For five more points (can not go over 10 total points): prove that the sequence $\{a_n\}$ defined by $a_n = \frac{n}{2n+1}$ converges to $\frac{1}{2}$.

1 10.3 # 19

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n2^n} &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \frac{1/2}{1-1/2} \\ &= 1\end{aligned}$$

Therefore by the comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges. Note that this series only converges to some value < to 1.

2 Limit Comparison Test

Let $\sum a_n$and $\sum b_n$ be positive termed seris such that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

- If $0 < L = c < \infty$, then either both series converge or both series diverge.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.

3 Example:

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{1/n^2}{1/n} = \lim_{n\to\infty} \frac{n}{n^2} = 0.$$

In this case $\sum a_n$ converges, but $\sum b_n$ diverges.

4 10.3 #39:

$\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$. Guess: converges. Consider the series $\sum_{n=2}^{\infty} \frac{n^2}{n^4}$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{n^2}{n^4}} &= \lim_{n\to\infty} \frac{n^2 n^4}{n^2(n^4-1)} \\
&= 1
\end{aligned}
$$

By limit comparison test: both series converge or both series diverge. Note

$$
\sum_{n=2}^{\infty} \frac{n^2}{n^4} = \sum_{n=2}^{\infty} \frac{1}{n^2}
$$

converges by the p-series with $p = 2$. Therefore the original series converges.

5 10.3 # 52:

$\displaystyle\sum_{n=1}^{\infty} \frac{n-\cos(n)}{n^3}$ Gues: convergent. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3}$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n-\cos(n)}{n^3}}{\frac{n}{n^3}} &= \lim_{n\to\infty} \frac{n-\cos(n)}{n} \\
&= 1
\end{aligned}
$$

by the squeeze theorem. Therefore by the limit comparison test: either both series converge or both series diverge.

$$
\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
$$

by the p-series with $p = 2$. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n-\cos(n)}{n^3}$ also converges.