j(\bar{R}))' \cong (\bar{R}/j(\bar{R}))^{\bar{A}\bar{A}} \cong \bar{R}^{\bar{A}\bar{A}}$.*

Proof. Let $\bar{\mathfrak{p}}$ be a prime of height less than or equal to one in $\bar{A}$. Recall that $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{\mathfrak{p}}}$ is étale, by Lemma 2.5, therefore, $\bar{R}_{\bar{\mathfrak{p}}}$ is free over $\bar{A}_{\bar{\mathfrak{p}}}$. The maximal ideals of $\bar{R}_{\bar{\mathfrak{p}}}$ are of the form $\bar{P}\bar{R}_{\bar{\mathfrak{p}}}$, where $\bar{P} \cap \bar{A} = \bar{\mathfrak{p}}$, thus $\mathrm{ht}_{\bar{R}} \bar{P} = 1$ by Lemma 2.4. Since $(\bar{R}_{\bar{\mathfrak{p}}})_{\bar{P}\bar{R}_{\bar{\mathfrak{p}}}} \cong \bar{R}_{\bar{P}}$ and $\bar{R}$ satisfies the (R_1) condition, $\bar{R}_{\bar{P}}$ is a regular local ring. Therefore, $\bar{R}_{\bar{\mathfrak{p}}}$ is a regular ring. Moreover, $j(\bar{R}_{\bar{P}}) = 0$ since $\bar{R}_{\bar{P}}$ is a domain; hence, $j(\bar{R}_{\bar{\mathfrak{p}}})_{\bar{P}\bar{R}_{\bar{\mathfrak{p}}}} = 0$ for all $\bar{P}\bar{R}_{\bar{\mathfrak{p}}}$ (as in the proof of Lemma 2.7). We conclude that $j(\bar{R})_{\bar{\mathfrak{p}}} = 0$. Consequently, $(\bar{R}/j(\bar{R}))_{\bar{\mathfrak{p}}} = \bar{R}_{\bar{\mathfrak{p}}}$. Since $\bar{R}_{\bar{\mathfrak{p}}}$ is regular, hence normal, we deduce that $((\bar{R}/j(\bar{R}))')_{\bar{\mathfrak{p}}} = (\bar{R}_{\bar{\mathfrak{p}}}/j(\bar{R})_{\bar{\mathfrak{p}}})' = \bar{R}_{\bar{\mathfrak{p}}}$.

It follows that the $\bar{A}$-homomorphisms $\bar{R}^{\bar{A}\bar{A}} \to (\bar{R}/j(\bar{R}))^{\bar{A}\bar{A}} \to ((\bar{R}/j(\bar{R}))')^{\bar{A}\bar{A}}$ are isomorphisms in codimension one. Note that $\bar{R}/j(\bar{R})$ is complete, local, and reduced, since it was shown in Lemma 2.6 that $j(\bar{R}) = \mathrm{nil}(\bar{R})$. Therefore, $(\bar{R}/j(\bar{R}))'$ is finite over $\bar{R}/j(\bar{R})$, (see, e.g. [18, p. 263]), and hence finitely generated over $\bar{A}$. It follows that the maps $\bar{R}^{\bar{A}\bar{A}} \to (\bar{R}/j(\bar{R}))^{\bar{A}\bar{A}} \to ((\bar{R}/j(\bar{R}))')^{\bar{A}\bar{A}}$ are homomorphisms of reflexive $\bar{A}$-modules and isomorphisms in codimension one. Since reflexive implies torsion free (see, e.g. [7, Corollary 3.7]), these maps are isomorphisms, as per Fact 1.23.

Lemma 2.9 says that $(\bar{R}/j(\bar{R}))'$ satisfies the (S_2) condition as an $\bar{A}$-module. Hence it is reflexive as an $\bar{A}$-module, by [7, Theorem 3.6]. Thus, we have $(\bar{R}/j(\bar{R}))' \cong ((\bar{R}/j(\bar{R}))')^{\bar{A}\bar{A}} \cong (\bar{R}/j(\bar{R}))^{\bar{A}\bar{A}}$ as $\bar{A}$-modules. □

Remark 2.11. The upshot of the previous lemma is that $(\bar{R}/j(\bar{R}))'$ is the S_2-ification of $\bar{R}$ (see Definition 1.10). Therefore, $(\bar{R}/j(\bar{R}))'$ is local, by assumption.

Lemma 2.12. *The map $\bar{A} \to (\bar{R}/j(\bar{R}))'$ is étale in codimension one.*

Proof. Recall the composition $\bar{A} \to \bar{R} \to \bar{R}/j(\bar{R}) \to (\bar{R}/j(\bar{R}))'$. Let $\bar{\mathfrak{p}}$ be a prime ideal of $\bar{A}$ with height less than or equal to one. By Lemma 2.5, $\bar{A}_{\bar{\mathfrak{p}}} \to \bar{R}_{\bar{\mathfrak{p}}}$ is étale. Next, $((\bar{R}/j(\bar{R}))')_{\bar{\mathfrak{p}}} = (\bar{R}_{\bar{\mathfrak{p}}}/j(\bar{R}_{\bar{\mathfrak{p}}}))' = \bar{R}_{\bar{\mathfrak{p}}} \cong (\bar{R}/j(\bar{R}))_{\bar{\mathfrak{p}}}$ since $j(\bar{R})_{\bar{\mathfrak{p}}} = j(\bar{R}_{\bar{\mathfrak{p}}}) = 0$; see the proof of Lemma 2.10. Hence the composition $\bar{A}_{\bar{\mathfrak{p}}} \to ((\bar{R}/j(\bar{R}))')_{\bar{\mathfrak{p}}}$ is étale.

Now, let $\tilde{P}$ be a prime ideal of $(\bar{R}/j(\bar{R}))'$ with height less than or equal to one, and set $\bar{\mathfrak{p}} = \tilde{P} \cap \bar{A}$. Lemma 2.9 implies that $\mathrm{ht}\, \bar{\mathfrak{p}} \leq 1$, so the map $\bar{A}_{\bar{\mathfrak{p}}} \to ((\bar{R}/j(\bar{R}))')_{\tilde{P}}$ is étale, by the previous paragraph. □

Lemma 2.13. *The ring $(\bar{R}/j(\bar{R}))'$ is a normal domain.*

Proof. Lemma 2.9 implies that $(\bar{R}/j(\bar{R}))'$ is (S_2) as a ring. We claim that it is also (R_1) as a ring. To this end, let $\tilde{P}$ be a height one prime of $(\bar{R}/j(\bar{R}))'$, and set $\bar{\mathfrak{p}} = \tilde{P} \cap \bar{A}$. Then we have $\mathrm{ht}\, \bar{\mathfrak{p}} = 1$, again by Lemma 2.9, so $\bar{A}_{\bar{\mathfrak{p}}}$ is regular. From Lemma 2.12, we know that $((\bar{R}/j(\bar{R}))')_{\tilde{P}}$ is étale over $\bar{A}_{\bar{\mathfrak{p}}}$. Since $\bar{A}_{\bar{\mathfrak{p}}}$ is regular, this implies that $((\bar{R}/j(\bar{R}))')_{\tilde{P}}$ is regular.

Using Serre's criterion for normality, it follows that $(\bar{R}/j(\bar{R}))'$ is a finite direct product of normal domains; see [18, p. 64]. However, since $(\bar{R}/j(\bar{R}))'$ is local, by assumption, it cannot decompose into a non-trivial product of domains. Hence it is a normal domain. □

Lemma 2.14. *The extension $\bar{A} \hookrightarrow (\bar{R}/j(\bar{R}))'$ is étale.*

Proof. Lemma 2.13 implies that $(\bar{R}/j(\bar{R}))'$ is a normal domain. The extension $\bar{A} \to (\bar{R}/j(\bar{R}))'$ between normal domains is module finite, free, and étale in codimension one by Lemmas 2.8, 2.10 and 2.12. Thus, it is étale by purity of branch locus, specifically, Fact 1.21 applies since $(\bar{R}/j(\bar{R}))'$ is a free $\bar{A}$ module, and hence projective over $\bar{A}$. □

Conclusion of proof of Theorem C. Let S denote $(\bar{R}/j(\bar{R}))'$. Since the extension $\bar{A} \hookrightarrow S$ is étale and local, the induced field extension $\bar{A}/\bar{\mathfrak{m}} \to S/\bar{\mathfrak{m}}S$ is separable and algebraic. Since $\bar{A}/\bar{\mathfrak{m}}$ is separably closed, this induced map is an isomorphism. Recall that S is free over $\bar{A}$ of rank e. Consequently, there are isomorphisms

$$\bar{A}/\bar{\mathfrak{m}} \cong S/\bar{\mathfrak{m}}S \cong (\bar{A}/\bar{\mathfrak{m}})^e.$$

It follows that $e = 1$, but this contradicts the assumptions in paragraph 2.1. □

Additional details in the case that $I = (t)$

Recall that the proofs of Theorem C and [10, Theorem 1.2] hinge on the fact that a certain S_2-ification is local. We provide additional background and details of this fact in the case that the quotient ring is determined by a principal ideal.

Remark 2.15. In this subsection, we are no longer assuming the conditions of paragraph 2.1. All assumptions will be expressly stated.

Proposition 2.16. *Let $C \hookrightarrow D$ be a module finite extension of local normal domains which is unramified in codimension one. If t is an element in the maximal ideal of C such that $\bar{C} = C/tC$ satisfies the condition (R_1), then $\bar{D} = D/tD$ satisfies the condition (R_1) as well, and $\bar{C} \hookrightarrow \bar{D}$ is unramified in codimension one.*

Proof. The proof uses elements established in the lemmas above. Let $\bar{P} \subset \bar{D}$ be a prime of height less than or equal to one, and set $\bar{\mathfrak{p}} = \bar{P} \cap \bar{C}$. The extension $\bar{C} \hookrightarrow \bar{D}$ is module finite, hence going-up holds, and as a result, $\operatorname{ht} \bar{\mathfrak{p}} \geq \operatorname{ht} \bar{P}$. Along the lines of Lemma 2.4, there is a commutative diagram, as shown below, where the map $C \hookrightarrow D$ satisfies going-down since C is normal and D is a domain.

$$\begin{array}{ccc} C & \longrightarrow & D \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \pi} \\ \bar{C} & \longrightarrow & \bar{D} \end{array}$$

As before, the vertical maps are the natural surjections, and the bottom horizontal map is induced by the inclusion (i.e. the top horizontal map). Set $P = \pi^{-1}(\bar{P})$ and $\mathfrak{p} = \tau^{-1}(\bar{\mathfrak{p}}) = P \cap C$. Moreover, the argument of Lemma 2.4 establishes going-down, hence $\operatorname{ht} \bar{\mathfrak{p}} = \operatorname{ht} \bar{P}$. Consequently, $\operatorname{ht}_C \mathfrak{p} \leq 2$. Since $\bar{C}$ satisfies (R_1), the ring $\bar{C}_{\bar{\mathfrak{p}}}$ is regular local, hence $C_{\mathfrak{p}}$ is local and t is a minimal generator of $\mathfrak{p}C_{\mathfrak{p}}$. Thus, $C_{\mathfrak{p}}$ is a regular local ring as well. As a result, the modules $D_{\mathfrak{p}} = C_{\mathfrak{p}} \otimes_C D$ and D_P have finite projective dimension over $C_{\mathfrak{p}}$. By [16, Theorem 2.1], the extensions $C_{\mathfrak{p}} \to D_{\mathfrak{p}}$ and $C_{\mathfrak{p}} \to D_P$ are both étale. The remainder of the proof now dovetails with that of Lemma 2.5. □

Corollary 2.17. *Under the same hypotheses as in the above proposition, if the rings are complete, then the integral closures* $(\bar{C})'$ *and* $(\bar{D})'$ *are local integral domains.*

Proof. This follows immediately from Fact 1.13 since complete noetherian local rings are excellent. □

The next corollary is another easy consequence and an important component of our argument.

Corollary 2.18. *Under the same hypotheses as in the previous corollary, the extension* $(\bar{C})' \to (\bar{D})'$ *is module finite and unramified in codimension one.*

Proof. In codimension one, the extension $(\bar{C})' \to (\bar{D})'$ coincides with $\bar{C} \to \bar{D}$. Additionally, the fact that the extension is module finite follows from the commutative diagram below since the other three maps are module finite. Note that the vertical maps are module finite since $\bar{C}$ and $\bar{D}$ are complete domains (see, e.g. [18, p. 263]).

$$\begin{array}{ccc} \bar{C} & \longrightarrow & \bar{D} \\ \downarrow & & \downarrow \\ (\bar{C})' & \longrightarrow & (\bar{D})' \end{array}$$

□

Conclusion of proof of Theorem A. Assume the hypotheses and notation of Theorem A. The condition of excellence allows one to assume that A is complete.[a] Let $\mathfrak{a}$ be a non-zero reflexive ideal in the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ of order $e > 1$. Set R to be the truncated Rees algebra $R = A \oplus \mathfrak{a} \oplus \mathfrak{a}^{(2)} \oplus \cdots \oplus \mathfrak{a}^{(e-1)}$, which is a complete local normal domain. We pick up the proof of Theorem A at [10, p. 477, line –10]. Taking $A = C$ and $R = D$ in Proposition 2.16, we deduce that R/tR satisfies (R_1) and that the morphism $A/tA \to R/tR$ is module finite and unramified in codimension one. In the notation of Theorem A, $B = (A/tA)'$. Set $\bar{R} = R/tR$. By Corollary 2.17, both B and $(\bar{R})'$ are local integral domains, the latter of which is the key fact.

At this point, the argument dovetails with the proof of Theorem C above, starting from Lemma 2.6, by replacing $\bar{A}$ with B, and with the understanding that $I = (t)$. Specifically, $j(\bar{R}) = 0$ in this case, and since the domain $\bar{R}$ satisfies (R_1), its integral closure $(\bar{R})'$ in its total ring of quotients is its S_2-ification, by Lemma 1.12. Next, as per the proof of Lemma 2.8, the B-module $(R \otimes_A B)^{BB} = (\bar{R} \otimes_A B)^{BB}$ is

[a]In the proof of Theorem B in the next section, we provide the details for this, and other, reductions.

free of rank e and thus satisfies (S_2) over B. An analogous result to Lemma 2.9 also holds: $(\bar{R})'$ satisfies (S_2) as a B-module and as a ring. (We make a couple of applications of [14, Proposition 3.5(a)]; use elements of the proof of Proposition 2.16; apply [12, Corollaire (5.7.11)].) Next, we note that there is an isomorphism of B-modules $(\bar{R})' \cong (\bar{R} \otimes_A B)^{BB}$ à la Lemma 2.10: since the extension $B \to (\bar{R})'$ is module finite and unramified in codimension one, as established in Corollary 2.18, it follows that the map $(\bar{R} \otimes_A B)^{BB} \to ((\bar{R})')^{BB}$ is a homomorphism of reflexive B-modules and an isomorphism in codimension one. As noted previously, reflexive implies torsion free, hence the map is an isomorphism (see, e.g. Fact 1.23). Since $(\bar{R})'$ satisfies the (S_2) condition as a B-module, it is a reflexive B-module, by [7, Theorem 3.6]. Thus, we have $(\bar{R})' \cong (\bar{R} \otimes_A B)^{BB}$. It is not difficult to see that Lemmas 2.12–2.14 also hold with respect to B and $(\bar{R})'$; i.e. that the extension $B \to (\bar{R})'$ is étale. The conclusion, as above, now follows. □

3. Proof of Theorem B

3.1. Let $(A, \mathfrak{m})$ be an excellent, local, normal domain and I a prime ideal of A with finite projective dimension such that:

(i) $\bar{A} = A/I$ is normal;
(ii) I is a complete intersection on the punctured spectrum of A; and
(iii) $\mu(I) \leq \dim A - 2$.

Let $e > 1$ be an integer which represents a unit in A. Let $\mathfrak{a}$ be a non-zero reflexive ideal of A such that $[(\mathfrak{a} \otimes_A \bar{A})^{\bar{A}\bar{A}}]$ is trivial in $\mathrm{Cl}(\bar{A})$, and suppose that the order of $[\mathfrak{a}]$ in $\mathrm{Cl}(A)$ is e. Thus, $\mathfrak{a}^{(e)} = aA$ for some $a \in \mathfrak{a}$. Set K to be the fraction field of A.

The theorem is trivial if $I = (0)$, therefore assume I is non-zero. If $\dim A \leq 2$, then the (R_1) condition on $\bar{A}$ forces the factor ring to be regular, and hence A must itself be regular by [3, Theorem 6.1(1)]. Consequently, $\mathrm{Cl}(A) = 0$, which again provides a trivial result. Therefore, assume that $\dim A \geq 3$. We make a series of reductions to reduce to the case of Theorem C, and as in the previous proof, ultimately obtain our conclusion by way of contradiction.

The prime ideal I has finite projective dimension, therefore

$$\mathrm{pd}_{A_I}(A_I/IA_I) \leq \mathrm{pd}_A \bar{A} < \infty.$$

It follows that the localization A_I is regular and hence a unique factorization domain. This implies that $\mathfrak{a}A_I = bA_I$ for some b in $\mathfrak{a}$, and hence

$$b^e A_I = (bA_I)^{(e)} = (\mathfrak{a}A_I)^{(e)} = \mathfrak{a}^{(e)} A_I = aA_I.$$

Lemma 3.2. *A can be assumed to be complete.*

Proof. Because A is an excellent local normal domain, the completion $\hat{A}$ is a complete local normal domain by [20, Corollary 37.6]. Next, $\mathrm{pd}_{\hat{A}} I\hat{A} < \infty$ since the map $A \to \hat{A}$ is flat. Moreover, because the map is also local, $\mu_{\hat{A}}(I\hat{A}) = \mu_A(I) \leq \dim A - 2 = \dim \hat{A} - 2$. We need to show that $I\hat{A}$ is a complete intersection on the punctured spectrum of $\hat{A}$, denoted $\operatorname{Spec}^\circ(\hat{A})$.

Let $P \in \operatorname{Spec}^\circ(\hat{A})$ and set $\mathfrak{p} = P \cap A$. Since the closed fiber $\hat{A}/\mathfrak{m}\hat{A}$ is isomorphic to $A/\mathfrak{m}$, the fact that P is not maximal implies that $\mathfrak{p} \neq \mathfrak{m}$. Thus, the ideal $IA_\mathfrak{p}$ is either $A_\mathfrak{p}$ or generated by an $A_\mathfrak{p}$-regular sequence. If $IA_\mathfrak{p} = A_\mathfrak{p}$, then $I\hat{A}_P = IA_\mathfrak{p} \cdot \hat{A}_P = \hat{A}_P$. If $IA_\mathfrak{p}$ is generated by an $A_\mathfrak{p}$-regular sequence, then $I\hat{A}_P = IA_\mathfrak{p} \cdot \hat{A}_P$ is generated over $\hat{A}_P$ by the same sequence, which is $\hat{A}_P$-regular since the induced map $A_\mathfrak{p} \to \hat{A}_P$ is flat and local.

The left-most diagram below is commutative and each map has finite flat dimension:

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{\bar{A}} \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{Cl}(A) & \longrightarrow & \operatorname{Cl}(\bar{A}) \\ \downarrow & & \downarrow \\ \operatorname{Cl}(\hat{A}) & \longrightarrow & \operatorname{Cl}(\hat{\bar{A}}). \end{array} \tag{3.2.1}$$

It follows that the composition $A \to \hat{\bar{A}}$ also has finite flat dimension, so [23, Theorem 1.14] implies that the second diagram above also commutes.

Suppose the result holds on $\operatorname{Cl}(\hat{A}) \to \operatorname{Cl}(\hat{\bar{A}})$. Since $[\mathfrak{a}] \in \operatorname{Ker}(\operatorname{Cl}(A) \to \operatorname{Cl}(\bar{A}))$, it follows that $\operatorname{im}([\mathfrak{a}]) \in \operatorname{Ker}(\operatorname{Cl}(\hat{A}) \to \operatorname{Cl}(\hat{\bar{A}}))$. By properties of group monomorphisms, the order of $\operatorname{im}([\mathfrak{a}])$ is e, which is a unit in $\operatorname{Cl}(\hat{A})$. The condition $e > 1$ contradicts the complete case. □

Lemma 3.3. *A can be assumed to contain a primitive eth root of unity.*

Proof. Suppose that A does not contain a primitive eth root of unity and let ζ be a primitive eth root of unity in the algebraic closure of K, the fraction field of A. Note that ζ exists because e is a unit in A. By assumption, A is a complete local normal domain.

As per [21, Proposition VI.1], the extension $A \to A[T]/(T^e - 1)$ is étale. Let $\Phi_e(T) \in K[T]$ be the minimal polynomial of ζ over K. Since A is integrally closed and the coefficients of $\Phi_e(T)$ are integral over A, it follows that $\Phi_e(T) \in A[T]$. Using the Division Algorithm in $A[T]$, we have $A[\zeta] \cong A[T]/\Phi_e(T)$, and $\Phi_e(T)$ divides $(T^e - 1)$ in $A[T]$. Set $\mathrm{d} = \deg \Phi_e(T)$. Note that $A[T]/\Phi_e(T) \cong \bigoplus_{i=0}^{d-1} A$ as A-modules, hence the extension $A \to A[T]/\Phi_e(T)$ is faithfully flat. The maps $A \to A[T]/(T^e - 1)$ and $A[T]/(T^e - 1) \to A[T]/\Phi_e(T)$ are both unramified, thus the composition is unramified; i.e. the extension $A \to A[T]/\Phi_e(T)$ is étale.

The ring $A[\zeta]$ is excellent since excellence is preserved under finitely generated extensions. Moreover, $A[\zeta]$ is a domain since it is a subring of $K[\zeta]$. Since A is a normal domain and the finite extension $A \to A[\zeta]$ is étale, $(A[\zeta])_Q$ is a normal domain for each prime Q of $A[\zeta]$ by [18, Theorem 23.9].

Let Q be a maximal ideal of $A[\zeta]$. Since $A \to A[\zeta]$ is finite, hence $Q \cap A$ is maximal; that is, $Q \cap A = \mathfrak{m}$. We next note that the ring $\widehat{A[\zeta]_Q}$ and the ideal

$I\widehat{A[\zeta]_Q}$ satisfy the hypothesis of the theorem. Indeed, since $A[\zeta]_Q$ is an excellent local normal domain, it follows that $\widehat{A[\zeta]_Q}$ is a complete normal local domain.

The maps in the sequence $A \to A[\zeta] \to A[\zeta]_Q \to \widehat{A[\zeta]_Q}$ are all flat, hence the composition $A \to \widehat{A[\zeta]_Q}$ is flat. It follows that the extension $I\widehat{A[\zeta]_Q}$ has finite projective dimension over $\widehat{A[\zeta]_Q}$. It remains to show that $\widehat{A[\zeta]_Q}/I\widehat{A[\zeta]_Q}$ is a normal domain. To this end, note that $IA[\zeta]_Q$ has finite projective dimension over $A[\zeta]_Q$. The map $A \to A[\zeta]_Q$ is flat and unramified because $A \to A[\zeta]$ is flat and unramified. Therefore, $A/I \to A[\zeta]_Q/IA[\zeta]_Q$ is flat and unramified. The fact that A/I is normal implies that $A[\zeta]_Q/IA[\zeta]_Q$ is normal; since $A[\zeta]_Q/IA[\zeta]_Q$ is local, it is also an excellent local normal domain. Hence its completion $\widehat{A[\zeta]_Q}/I\widehat{A[\zeta]_Q}$ a local normal domain.

As in Lemma 3.2, we have $\mu_{\widehat{A[\zeta]_Q}}(I\widehat{A[\zeta]_Q}) \leq \dim \widehat{A[\zeta]_Q} - 2$ and the fact that the map $A \to \widehat{A[\zeta]_Q}$ is flat and local, with closed fiber a field, implies that $I\widehat{A[\zeta]_Q}$ is a complete intersection on $\operatorname{Spec}^\circ(\widehat{A[\zeta]_Q})$. Likewise, as in Lemma 3.2, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(A) & \longrightarrow & \mathrm{Cl}(\bar{A}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Cl}(\widehat{A[\zeta]_Q}) & \longrightarrow & \mathrm{Cl}(\widehat{A[\zeta]_Q}/I\widehat{A[\zeta]_Q}) \end{array} \qquad (3.3.1)$$

and the same argument shows that we may replace A with $\widehat{A[\zeta]_Q}$ to assume that A contains a primitive eth root of unity. □

Lemma 3.4. $A/\mathfrak{m}$ *can be assumed to be separably closed.*

Proof. Set $\mathsf{k} = \mathsf{A}/\mathfrak{m}$ and let k^{sep} be a separable closure of k (i.e. the set of all separable elements in a fixed algebraic closure). This means that $\mathsf{k} \subseteq \mathsf{k}^{\text{sep}}$ is a separable algebraic extension, and k^{sep} has no nontrivial separable algebraic extensions. Grothendieck [11, Proposition (0.10.3.1)] shows that there is a flat local ring homomorphism $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ such that $\mathfrak{n} = \mathfrak{m}B$, the ring B is complete, and $B/\mathfrak{n} \cong \mathsf{k}^{\text{sep}}$.

It follows that the extension $A \to B$ is regular, in the terminology of Matsumura [18, pp. 255–256]. To see this, observe that the extension $\mathsf{k} \to \mathsf{k}^{\text{sep}}$ is 0-smooth by [18, Theorem 26.9]. Using [18, Theorem 28.10] it follows that B is $\mathfrak{m}B$-smooth over A, that is, that B is $\mathfrak{n}$-smooth over A. Now, apply the theorem of André from [1, Lemme II.57] to conclude that the extension $A \to B$ is regular. (See also [18, p. 260].)

From [18, Theorem 32.2] it follows that B is a normal domain. The same reasoning implies that the ring B/IB is a normal domain. The fact that the extension $A \to B$ is flat and local implies that $\operatorname{pd}_B(B/IB) = \operatorname{pd}_A(\bar{A}) < \infty$.

Again as in Lemma 3.2, we have $\mu_B(IB) \leq \dim B - 2$, and the fact that the map $A \to B$ is flat and local, with closed fiber a field, implies that IB is a complete intersection on $\operatorname{Spec}^\circ(B)$.

Now argue as in the proof of Lemma 3.2: there is a commutative diagram of divisor class groups as in (3.2.1) and (3.3.1), but with bottom row $\operatorname{Cl}(B) \to \operatorname{Cl}(B/IB)$. As $[\mathfrak{a}] \in \operatorname{Ker}(\operatorname{Cl}(A) \to \operatorname{Cl}(\bar{A}))$ has order e in $\operatorname{Cl}(A)$, the element $\operatorname{im}([\mathfrak{a}]) \in \operatorname{Ker}(\operatorname{Cl}(B) \to \operatorname{Cl}(B/IB))$ has order e as well. If the theorem holds for $B \to B/IB$, then we have a contradiction. □

Conclusion of proof of Theorem B. We have reduced the theorem to the case where A is complete with separably closed residue field and contains a primitive eth root of unity. Set R to be the truncated symbolic Rees algebra as in Eq. (2.1.1). It only remains to show that $\bar{R} = R/IR$ satisfies any of the equivalent conditions in Fact 1.14. The result will then follow from Theorem C.

We will show that the S_2-ification of $\bar{R}$ is local. To this end, note that R is a local complete domain, hence equidimensional; see Lemma 2.3. Set $n = \dim A = \dim R$, where the equality is from the fact that R is finite over A.

By way of contradiction, suppose the S_2-ification of $\bar{R}$ is not local. Then there is a prime ideal $\bar{P} = P/IR \subset \bar{R}$ of height at least two such that $\operatorname{Spec}^\circ(\bar{R}_{\bar{P}})$ is disconnected. (This fact is implicit in [14, proof of Proposition 3.9(c)]. We provide the details in Lemmas 3.5–3.7.) Now R_P has a canonical module (since it is a homomorphic image of a regular local ring). Moreover, it is (S_2), therefore $\mathrm{H}^{\dim R_P}_{PR_P}(R_P)$ is indecomposable by [14, Theorem 3.7].

Case 1. If P is maximal in R, then $\bar{R}_{\bar{P}} = \bar{R}$, and $\mu(IR_P) \leq \mu(I) \leq n-2 = \dim R - 2 = \dim R_P - 2$, by condition (ii) of the hypotheses. Thus, $\mathrm{H}^n_P(R)$ is indecomposable. By [14, Theorem 3.3], $\operatorname{Spec}^\circ(\bar{R}_{\bar{P}}) = \operatorname{Spec}^\circ(\bar{R})$ is connected, contradicting our choice of $\bar{P}$.

Case 2. If P is not maximal in R, then $\mathfrak{p} = P \cap A$ is not maximal in A since R is finite over A. Furthermore, we have $\mathfrak{p} \supseteq IR \cap A \supseteq I$ since $P \supseteq IR$. Therefore $I_\mathfrak{p}$ is generated by an $A_\mathfrak{p}$-regular sequence $\mathbf{x} = x_1, \ldots, x_c \in I_\mathfrak{p}$ by condition (iii) of the hypotheses.

We claim that $\mathbf{x}$ is part of a system of parameters for R_P. (Then [14, Theorem 3.9(c)] implies that $\operatorname{Spec}^\circ(\bar{R}_{\bar{P}})$ is connected, again contradicting our choice of $\bar{P}$.) Let $\mathbf{x}' = x_1, \ldots, x_c, \ldots, x_d \in \mathfrak{p}_\mathfrak{p}$ be a system of parameters for $A_\mathfrak{p}$. It follows that $A_\mathfrak{p}/(\mathbf{x}')$ has finite length. Since $R_\mathfrak{p}$ is finitely generated over $A_\mathfrak{p}$, it follows that $R_\mathfrak{p}/(\mathbf{x}')$ has finite length, that is, $R_\mathfrak{p}/(\mathbf{x}')$ is artinian. The ring $R_P/(\mathbf{x}')$ is a localization of $R_\mathfrak{p}/(\mathbf{x}')$, so it is also artinian. Thus, the fact that the sequence $\mathbf{x}' \in P_P$ has length $d = \dim(A_\mathfrak{p}) = \dim(R_P)$ by Lemma 2.2 implies that $\mathbf{x}'$ is a system of parameters for R_P, as claimed.

Thus, the S_2-ification of $\bar{R}$ is local. The result now follows from Theorem C. □

As mentioned above, the three lemmas below give the explicit details, implicit in [14, proof of Proposition 3.9(c)], as to the fact that if the S_2-ification of $\bar{R}$ is not local, then there exists a prime ideal $\tilde{P}$ of $\bar{R}$ of height at least two such that the punctured spectrum of $\bar{R}_{\tilde{P}}$ is disconnected. (It should be noted that all assumptions in these lemmas are as stated and do not depend upon previously set notation in (3.1).)

Lemma 3.5. *Let B be a complete local equidimensional ring. Assume that the S_2-ification of B is not local. Then there exist ideals $K_1, K_2 \subseteq B$ such that*

(1) $\operatorname{ht}(K_1 + K_2) \geq 2$,
(2) $K_1 \cap K_2 \subseteq \operatorname{nil}(B)$, *and*
(3) $K_j \not\subseteq \operatorname{nil}(B)$ *for* $j = 1, 2$.

Proof. Since the S_2-ification of B is not local, we know from [13, Theorem 3.6] that there is an ideal $J \subseteq B$ such that $\operatorname{ht} J \geq 2$ and $\operatorname{Spec}(B) - V(J)$ is disconnected. Note that we are using the subspace topology on $\operatorname{Spec}(B) - V(J)$ induced from the Zariski topology on $\operatorname{Spec}(B)$. It follows that there are non-empty disjoint open subsets $U_1, U_2 \subsetneq \operatorname{Spec}(B) - V(J)$ such that $\operatorname{Spec}(B) - V(J) = U_1 \cup U_2$. Since each U_j is open in $\operatorname{Spec}(B) - V(J)$, it is of the form $U_j = [\operatorname{Spec}(B) - V(J)] \cap W_j$ for some open subset $W_j \subseteq \operatorname{Spec}(B)$. Thus, there are ideals $I_j \subseteq B$ such that $W_j = \operatorname{Spec}(B) - V(I_j)$, and it follows that

$$\begin{aligned} U_j &= [\operatorname{Spec}(B) - V(J)] \cap [\operatorname{Spec}(B) - V(I_j)] \\ &= \operatorname{Spec}(B) - [V(J) \cup V(I_j)] \\ &= \operatorname{Spec}(B) - V(JI_j). \end{aligned}$$

For $j = 1, 2$, set $K_j = JI_j$. This implies that $U_j = \operatorname{Spec}(B) - V(K_j)$ for $j = 1, 2$.

The condition $\operatorname{Spec}(B) - V(J) = U_1 \cup U_2$ implies that

$$\begin{aligned} \operatorname{Spec}(B) - V(J) &= [\operatorname{Spec}(B) - V(K_1)] \cup [\operatorname{Spec}(B) - V(K_2)] \\ &= \operatorname{Spec}(B) - [V(K_1) \cap V(K_2)] \\ &= \operatorname{Spec}(B) - V(K_1 + K_2). \end{aligned}$$

So we have

$$\begin{aligned} \operatorname{ht}(K_1 + K_2) &= \min\{\operatorname{ht} Q \mid Q \in V(K_1 + K_2)\} \\ &= \min\{\operatorname{ht} Q \mid Q \in V(J)\} \\ &= \operatorname{ht} J \\ &\geq 2. \end{aligned}$$

This explains condition (1) from the statement of the lemma.

Next, we use the fact that U_1 and U_2 are disjoint:

$$\begin{aligned} \emptyset &= U_1 \cap U_2 \\ &= [\operatorname{Spec}(B) - V(K_1)] \cap [\operatorname{Spec}(B) - V(K_2)] \end{aligned}$$

$$= \operatorname{Spec}(B) - [V(K_1) \cup V(K_2)]$$
$$= \operatorname{Spec}(B) - V(K_1 \cap K_2)$$

so that $V(K_1 \cap K_2) = \operatorname{Spec}(B)$. It follows that, for each $Q \in \operatorname{Spec}(B)$ we have $K_1 \cap K_2 \subseteq Q$. It follows that $K_1 \cap K_2 \subseteq \operatorname{nil}(B)$, i.e. we have condition (2) from the statement of the lemma.

For condition (3), we argue by contradiction. Suppose that $K_j \subseteq \operatorname{nil}(B)$. This implies that $V(K_j) = \operatorname{Spec}(B)$, so $U_j = \operatorname{Spec}(B) - V(K_j) = \emptyset$. This contradicts our choice of U_j. Thus, condition (3) is satisfied. □

Lemma 3.6. *Let B be a complete local equidimensional ring. Assume that the S_2-ification of B is not local. Then there exist ideals $L_1, L_2 \subseteq B$ such that*

(i) $\operatorname{ht}(L_1 + L_2) \geq 2$,
(ii) $L_1 \cap L_2 = \operatorname{nil}(B)$,
(iii) $L_j \nsubseteq \operatorname{nil}(B)$ *for* $j = 1, 2$, *and*
(iv) $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j}$ *for* $j = 1, 2$, *such that* $t_j \geq 1$ *and each* $P_{j,k} \in \min(B)$.

Proof. Let $K_1, K_2 \subseteq B$ be as in Lemma 3.5. It follows that

(a) $\operatorname{ht}(\operatorname{rad} K_1 + \operatorname{rad} K_2) \geq \operatorname{ht}(K_1 + K_2) \geq 2$,
(b) $\operatorname{rad} K_1 \cap \operatorname{rad} K_2 = \operatorname{rad}(K_1 \cap K_2) \subseteq \operatorname{rad} \operatorname{nil}(B) = \operatorname{nil}(B)$,
(c) $\operatorname{rad} K_j \nsubseteq \operatorname{nil}(B)$ for $j = 1, 2$, and
(d) each $\operatorname{rad} K_j$ is an intersection of primes of B.

Thus, we may replace K_j with $\operatorname{rad} K_j$ to assume that each K_j is an intersection of primes of B.

Claim. *Each K_j is contained in a minimal prime of B. We prove this by contradiction. By symmetry, suppose that K_1 is not contained in any minimal prime of B. Then for each $Q \in \min(B)$, we have*

$$K_1 K_2 \subseteq K_1 \cap K_2 \subseteq \operatorname{nil}(B) \subseteq Q.$$

Since Q is prime, we have $K_j \subseteq Q$ for some $j = 1, 2$. But $K_1 \nsubseteq Q$ by assumption, so we must have $K_2 \subseteq Q$. Since Q was chosen arbitrarily from $\min(B)$, we conclude that K_2 is contained in the intersection of the minimal primes of B, that is, $K_2 \subseteq \operatorname{nil}(B)$. This contradicts condition (3) from Lemma 3.5. Thus, the claim is established.

By condition (d) above, for $j = 1, 2$ we can write $K_j = P_{j,1} \cap \cdots \cap P_{j,n_j}$ where each $P_{j,k}$ is prime. Re-order the $P_{j,k}$ if necessary to assume that $P_{j,1}, \ldots, P_{j,t_j} \in \min(B)$ and $P_{j,t_j+1}, \ldots, P_{j,n_j} \notin \min(B)$. Note that the claim above implies that $t_j \geq 1$. For $j = 1, 2$ set $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \supseteq K_j$. By definition of L_j, we have condition (iv) from the statement of the lemma. Furthermore, the condition $L_j \supseteq K_j$ implies that

$$\operatorname{ht}(L_1 + L_2) \geq \operatorname{ht}(K_1 + K_2) \geq 2$$

and $L_j \nsubseteq \operatorname{nil}(B)$, so conditions (i) and (iii) are satisfied.

We conclude the proof by verifying condition (ii). Since each L_j is an intersection of primes of B, we have $L_j \supseteq \operatorname{nil}(B)$, and hence $L_1 \cap L_2 \supseteq \operatorname{nil}(B)$. To show the reverse containment, let $Q \in \min(B)$; it suffices to show that $L_1 \cap L_2 \subseteq Q$. We know that

$$P_{1,1} \cap \cdots \cap P_{1,n_1} \cap P_{2,1} \cap \cdots \cap P_{2,n_2} = K_1 \cap K_2 \subseteq \operatorname{nil}(B) \subseteq Q.$$

Thus, we have $P_{j,k} \subseteq Q$ for some j, k. Since Q is minimal, we must have $P_{j,k} = Q$ so $k \leq t_j$. From this, we have $L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \subseteq P_{j,k} = Q$, as desired. □

Lemma 3.7. *Let B be a complete local equidimensional ring. Assume that the S_2-ification of B is not local. Then there is a prime ideal P of B such that* $\operatorname{ht} P \geq 2$ *and* $\operatorname{Spec}^\circ(B_P)$ *is disconnected.*

Proof. Let L_1, L_2 be as in Lemma 3.6, and let P be minimal in $V(L_1 + L_2)$. It follows that $\operatorname{ht} P \geq \operatorname{ht}(L_1 + L_2) \geq 2$. We show that $\operatorname{Spec}^\circ(B_P)$ is disconnected. For $j = 1, 2$ we set

$$V^\circ(L_j B_P) = V(L_j B_P) \cap \operatorname{Spec}^\circ(B_P) = V(L_j B_P) - \{P B_P\}.$$

Since we are using the subspace topology on $\operatorname{Spec}^\circ(B_P)$ induced from the Zariski topology on $\operatorname{Spec}(B_P)$, the sets $V^\circ(L_j B_P)$ are closed in $\operatorname{Spec}^\circ(B_P)$. We show that the closed sets $V^\circ(L_j B_P)$ give a disconnection of $\operatorname{Spec}^\circ(B_P)$.

Claim 1: $\operatorname{Spec}^\circ(B_P) = V^\circ(L_1 B_P) \cup V^\circ(L_2 B_P)$. *The containment $\supseteq$ is by definition of $V^\circ(L_j B_P)$. For the reverse containment, let $QB_P \in \operatorname{Spec}^\circ(B_P)$. Thus, we have $Q \in \operatorname{Spec}(B)$ and thus $L_1 L_2 \subseteq L_1 \cap L_2 = \operatorname{nil}(B) \subseteq Q$, by condition* (ii) *of Lemma* 3.6. *It follows that $L_j \subseteq Q$ for some j, hence $L_j B_P \subseteq QB_P$, so*

$$QB_P \in V(L_j B_P) \cap \operatorname{Spec}^\circ(B_P) = V^\circ(L_j B_P) \subseteq V^\circ(L_1 B_P) \cup V^\circ(L_2 B_P)$$

as desired.

Claim 2: $V^\circ(L_1 B_P) \cap V^\circ(L_2 B_P) = \emptyset$. *By way of contradiction, suppose that $V^\circ(L_1 B_P) \cap V^\circ(L_2 B_P) \neq \emptyset$, and let $QB_P \in V^\circ(L_1 B_P) \cap V^\circ(L_2 B_P)$. It follows that $Q \subsetneq P$ and $L_j B_P \subseteq QB_P$ for $j = 1, 2$. It follows that*

$$L_j \subseteq B \cap L_j B_P \subseteq B \cap QB_P = Q \subsetneq P$$

for $j = 1, 2$. (*Given an ideal $I \subseteq B$, we use the notation $B \cap IB_P$ to denote the contraction of IB_P along the natural map $B \to B_P$.*) *From this, we have $L_1 + L_2 \subseteq Q \subsetneq P$, contradicting the fact that P is minimal in $V(L_1 + L_2)$.*

To complete the proof, it suffices to show that $V^\circ(L_j B_P) \neq \emptyset$ for $j = 1, 2$. For this, it suffices to show that there is a prime $P_{j,k}$ in B such that $L_j \subseteq P_{j,k} \subsetneq P$. Using condition (iv) of Lemma 3.6, we have

$$P_{j,1} \cap \cdots \cap P_{j,t_j} = L_j \subseteq L_1 + L_2 \subseteq P$$

so we find that $P_{j,k} \subseteq P$ for some k. By construction, we have

$$L_j = P_{j,1} \cap \cdots \cap P_{j,t_j} \subseteq P_{j,k} \subseteq P.$$

Moreover, we have $\operatorname{ht}(P_{j,k}) = 0 < 2 = \operatorname{ht} P$, so $P_{j,k} \subsetneq P$, as desired. □

4. Corollaries and an Example

We begin this section by partially recovering [10, Theorem 1.2], as described in the introduction.

Corollary 4.1. *Let* $(A, \mathfrak{m})$ *be an excellent, local, normal domain and* I *a prime ideal of* A *generated by an* A*-regular sequence such that* $\bar{A} = A/I$ *is normal. For any integer* $e > 1$ *which represents a unit in* A, *the kernel of the homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(\bar{A})$ *contains no element of order* e.

Proof. Case 1. $\dim(A/I) \leq 1$. In this case, the fact that A/I is normal (hence, it satisfies (R_1)) implies that A/I is regular. Since I is generated by an A-regular sequence, it follows from [3, Theorem 6.1(1)] that A is regular, so the result follows in this case.

Case 2. $\dim(A/I) \geq 2$. In this case, we have $\mu(I) = \dim(A) - \dim(A/I) \leq \dim(A) - 2$ since I is generated by an A-regular sequence. Thus, the desired conclusion follows from Theorem B in this case. □

The next two results follow from Theorem B, as in [10].

Corollary 4.2 (see [10, Corollary 1.3]). *Under the same assumptions as in Theorem* B, *with the additional hypothesis that* A *is* $\mathbb{Q}$*-algebra, the kernel of the homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(\bar{A})$ *is torsion free.*

Corollary 4.3 (see [10, Corollary 1.4]). *If the local hypothesis on* A *is removed in Theorem* B, *and instead it is assumed that* $\mathrm{Pic}(A) = 0$, *and* I *is a prime ideal of* A *in the Jacobson radical of* A *with finite projective dimension satisfying the three conditions of Theorem* B, *then for any integer* $e > 1$ *representing a unit in* A, *the kernel of the homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(\bar{A})$ *contains no element of order* e.

We conclude with an example where the hypotheses of Theorem B are satisfied by an ideal I that is not generated by an A-regular sequence.

Example 4.4. Let $(A, \mathfrak{m})$ be an excellent normal local integral domain, and let the positive integer e represent a unit in A. Assume that A has an A-regular sequence $f_1, \ldots, f_6 \in \mathfrak{m}$ such that $A/(f_1, \ldots, f_6)A$ is a normal domain. (Examples of such rings are constructed in [23, Example 3.4].) In particular, we have $6 \leq \operatorname{depth} A \leq \dim A$. Localize at a minimal prime of the ideal $J = (f_1, \ldots, f_6)A$, if necessary, to assume that J is $\mathfrak{m}$-primary. It follows that A is Cohen–Macaulay of dimension 6. Arrange the sequence $f_1, \ldots, f_6$ in a 2×3 matrix $F = \left(\begin{smallmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \end{smallmatrix}\right)$, and consider the ideal $I = I_2(F)$ generated by the 2×2 minors $d_1 = f_1 f_5 - f_2 f_4$, $d_2 = f_2 f_6 - f_3 f_5$, $d_3 = f_1 f_6 - f_3 f_4$.

We claim that A and I satisfy the hypotheses of Theorem B. The proof of [23, Proposition 3.1] shows that I is a height-2 prime ideal of A of finite projective dimension generated by three elements (so $\mu(I) < \dim A - 2$) such that A/I is a normal domain. Thus, we need only show that I is a complete intersection on the punctured spectrum. Let $\mathfrak{p} \in \operatorname{Spec}^{\circ}(A)$. We need to assume that $I \subseteq \mathfrak{p}$ and show that $IA_{\mathfrak{p}}$ is generated by an $A_{\mathfrak{p}}$-regular sequence.

Since J is $\mathfrak{m}$-primary and $\mathfrak{p} \neq \mathfrak{m}$, we have $J \not\subseteq \mathfrak{p}$, so some generator of J is not in $\mathfrak{p}$. By symmetry, assume that the generator f_1 is not in $\mathfrak{p}$. It follows that f_1 is a unit in $A_{\mathfrak{p}}$. Thus, a routine computation shows that in $A_{\mathfrak{p}}$ we have

$$d_2 = f_2 f_6 - f_3 f_5 = \frac{f_2}{f_1} d_3 - \frac{f_3}{f_1} d_1$$

so we conclude that $IA_{\mathfrak{p}} = (d_1, d_3)A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is Cohen–Macaulay and $\operatorname{ht}(IA_{\mathfrak{p}}) = 2$, it follows that the sequence d_1, d_3 is $A_{\mathfrak{p}}$-regular (e.g. by [18, Theorem 17.4]), as desired.

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