1 Questions:

1: $f(x) = e^x$. Mac series is defined to be:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

2: Write a power series for $\dfrac{e^{-x^2} - 1}{x}$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\begin{aligned}
\frac{e^{-x^2} - 1}{x} &= \frac{e^{-x^2}}{x} - \frac{1}{x} \\
&= \frac{1}{x} e^{-x^2} - \frac{1}{x} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} - \frac{1}{x} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{n!} - \frac{1}{x} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{n!} + x^{-1} - \frac{1}{x} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{n!}
\end{aligned}$$

3: $\sum_{n=0}^{\infty} n(n+2)x^n$. We would like to find a geometric series (somehow). Consider the series

$$\sum_{n=1}^{\infty} (n)(n+1)x^{n-1} : \int \sum_{n=1}^{\infty} (n)(n+1)x^{n-1}$$

$$\begin{aligned}
& A + \sum_{n=1}^{\infty} \frac{(n)(n+1)x^n}{n} = A + \sum_{n=1}^{\infty} (n+1)x^n \\
& \int \left[A + \sum_{n=1}^{\infty} (n+1)x^n \right] dx \\
& = Ax + B + \sum_{n=1}^{\infty} x^{n+1} = Ax + B + \sum_{n=0}^{\infty} x^2 x^n \\
& = Ax + B + x^2 \frac{1}{1-x} = Ax + B + \frac{x^2}{1-x} \\
& \frac{d^2}{dx^2} \left[Ax + B + \frac{x^2}{1-x} \right] = \sum_{n=1}^{\infty} n(n+1)x^{n-1} \\
& = \frac{(2-2x)(1-x) + 2x - x^2}{(1-x)^3}
\end{aligned}$$