Homework # 3 Solutions

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Solution (3.10.13)**.**

(a) Since $y = \frac{u+1}{u-1}$, the quotient rule tells us that

$$\frac{dy}{du} = \frac{(u+1)'(u-1) - (u+1)(u-1)'}{(u-1)^2} = \frac{(u-1) - (u+1)}{(u-1)^2} = -\frac{2}{(u-1)^2}.$$

Therefore

$$dy = -\frac{2}{(u-1)^2}du.$$

(b) Since $y = (1+r^3)^{-2}$, the chain rule tells us that

$$\frac{dy}{dr} = -2(1+r^3)^{-3}(1+r^3)' = -2(1+r^3)^{-3}(3r^2) = \frac{-6r^2}{(1+r^3)^3}.$$

Solution (3.10.17)**.**

(a) Since $y = \tan(x)$, we have that $\frac{dy}{dx} = \sec^2(x)$, and therefore

$$dy = \sec^2(x)dx.$$

(b) For $x = \pi/4$ and $dx = -0.1$, we obtain

$$dy = \sec^2(\pi/4)(-0.1) = 2 * (-0.1) = -0.2.$$

Solution (3.10.27)**.** We wish to estimate $\tan(\theta)$ for $\theta = 44°$. In terms of radians, $44° = 44\pi/180$. We note that $44\pi/180$ is close to the value $45\pi/180 = \pi/4$. Moreover, $\tan(\pi/4) = 1$, $(\tan(x))' = \sec^2(x)$ and $\sec^2(\pi/4) = 2$. The linearization of $\tan(\theta)$ at $\theta = \pi/4$ gives us the estimate

$$\tan(x) \approx 1 + 2(x - \frac{\pi}{4}) \qquad (x \text{ close to } \pi/4.)$$

Therefore

$$\tan(44°) \approx 1 + 2(\frac{44\pi}{180} - \frac{\pi}{4}) = 1 - \frac{\pi}{90} \approx 0.9651.$$

Solution (3.10.33)**.** Let x be the length of an edge of the cube.

(a) Let V be the volume of the cube. Then $V = x^3$, from which it follows that $\frac{dV}{dx} = 3x^2$ and therefore

$$dV = 3x^2 dx.$$

Since the length of a side of the cube was measured to be 30 cm, with a possible error in the measurement of 0.1 cm, we take $x = 30$ and $dx = 0.1$. Then $dV = 3(30)^2(0.1) = 270$. Additionally, $V = (30)^3 = 27000$, so we find:

MAX. ERROR:	270 cm^3
REL. ERROR:	$270/27000 = 0.01$
PER. ERROR:	$(0.01 * 100)\% = 1\%$

(b) Let A be the surface area of the cube. Then $A = 6x^2$, from which it follows that $\frac{dA}{dx} = 12x$ and therefore

$$dA = 12xdx.$$

Since the length of a side of the cube was measured to be 30 cm, with a possible error in the measurement of 0.1 cm, we take $x = 30$ and $dx = 0.1$. Then $dA = 12(30)(0.1) = 36$. Additionally, $A = 6(30)^2 = 5400$, so we find:

MAX. ERROR:	36 cm^3
REL. ERROR:	$36/5400 \approx 0.0067$
PER. ERROR:	$(0.0067 * 100)\% = 0.67\%$

Solution (3.11.27)**.** We will prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

using two different methods demonstrated in (a) and (b) below.

(a) Let $y = \tanh^{-1}(x)$. Then $x = \tanh(y)$ and

$$\tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{\left(\frac{e^y - e^{-y}}{2}\right)}{\left(\frac{e^y + e^{-y}}{2}\right)} = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Thus,

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Multiplying both sides of the equality by $e^y + e^{-y}$, we obtain

$$xe^y + xe^{-y} = e^y - e^{-y}.$$

Now multiplying both sides of the equality by e^y, we find

$$xe^{2y} + x = e^{2y} - 1.$$

Rearranging the terms, this expression becomes

$$e^{2y} - xe^{2y} = 1 + x,$$

or rather

$$(1-x)e^{2y} = 1+x.$$

Now dividing both sides of the equality by $(x-1)$, we find

$$e^{2y} = \frac{1+x}{1-x}.$$

Taking the natural log of both sides, we obtain

$$2y = \ln\left(\frac{1+x}{1-x}\right).$$

Lastly, dividing by 2, and replacing y with $\tanh^{-1}(x)$ we find

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

(b) Again let $y = \tanh^{-1}(x)$. Then we have that $\tan(y) = x$. Moreover, the result of Exercise 3.11.18 (using y in place of x) is that

$$\frac{1+\tanh(y)}{1-\tanh(y)} = e^{2y}.$$

Replacing $\tanh(y)$ with x in the above, we find

$$\frac{1+x}{1-x} = e^{2y}.$$

Taking the natural log of both sides, we obtain

$$\ln\left(\frac{1+x}{1-x}\right) = 2y.$$

Lastly, dividing by 2, and replacing y with $\tanh^{-1}(x)$ we find

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Solution (3.11.39)**.** Recall that $(\arctan(x))' = \frac{1}{1+x^2}$ and $(\tanh(x))' = \operatorname{sech}^2(x)$. For $y = \arctan(\tanh(x))$, the chain rule tells us that

$$\frac{dy}{dx} = \frac{1}{1+\tanh^2(x)}(\tanh(x))' = \frac{1}{1+\tanh^2(x)}\operatorname{sech}^2(x) = \frac{\operatorname{sech}^2(x)}{1+\tanh^2(x)}.$$

Solution (3.11.55)**.** Consider the curve $y = \cosh(x)$. The slope of the line tangent to the curve is $y' = \sinh(x)$. If the tangent line has slope 1, then $\sinh(x) = 1$, meaning that $x = \sinh^{-1}(1)$. Recalling that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$, this means that

$$x = \sinh^{-1}(1) = \ln(1+\sqrt{(1)^2+1}) = \ln(1+\sqrt{2}).$$

Moreover, when $x = \ln(1+\sqrt{2})$, we have that

$$
\begin{aligned}
y = \cosh(\ln(1+\sqrt{2})) &= \frac{e^{\ln(1+\sqrt{2})} + e^{-\ln(1+\sqrt{2})}}{2} = \frac{(1+\sqrt{2}) + (1+\sqrt{2})^{-1}}{2} \\
&= \frac{4+2\sqrt{2}}{2(1+\sqrt{2})} = \frac{2+\sqrt{2}}{1+\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2}+1)}{1+\sqrt{2}} = \sqrt{2}.
\end{aligned}
$$

Thus the point on the curve $y = \cosh(x)$ at which the tangent line has slope 1 is $(\ln(1+\sqrt{2}), \sqrt{2})$.

Solution (4.1.33)**.** The critical numbers of $s(t) = 3t^4 + 4t^3 - 6t^2$ are those values of t for which $s'(t) = 0$ or $s'(t)$ does not exist. Since $s'(t) = 12t^3 + 12t^2 - 12t$ exists everywhere, the critical points must be those values of t for which

$$12t^3 + 12t^2 - 12t = 0.$$

Dividing both sides by 12, we find

$$t^3 + t^2 - t = 0.$$

Thus either $t = 0$ or $t^2 + t - 1 = 0$. In the latter case, the quadratic formula tells us that $t = \frac{-1\pm\sqrt{5}}{2}$. Therefore the critical points are 0, $\frac{1+\sqrt{5}}{2}$, and $\frac{-1+\sqrt{5}}{2}$.

Solution (4.1.57)**.** We wish to find the absolute minimum and maximum values of $f(t) = 2\cos(t) + \sin(2t)$ on the interval $[0, \pi/2]$. To do so, we first examine the derivative of $f(t)$ to find the critical points of f. We have that $f'(t) = -2\sin(t) + 2\cos(2t)$. Therefore the derivative exists everywhere and the critical points are all those values of t for which $f'(t) = 0$, that is

$$-2\sin(t) + 2\cos(2t) = 0.$$

Dividing by 2, this becomes

$$-\sin(t) + \cos(2t) = 0.$$

Using the double angle formula, $\cos(2t) = 1 - 2\sin^2(t)$, we then find

$$-2\sin^2(t) - \sin(t) + 1 = 0,$$

Factoring this, we find

$$-(2\sin(t) - 1)(\sin(t) + 1) = 0.$$

Thus $\sin(t) = \frac{1}{2}$ or $\sin(t) = -1$. In the first case, since $t \in [0, \pi/2]$, we must have that $t = \pi/6$. In the second case, since $\sin(t)$ is nonnegative in the interval $[0, \pi/2]$, there is no value of t in $[0, \pi/2]$ for which $\sin(t) = -1$. Therefore the only critical point of f in the interval $[0, \pi/2]$ is at $t = \pi/6$. The boundary points of the interval are 0 and $\pi/2$. Since $f(0) = 2$, $f(\pi/2) = 0$, and $f(\pi/6) = \frac{3}{2}\sqrt{3}$ we conclude

$$
\begin{aligned}
\text{ABSOLUTE MIN: } & \text{occurs at } x = \pi/2 \text{ with value } 0 \\
\text{ABSOLUTE MAX: } & \text{occurs at } x = \pi/4 \text{ with value } \frac{3}{2}\sqrt{3}.
\end{aligned}
$$

Solution (4.1.71)**.** We wish to maximize the function $S(t)$ on the interval $[0, 10]$. The derivative of $S(t)$ is

$$S'(t) = -0.00016185t^4 + 0.0036148t^3 - 0.026868t^2 + .072580t - 0.4458.$$

The roots of this polynomial are approximately the values 0.854778, 4.61772, 7.29191, and 9.56986, so these must be the critical points of $S(t)$. We have that $S(0.854778) = 0.390683$, $S(4.61772) = 0.436446$, $S(7.29191) = 0.427119$, $S(9.56986) = 0.436414$. Additionally, then endpoints of the interval $[0, 10]$ are 0 and 10 and $S(0) = 0.4074$ and $S(10) = 0.4346$. It follows that

$$\begin{aligned} \text{ABSOLUTE MIN: occurs at } \; t = 0.854778 \; \text{ with value } \; 0.390683 \\ \text{ABSOLUTE MAX: occurs at } \; t = 4.61772 \; \text{ with value } \; 0.436446. \end{aligned}$$

Thus sugar was cheapest at $t = 0.854778$ (June 1994) and most expensive at $t = 4.61772$ (March 1998).