(A SOMEWHAT GENTLE INTRODUCTION TO) DIFFERENTIAL GRADED COMMUTATIVE ALGEBRA

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Abstract. Differential graded (DG) commutative algebra provides powerful techniques for proving theorems about modules over commutative rings. These notes are a somewhat colloquial introduction to these techniques, introduced in the context of a recent result of Nasseh and Sather-Wagstaff so as to provide some motivation for commutative algebraists who are wondering about the benefits of learning and using these techniques.

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1. Introduction

Convention. The term "ring" is short for "commutative noetherian ring with identity", and "module" is short for "unital module". Let R be a ring.

These are notes for the course "Differential Graded Commutative Algebra" that is/was part of the Workshop on Connections Between Algebra and Geometry held at the University of Regina, May 29–June 1, 2012. They represent our attempt to provide a small amount of (1) motivation for commutative algebraists who are wondering about the benefits of learning and using Differential Graded (DG) techniques, and (2) actual DG techniques.

DG Algebra. DG commutative algebra provides a useful framework for proving theorems about rings and modules, the statements of which have no reference to the DG universe. For instance, a standard theorem (due to Bass or Grothendieck?) says the following:

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Theorem 1.1. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism, that is, a ring homomorphism making S into a flat R-module such that $\mathfrak{m}S \subseteq \mathfrak{n}$. Then S is Gorenstein if and only if R and $S/\mathfrak{m}S$ are Gorenstein. Moreover, there is an equality of Bass series $I^S(t) = I^R(t)I^{S/\mathfrak{m}S}(t)$.*

(See Definition A.1 for the term "Bass series".) Of course, the flat hypothesis is very important here. On the other hand, the use of DG algebras allows for a slight (or vast, depending on your perspective) improvement of this:

Theorem 1.2 (cite Avramov and Foxby)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism of finite flat dimension, that is, a local ring homomorphism such that S has a bounded resolution by flat R-module. Then there is a formal Laurent series $I^\varphi(t)$ with non-negative integer coefficients such that $I^S(t) = I^R(t)I^\varphi(t)$. In particular, if S is Gorenstein, then so is R.*

In this result, the series $I^\varphi(t)$ is the *Bass series* of φ. It is the Bass series of the "homotopy closed fibre" of φ (instead of the usual closed fibre $S/\mathfrak{m}S$ of φ that is used in Theorem 1.1) which is the commutative DG algebra $S \otimes^{\mathbf{L}}_R R/\mathfrak{m}$. In particular, when S is flat over R, this is the usual closed fibre $S/\mathfrak{m}S \cong S \otimes_R R/\mathfrak{m}$, so we recover Theorem 1.1 as a corollary of Theorem 1.2.

Furthermore, DG algebra comes equipped with constructions that can be used to replace your given ring with one that is nicer in some sense. To see how this works, consider the following formula for using completions.

To prove a theorem about a given local ring R, first show that the assumptions ascend to $\widehat{R}$, then prove the result for the complete ring $\widehat{R}$, and finally show how the conclusion for $\widehat{R}$ implies the desired conclusion for R. This technique is useful since frequently $\widehat{R}$ is nicer then R. For instance, $\widehat{R}$ is a homomorphic image of a power series ring over a field or a complete discrete valuation ring, so it is universally catenary (whatever that means) and it has a dualizing complex (whatever that is), while the original ring R may not have either of these properties.

When R is Cohen-Macaulay, a similar formula sometimes allows one to mod out by a maximal R-regular sequence to assume that R is artinian. The regular sequence assumption is like the flat condition for $\widehat{R}$ in that it (sometimes) allows for the transfer of hypotheses and conclusions between R and the quotient $\overline{R}$. The artinian hypothesis is particularly nice, for instance, when R contains a field because then $\overline{R}$ is a finite dimensional algebra over a field.

The DG universe contains a construction $\widetilde{R}$ that is similar $\overline{R}$, with an advantage and a disadvantage. The advantage is that it is more flexible than $\overline{R}$ because it does not require the ring to be Cohen-Macaulay, and it produces a finite dimensional algebra over a field regardless of whether or not R contains a field. The disadvantage is that $\widetilde{R}$ is a DG algebra instead of just an algebra, so it is graded commutative (almost but not quite commutative) and there is a bit more data to keep track of when working with $\widetilde{R}$. However, the advantages outweigh the disadvantages in that $\widetilde{R}$ allows us to prove results for arbitrary local rings that can only be proved (as we understand things today) in special cases using $\overline{R}$. One such result is the following:

Theorem 1.3 ([30, Theorem A])**.** *A local ring has only finitely many semidualizing modules up to isomorphism.*

Even if you don't know what a semidualizing module is, you can see the point. Without DG techniques, we only know how to prove this result for Cohen-Macaulay rings that contain a field; see [14]. With DG techniques, you get the unqualified result, which answers a question of Vasconcelos [33] from 1974.

What These Notes Are. Essentially, these notes contain a colloquial description of the proof of Theorem 1.3. Our first goal is to provide a big-picture view of some of the tools and techniques in DG algebra (and other ares) needed to get a basic understanding of the proof of this result; see 5.26, 7.21, and 8.16 below. Also, since our motivation comes from the study of semidualizing modules, we provide a bit of motivation for the study of those gadgets in Appendix A. (For instance, they allow for a version of Theorem 1.2 where φ is only assumed to have finite G-dimension [5].) In particular, we do not assume that the reader is familiar with the semidualizing world.

What These Notes Are Not. These notes do not contain a great number of details about the tools and techniques in DG algebra. There are already excellent sources available for this, particularly, the seminal works [4, 6, 8]. The interested reader is encouraged to dig into these sources for proofs and constructions not given here. Our goal is to give some idea of what the tools look like and how they get used to solve problems. (To help readers in their digging, we provide many references for properties that we use.)

Exercises. Since these notes are based on a course, it contains many exercises. It also contains a number of examples and facts that are presented without proof. A diligent reader may wish to consider each of these as exercises as well.

Notation. When it is convenient, we use notation from [9, 28]. Here we specify our conventions for some notions that have several notations:
$\mathrm{pd}_R(M)$: projective dimension
$\mathrm{id}_R(M)$: injective dimension
$\mathrm{len}_R(M)$: length
S_n: the symmetric group on $\{1, \ldots, n\}$.

2. Semidualizing Modules

This section contains background material on semidualizing modules. It also contains a special case of Theorem 1.3; see Theorem 2.9.

Definition 2.1. A finitely generated R-module C is *semidualizing* if the natural map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. A *dualizing* R-module is a semidualizing R-module such that $\mathrm{id}_R(C) < \infty$. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$.

Remark 2.2. The symbol $\mathfrak{S}$ is an S, as in `\mathfrak{S}`.

Fact 2.3. The free R-module R^1 is semidualizing. The ring R has a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring; when these conditions are satisfied, a dualizing R-module is the same as a "canonical" R-module.

Remark 2.4. To the best of our knowledge, semidualizing modules were first introduced by Foxby [18]. They have been rediscovered independently by several authors who seem to all use different terminology for them. A few examples of this, presented chronologically, are:

1. Foxby [18]: PG-module of rank 1. Context: commutative algebra.
2. Vasconcelos [33]: spherical module. Context: commutative algebra.
3. Golod [20]: suitable module.[1] Context: commutative algebra.
4. Wakamatsu [35]: generalized tilting module. Context: representation theory.
5. Christensen [13]: semidualizing module. Context: commutative algebra.
6. Mantese and Reiten [27]: Wakamatsu tilting module. Context: representation theory.

Fact 2.5. Assume that R is local, and let C be a semidualizing R-module. If R is Gorenstein, then $C \cong R$. The converse holds if D is a dualizing R-module. If $\mathrm{pd}_R(C) < \infty$, then $C \cong R$.

Fact 2.6. Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. (For example, this is so if φ is flat or surjective with kernel generated by an R-regular sequence.) If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module. The converse holds when φ is faithfully flat or local. The functor $S \otimes_R -$ induces a well-defined function $\mathfrak{S}(R) \to \mathfrak{S}(S)$ which is injective when φ is local.

Exercise 2.7. Prove some of the claims from Facts 2.5–2.6.

Hint for 2.5: The isomorphism $\mathrm{Hom}_R(C,C) \cong R$ implies that $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ and $\mathrm{Ass}_R(C) = \mathrm{Ass}_R(R)$. In particular, an element $x \in \mathfrak{m}$ is C-regular if and only if it is R-regular. Now, argue by induction on $\mathrm{depth}(R)$ to show that $\mathrm{depth}_R(C) = \mathrm{depth}(R)$, and apply the Auslander-Buchsbaum formula.

Suggestion for 2.6: Focus on two cases. Case 1: φ is (faithfully) flat. Case 2: R is local, and φ is surjective with kernel generated by an R-regular sequence. In Case 2, the hint for Fact 2.5 is useful.

Lemma 2.8. *Assume that $(R, \mathfrak{m}, k)$ is local and artinian. Then there is an integer $\rho = \rho(R)$ depending only on R such that for every semidualizing R-module C one has $\mathrm{len}_R(C) \leqslant \rho$.*

Proof. We show that the integer $\rho = \mathrm{len}_R(R)\mu_R^0$ satisfies the conclusion, where μ_R^0 is the 0th Bass number of R; see Definition A.1. Let C be a semidualizing R-module. Set $\beta = \mathrm{rank}_k(k \otimes_R C)$ and $\mu = \mathrm{rank}_k(\mathrm{Hom}_R(k, C))$. Since R is artinian and C is finitely generated, it follows that $\mu \geqslant 1$. Also, the fact that R is local implies that there is an R-module epimorphism $R^\beta \to C$, so we have $\mathrm{len}_R(C) \leqslant \mathrm{len}_R(R)\beta$. Thus it remains to show that $\beta \leqslant \mu_R^0$.

The following sequence of isomorphisms uses Hom-tensor adjointness and tensor cancellation:

$$\begin{aligned} k^{\mu_R^0} &\cong \mathrm{Hom}_R(k, R) \\ &\cong \mathrm{Hom}_R(k, \mathrm{Hom}_R(C, C)) \\ &\cong \mathrm{Hom}_R(C \otimes_R k, C) \\ &\cong \mathrm{Hom}_R(k \otimes_k (C \otimes_R k), C) \\ &\cong \mathrm{Hom}_k(C \otimes_R k, \mathrm{Hom}_R(k, C)) \\ &\cong \mathrm{Hom}_k(k^\beta, k^\mu) \\ &\cong k^{\beta\mu}. \end{aligned}$$

Since $\mu \geqslant 1$, it follows that $\beta \leqslant \beta\mu = \mu_R^0$, as desired. □

[1]Apparently, another translation of the Russian term Golod used is "comfortable" module.

We sketch the proof of Theorem 1.3 when R is Cohen-Macaulay and contains a field. This sketch serves to guide the proof of the result in general.

Theorem 2.9 ([14])**.** *Assume that $(R, \mathfrak{m}, k)$ is Cohen-Macaulay local and contains a field. Then R has only finitely many semidualizing modules up to isomorphism.*

Proof. Case 1: R is artinian, and k is algebraically closed. In this case, Cohen's structure theorem implies that R is a finite dimensional k-algebra. Since k is algebraically closed, a result of Happel [22] (or rather the proof of that result) says that for each $n \in \mathbb{N}$ the following set is finite.

$$T_n = \{\text{isomorphism classes of } R\text{-modules } N \mid \operatorname{Ext}^1_R(N, N) = 0 \text{ and } \operatorname{len}_R(N) = n\}$$

Lemma 2.8 implies that there is a $\rho \in \mathbb{N}$ such that $\mathfrak{S}_0(R)$ is contained in the finite set $\bigcup_{n=1}^{\rho} T_n$, so $\mathfrak{S}_0(R)$ is finite.

Case 2: k is algebraically closed. Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be a maximal R-regular sequence. Since R is Cohen-Macaulay, the quotient $R' = R/(\mathbf{x})$ is artinian. Also, R' has the same residue field as R, so Case 1 implies that $\mathfrak{S}_0(R')$ is finite. Since R is local, Fact 2.6 provides an injective function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$, so $\mathfrak{S}_0(R)$ is finite as well.

Case 3: the general case. Let $\overline{k}$ be an algebraic closure of k. A result of Grothendieck [21, Théorèm 19.8.2(ii)] provides a flat local ring homomorphism $R \to \overline{R}$ such that $\overline{R}/\mathfrak{m}\overline{R} \cong \overline{k}$. In particular, since R and $\overline{R}/\mathfrak{m}\overline{R}$ are Cohen-Macaulay, it follows that $\overline{R}$ is Cohen-Macaulay. The fact that R contains a field implies that $\overline{R}$ also contains a field. Hence, Case 2 shows that $\mathfrak{S}_0(\overline{R})$ is finite. Since R is local, Fact 2.6 provides an injective function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$, so $\mathfrak{S}(R)$ is finite as well. □

Remark 2.10. Happel's result uses some deep ideas from algebraic geometry and representation theory. The essential point comes from a theorem of Voigt [34] (see also Gabriel [19]). We'll need a souped-up version of it for the full proof of Theorem 1.3. This is the point of Section 8.

Remark 2.11. The proof of Theorem 2.9 uses the extra assumptions in crucial places. The Cohen-Macaulay assumption is used in the reduction to the artinian case. And the fact that R contains a field is used in order to invoke Happel's result. In order to remove these assumptions for the proof of Theorem 1.3, we find an algebra U that is finite dimensional over an algebraically closed field such that $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(U)$. The trick is that U is a DG algebra, and $\mathfrak{S}(U)$ is a set of equivalence classes of semidualizing DG U-modules. So, we need to understand

(a) What are DG algebras, and how is U constructed?
(b) What are semidualizing DG modules, and how is the map $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(U)$ constructed?
(c) Why is $\mathfrak{S}(U)$ finite?

This is the point of the rest of the notes. See Sections 5, 7, and 8.

3. Hom Complexes

This section has several purposes. First, we set some notation and terminology. Second, we make sure that the reader is familiar with some notions that we need later in the notes. One of the main points of this section is Fact 3.13.

Complexes.

Definition 3.1. An *R-complex*[2] is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

such that $\partial^X_i \partial^X_{i+1} = 0$ for all i. For each $x \in X_i$, the *degree* of x is $|x| := i$. The *ith homology module* of X is $\mathrm{H}_i(X) := \mathrm{Ker}(\partial^X_i)/\,\mathrm{Im}(\partial^X_{i+1})$.

Example 3.2. Let M be an R-module. We consider M as an R-complex "concentrated in degree 0":

$$M = \quad 0 \to M \to 0.$$

Given an augmented projective resolution

$$P^+ = \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \to 0$$

the truncated resolution

$$P = \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \to 0$$

is an R-complex such that $\mathrm{H}_0(P) \cong M$ and $\mathrm{H}_i(P) = 0$ for all $i \neq 0$.

The Hom Complex.

Definition 3.3. Let X and Y be R-complexes. The *Hom complex* $\mathrm{Hom}_R(X,Y)$ is defined as follows. For each index n, set $\mathrm{Hom}_R(X,Y)_n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X_p, Y_{p+n})$ and $\partial^{\mathrm{Hom}_R(X,Y)}_n(\{f_p\}) := \{\partial^Y_{p+n} f_p - (-1)^n f_{p-1}\partial^X_p\}$. A *chain map* $X \to Y$ is an element of $\mathrm{Ker}(\partial^{\mathrm{Hom}_R(X,Y)}_0)$. An element $\{f_p\} \in \mathrm{Hom}_R(X,Y)_0$ is *null-homotopic* if it is in $\mathrm{Im}(\partial^{\mathrm{Hom}_R(X,Y)}_1)$. An *isomorphism* $X \xrightarrow{\cong} Y$ is a chain map $X \to Y$ with a two-sided inverse. We sometimes write f in place of $\{f_p\}$.

Exercise 3.4. Let X and Y be R-complexes. Prove that $\mathrm{Hom}_R(X,Y)$ is an R-complex. Prove that a chain map $X \to Y$ is a sequence of R-module homomorphisms $\{f_p \colon X_p \to Y_p\}$ making the following diagram commute:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial^X_{i+1}} & X_i & \xrightarrow{\partial^X_i} & X_{i-1} & \xrightarrow{\partial^X_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \xrightarrow{\partial^Y_{i+1}} & Y_i & \xrightarrow{\partial^Y_i} & Y_{i-1} & \xrightarrow{\partial^Y_{i-1}} & \cdots .
\end{array}$$

Prove that if $\{f_p\} \in \mathrm{Hom}_R(X,Y)_0$ is null-homotopic, then it is a chain map. Prove that a sequence $\{f_p\} \in \mathrm{Hom}_R(X,Y)_0$ is null-homotopic if and only if there is a sequence $\{s_p \colon X_p \to Y_{p+1}\}$ of R-module homomorphisms such that $f_p = \partial^Y_{p+1} s_p + s_{p-1}\partial^X_p$ for all $p \in \mathbb{Z}$.

The next isomorphism is called "Hom cancellation".

Exercise 3.5. Let X be an R-complex. Prove that the map $\mathrm{Hom}_R(R,X) \to X$ given by $\{f_p\} \mapsto \{f_p(1)\}$ is an isomorphism of R-complexes.

[2]Readers more comfortable with notations like $X_\bullet$ or X_* for complexes should feel free to decorate their complexes as they see fit.

Exercise 3.6. Let $f\colon X \to Y$ be a chain map. Prove that for each $i \in \mathbb{Z}$, the chain map f induces a well-defined R-module homomorphism $\mathrm{H}_i(f)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ given by $\mathrm{H}_i(f)(\overline{x}) := \overline{f_i(x)}$. Prove that if f is null-homotopic, then $\mathrm{H}_i(f) = 0$ for all $i \in \mathbb{Z}$.

Definition 3.7. A chain map $f\colon X \to Y$ is a *quasiisomorphism* if for all $i \in \mathbb{Z}$ the induced map $\mathrm{H}_i(f)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism. We use the symbol $\simeq$ to identify quasiisomorphisms.

Exercise 3.8. Let X and Y be R-complexes. Prove that an isomorphism $X \to Y$ is a quasiisomorphism.

Exercise 3.9. Let M be an R-module with augmented projective resolution P^+ and augmented injective resolution ${}^+I$. Using the notation from Example 3.2, prove that τ and ϵ induce quasiisomorphisms $P \xrightarrow{\simeq} M \xrightarrow{\simeq} I$.

Remark 3.10. Let M and N be R-modules. The fact that $\mathrm{Ext}^i_R(M,N)$ can be computed using a projective resolution P of M or an injective resolution I of N is called the "balance" property for Ext. It can be proved by showing that there are quasiisomorphisms $\mathrm{Hom}_R(P,N) \xrightarrow{\simeq} \mathrm{Hom}_R(P,I) \xleftarrow{\simeq} \mathrm{Hom}_R(M,I)$.

Homotheties and Semidualizing Modules.

Exercise 3.11. Let X be an R-complex, and let $r \in R$. For each $p \in \mathbb{Z}$, let $\mu^{X,r}_p\colon X_p \to X_p$ be given by $x \mapsto rx$. (Such a map is a "homothety". When it is convenient, we denote this map as $X \xrightarrow{r} X$.)

Prove that $\mu^{X,r} := \{\mu^{X,r}_p\} \in \mathrm{Hom}_R(X,X)_0$ is a chain map. Prove that for all $i \in \mathbb{Z}$ the induced map $\mathrm{H}_i(\mu^{X,r})\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is multiplication by r.

Exercise 3.12. Let X be an R-complex. We use the notation from Exercise 3.11. Define $\chi^X_0\colon R \to \mathrm{Hom}_R(X,X)$ by the formula $\chi^X_0(r) := \{\mu^{X,r}_p\} \in \mathrm{Hom}_R(X,X)_0$. Prove that this determines a chain map $\chi^X\colon R \to \mathrm{Hom}_R(X,X)$. The chain map χ^X is the "homothety morphism" for X.

Fact 3.13. Let M be a finitely generated R-module. We use the notation from Exercise 3.12. The following conditions are equivalent:

(i) M is a semidualizing R-module.
(ii) For each projective resolution P of M, the chain map $\chi^P\colon R \to \mathrm{Hom}_R(P,P)$ is a quasiisomorphism.
(iii) For some projective resolution P of M, the chain map $\chi^P\colon R \to \mathrm{Hom}_R(P,P)$ is a quasiisomorphism.
(iv) For each injective resolution I of M, the chain map $\chi^I\colon R \to \mathrm{Hom}_R(I,I)$ is a quasiisomorphism.
(v) For some injective resolution I of M, the chain map $\chi^I\colon R \to \mathrm{Hom}_R(I,I)$ is a quasiisomorphism.

The point is that the homologies of the complexes $\mathrm{Hom}_R(P,P)$ and $\mathrm{Hom}_R(I,I)$ are exactly the modules $\mathrm{Ext}^i_R(M,M)$.

4. Tensor Products and the Koszul Complex

Tensor Product of Complexes.

Definition 4.1. Let X and Y be R-complexes. The *tensor product complex* $X \otimes_R Y$ is defined as follows. For each index n, set $(X \otimes_R Y)_n := \bigoplus_{p \in \mathbb{Z}} X_p \otimes_R Y_{n-p}$ and let $\partial_n^{X \otimes_R Y}$ be given on generators by the formula $\partial_n^{X \otimes_R Y}(\ldots, 0, x_p \otimes y_{n-p}, 0, \ldots) := (\ldots, 0, \partial_p^X(x_p) \otimes y_{n-p}, (-1)^p x_p \otimes \partial_{n-p}^Y(y_{n-p}), 0, \ldots)$.

Exercise 4.2. Let X, Y, and Z be R-complexes. Prove that $X \otimes_R Y$ is an R-complex. Prove that there are isomorphisms $R \otimes_R X \cong X$ and $X \otimes_R Y \cong Y \otimes_R X$ and $X \otimes_R (Y \otimes_R Z) \cong (X \otimes_R Y) \otimes_R Z$. Hint: Be careful with your signs! For instance the second isomorphism is given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.

Fact 4.3. Given R-complexes $X^1, \ldots, X^n$ an induction argument using the associativity isomorphism from Exercise 4.2 shows that the n-fold tensor product $X^1 \otimes_R \cdots \otimes_R X^n$ is well-defined (up to isomorphism).

The Koszul Complex.

Definition 4.4. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. For $i = 1, \ldots, n$ set

$$K^R(x_i) = \quad 0 \to R \xrightarrow{x_i} R \to 0.$$

Using Remark 4.3, we set

$$K^R(\mathbf{x}) = K^R(x_1, \ldots, x_n) = K^R(x_1) \otimes_R \cdots \otimes_R K^R(x_n).$$

Exercise 4.5. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Write out explicit formulas, using matrices for the differentials, for $K^R(\mathbf{x})$ in the cases $n = 2, 3$.

Exercise 4.6. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Prove that $K^R(\mathbf{x})_i \cong R^{\binom{n}{i}}$ for all $i \in \mathbb{Z}$. (Here we use the convention $\binom{n}{i} = 0$ for all $i < 0$ and $i > n$.)

Exercise 4.7. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Let $\sigma \in S_n$, and set $\mathbf{x}' = x_{\sigma(1)}, \ldots, x_{\sigma(n)}$. Prove that $K^R(\mathbf{x}) \cong K^R(\mathbf{x}')$.

Fact 4.8. Let $\mathbf{x} = x_1, \ldots, x_n \in R$, and consider the ideal $\mathfrak{a} = (\mathbf{x})R$. Then $\mathfrak{a} \operatorname{H}_i(K^R(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$. Sketch of proof: It suffices to show that for $j = 1, \ldots, n$ we have $x_j \operatorname{H}_i(K^R(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$. By symmetry (see Exercise 4.7) it suffices to show that $x_1 \operatorname{H}_i(K^R(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$ for $j = 1, \ldots, n$. Show that the map $K^R(x_1) \xrightarrow{x_1} K^R(x_1)$ is null-homotopic. Conclude that the induced map $K^R(\mathbf{x}) \xrightarrow{x_1} K^R(\mathbf{x})$ is null-homotopic, and apply Exercise 3.6.

Definition 4.9. Let X be an R-complex, and let $n \in \mathbb{Z}$. The *nth suspension* (or *shift*) of X is the complex $\mathbf{\Sigma}^n X$ such that $(\mathbf{\Sigma}^n X)_i := X_{i-n}$ and $\partial_i^{\mathbf{\Sigma}^n X} = (-1)^n \partial_{i-n}^X$. (In this form, this only gets used in Exercise 4.10.)

Fact 4.10. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Prove that $K^R(\mathbf{x})_i$ is "self-dual", that is, that there is an isomorphism of R-complexes $\operatorname{Hom}_R(K^R(\mathbf{x}), R) \cong \mathbf{\Sigma}^n K^R(\mathbf{x})$.

Alternate Description of the Koszul Complex. The alternate description of $K^R(\mathbf{x})$ is quite useful. It says that $K^R(\mathbf{x})$ is given by the "exterior algebra" on R^n.

Definition 4.11. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Fix a basis $e_1, \ldots, e_n \in R^n$. For $i > 1$, set $\bigwedge^i R^n := R^{\binom{n}{i}}$ with basis given by the set of formal symbols $e_{j_1} \wedge \cdots \wedge e_{j_i}$ such that $1 \leqslant j_1 < \cdots < j_i \leqslant n$. This extends to all $i \in \mathbb{Z}$ as follows: $\bigwedge^1 R^n = R^n$ with basis $e_1, \ldots, e_n$ and $\bigwedge^0 R^n = R^1$ with basis 1; for $i < 0$, set $\bigwedge^i R^n = R^{\binom{n}{i}} = 0$.

Define $\widetilde{K}^R(\mathbf{x})$ as follows. For all $i \in \mathbb{Z}$ set $\widetilde{K}^R(\mathbf{x})_i = \bigwedge^i R^n$, and let $\partial_i^{\widetilde{K}^R(\mathbf{x})}$ be given on basis vectors by the formula

$$\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{p=1}^{i} (-1)^{p+1} x_{j_p} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_p}} \cdots \wedge e_{j_i}$$

where the notation $\widehat{e_{j_p}}$ indicates that e_{j_p} has been removed from the list. In the case $i = 1$, the formula reads as $\partial_1^{\widetilde{K}^R(\mathbf{x})}(e_j) = x_j$.

Remark 4.12. Our definition of $\bigwedge^i R^n$ is *ad hoc*. A better way to think about it is in terms of a universal mapping property for alternating multilinear maps. A basis-free construction can be given in terms of a certain quotient of the i-fold tensor product $R^n \otimes_R \cdots \otimes_R R^n$.

Exercise 4.13. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Write out explicit formulas, using matrices for the differentials, for $\widetilde{K}^R(\mathbf{x})$ in the cases $n = 2, 3$.

Fact 4.14. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. There is an isomorphism of R-complexes $K^R(\mathbf{x}) \cong \widetilde{K}^R(\mathbf{x})$. (For perspective on this, compare the answers to Exercises 4.5 and 4.13.)

Remark 4.15. A third description of $K^R(\mathbf{x})$ involves the mapping cone, but I don't think we need it, even though it is extremely useful, so I'm not including it. One consequence of this description is the fact that, if $\mathbf{x}$ is R-regular, then $K^R(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$. That does get used later, but we really don't have the time to prove it.

Algebra Structure on the Koszul Complex. In our estimation, the Koszul complex is one of the most important constructions in commutative algebra. When the sequence $\mathbf{x}$ is R-regular, it is an R-free resolution of $R/(\mathbf{x})$. In general, it detects depth and has all scads of other magical properties. For us, one its most important features is its (DG) algebra structure, which we describe next.

Definition 4.16. Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_n \in R^n$ be a basis. In $\bigwedge^2 R^n$, define

$$e_{j_2} \wedge e_{j_1} := \begin{cases} -e_{j_1} \wedge e_{j_2} & \text{whenever } 1 \leqslant j_1 < j_2 \leqslant n \\ 0 & \text{whenever } 1 \leqslant j_1 = j_2 \leqslant n. \end{cases}$$

Extending this bilinearly, we define $\alpha \wedge \beta$ for all $\alpha, \beta \in \bigwedge^2 R^n$: write $\alpha = \sum_p \alpha_p e_p$ and $\beta = \sum_q \beta_q e_q$, and define

$$\alpha \wedge \beta = \left(\sum_p \alpha_p e_p\right) \wedge \left(\sum_q \beta_q e_q\right) = \sum_{p,q} \alpha_p \beta_q e_p \wedge e_q = \sum_{p<q} (\alpha_p \beta_q - \alpha_q \beta_p) e_p \wedge e_q.$$

This extends (by induction on $i+j$) to a multiplication $\bigwedge^i R^n \times \bigwedge^j R^n \to \bigwedge^{i+j} R^n$ which we denote as $(\alpha, \beta) \mapsto \alpha \wedge \beta$. (When $i = 0$ this is defined as the usual scalar multiplication $\bigwedge^0 R^n \times \bigwedge^j R^n \to \bigwedge^j R^n$, and similarly when $j = 0$.) This further extends to a well-defined multiplication on $\bigwedge R^n := \bigoplus_i \bigwedge^i R^n$.

Exercise 4.17. Write out multiplication tables (for basis vectors only) for $\bigwedge R^n$ with $n = 1, 2, 3, 4$.

Exercise 4.18. Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_n \in R^n$ be a basis. Prove that for any sequence $j_1, \ldots, j_i \in \{1, \ldots, n\}$ the element $e_{j_1} \wedge \cdots \wedge e_{j_i} \in \bigwedge^i R^n$ is well-defined. Specifically, prove that

$$e_{j_1} \wedge \cdots \wedge e_{j_i} = \begin{cases} 0 & \text{if } j_p = j_q \text{ for some } p \neq q \\ (-1)^{\operatorname{sgn}(\iota)} e_{\iota(j_1)} \wedge \cdots \wedge e_{\iota(j_i)} & \text{if } \iota \in S_n \text{ where } \iota(j_1) < \cdots < \iota(j_i). \end{cases}$$

Exercise 4.19. Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_n \in R^n$ be a basis. Prove that the multiplication from Definition 4.16 makes $\bigwedge R^n$ into a graded commutative R-algebra. That is:

(a) multiplication in $\bigwedge R^n$ is associative, distributive, and unital;
(b) for elements $\alpha \in \bigwedge^i R^n$ and $\beta \in \bigwedge^j R^n$, we have $\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$;
(c) for $\alpha \in \bigwedge^i R^n$, if i is odd, then $\alpha \wedge \alpha = 0$; and
(d) the composition $R \xrightarrow{\cong} \bigwedge^0 R^n \xrightarrow{\subseteq} \bigwedge R^n$ is a ring homomorphism whose image is contained in the center of $\bigwedge R^n$.

Hint: The distributive law holds essentially by definition. For the other properties in (a) and (b), prove the desired formula for basis vectors, then verify it for general elements using distributivity.

Exercise 4.20. Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_n \in R^n$ be a basis. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Prove that the multiplication from Definition 4.16 satisfies the "Leibniz rule": for elements $\alpha \in \bigwedge^i R^n$ and $\beta \in \bigwedge^j R^n$, we have

$$\partial_{i+j}^{\widetilde{K}^R(\mathbf{x})}(\alpha \wedge \beta) = \partial_i^{\widetilde{K}^R(\mathbf{x})}(\alpha) \wedge \beta + (-1)^i \alpha \wedge \partial_j^{\widetilde{K}^R(\mathbf{x})}(\beta).$$

Hint: Prove the formula for basis vectors and verify it for general elements using distributivity and linearity.

Tensor Products and Chain Maps (Functoriality).

Definition 4.21. Given a chain map $f \colon X \to Y$ and an R-complex Z, define $Z \otimes_R f \colon Z \otimes_R X \to Z \otimes_R Y$ by the formula $z \otimes y \mapsto z \otimes f(y)$. Define the map $f \otimes_R Z \colon X \otimes_R Z \to Y \otimes_R Z$ similarly.

Exercise 4.22. Given a chain map $f \colon X \to Y$ and an R-complex Z, the maps $Z \otimes_R f \colon Z \otimes_R X \to Z \otimes_R Y$ and $f \otimes_R Z \colon X \otimes_R Z \to Y \otimes_R Z$ are chain maps.

Fact 4.23. Let $f \colon X \xrightarrow{\simeq} Y$ be a quasiisomorphism, and let Z be an R-complex. In general, the chain map $Z \otimes_R f \colon Z \otimes_R X \to Z \otimes_R Y$ is not a quasiisomorphism. However, if Z is a complex of projective R-modules such that $Z_i = 0$ for $i \ll 0$, then $Z \otimes_R f$ is a quasiisomorphism.

5. DG Algebras and DG Modules I

Here we introduce the framework for the proof of Theorem 1.3.

DG Algebras.

Definition 5.1. A *commutative differential graded algebra over R* (*DG R-algebra* for short) is an R-complex A equipped with a binary operation $(a, b) \mapsto ab$ (the *product* on A) satisfying the following properties:[3]

[3]We assume that readers of notes at this level are familiar with associative laws and the like. However, given that the DG universe is riddled with sign conventions, we explicitly state these laws for the sake of clarity.

associative: for all $a, b, c \in A$ we have $(ab)c = a(bc)$;
distributive: for all $a, b, c \in A$ such that $|a| = |b|$ we have $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$);
unital: there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
graded commutative: for all $a, b \in A$ we have $ba = (-1)^{|a||b|}ab \in A_{|a|+|b|}$, and $a^2 = 0$ when $|a|$ is odd;
positively graded: $A_i = 0$ for $i < 0$; and
Leibniz Rule: for all $a, b \in A$ we have $\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b)$.

Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i=0}^{\infty} A_i$. When R is a field and $\operatorname{rank}_R(\oplus_{i\geqslant 0} A_i) < \infty$, we say that A is *finite-dimensional* over R.

Example 5.2. The ring R, considered as a complex concentrated in degree 0, is a DG R-algebra such that $R^\natural = R$.

Example 5.3. Given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, the Koszul complex $K = K^R(\mathbf{x})$ is a DG R-algebra such that $K^\natural = \bigwedge R^n$; see Exercises 4.19 and 4.20. In particular, if $n = 1$, then $K^\natural \cong R[X]/(X^2)$.

Exercise 5.4. Let A be a DG R-algebra. Prove that there is a well-defined chain map $\mu^A\colon A \otimes_R A \to A$ given by $\mu^A(a \otimes b) = ab$, and that A_0 is an R-algebra.

Definition 5.5. A *morphism* of DG R-algebras is a chain map $f\colon A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$.

Exercise 5.6. Let A be a DG R-algebra. Prove that the map $R \to A$ given by $r \mapsto r \cdot 1$ is a morphism of DG R-algebras. Prove that the natural inclusion map $A_0 \to A$ is a morphism of DG R-algebras. As a special case, given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, the natural map $R \to K^R(\mathbf{x})$ given by $r \mapsto r \cdot 1$ is a morphism of DG R-algebras.

Example 5.7. Let A be a DG R-algebra. Prove that the condition $A_{-1} = 0$ implies that A_0 surjects onto $\mathrm{H}_0(A)$ and that $\mathrm{H}_0(A)$ is an A_0-algebra. Prove that the R-module A_i is an A_0-module, and $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 5.8. Let A be a DG R-algebra. We say that A is *noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$.

Exercise 5.9. Given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, prove that the Koszul complex $K^R(\mathbf{x})$ is a noetherian DG R-algebra. Moreover, prove that any DG R-algebra A such that each A_i is finitely generated over R is noetherian.

DG Modules. In the passage from rings to DG algebras, modules change to DG modules, which we describe next.

Definition 5.10. Let A be a DG R-algebra. A *differential graded module over A* (*DG A-module* for short) is an R-complex M equipped with an operation $A \times M \to M$, written as $(a, m) \mapsto am$ and called the *scalar multiplication* of A on M, satisfying the following properties:

associative: for all $a, b \in A$ and $m \in M$ we have $(ab)m = a(bm)$;
distributive: for all $a, b \in A$ and $m, n \in M$ such that $|a| = |b|$ and $|m| = |n|$, we have $(a + b)m = am + bm$ and $a(m + n) = am + an$);

unital: for all $m \in M$ we have $1m = m$;
graded: for all $a \in A$ and $m \in M$ we have $am \in M_{|a|+|m|}$;
Leibniz Rule: for all $a \in A$ and $m \in M$ we have $\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$.

The *underlying* $A^\natural$*-module* associated to M is the $A^\natural$-module $M^\natural = \oplus_{i=-\infty}^{\infty} M_i$.

Exercise 5.11. Prove that DG R-module is just an R-complex. Given a DG R-algebra A, prove that the complex A is a DG A-module where the scalar multiplication is just the internal multiplication on A.

Exercise 5.12. Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$. Given an R-module M, prove that the complex $K \otimes_R M$ is a DG K-module via the multiplication $a(b \otimes m) := (ab) \otimes m$. More generally, given an R-complex X and a DG R-algebra A, prove that the complex $A \otimes_R X$ is a DG A-module via the multiplication $a(b \otimes x) := (ab) \otimes x$.

Prove that the quotient $R/(\mathbf{x})$ is a DG K-module (concentrated in degree 0) by the natural action of $K_0 = R$ in degree 0 and such that $K_i R/(\mathbf{x}) = 0$ for $i \neq 0$. (One can check this directly, or use the fact that the subcomplex

$$I \quad 0 \to K_n \xrightarrow{\partial^K_n} \cdots \xrightarrow{\partial^K_2} K_1 \xrightarrow{(x_1 \ldots x_n)} (\mathbf{x}) \to 0$$

is a "DG ideal" of K such that $K/I \cong R/(\mathbf{x})$.) Prove that the natural map $K \to R/(\mathbf{x})$ is a morphism of DG R-algebras that is a quasiisomorphism if $\mathbf{x}$ is R-regular; see Remark 4.15.

Exercise 5.13. Let A be a DG R-algebra, and let M be a DG A-module. Prove that there is a well-defined chain map $\mu^M : A \otimes_R M \to M$ given by $\mu^M(a \otimes m) = am$.

We consider the following example throughout these notes. It is simple but demonstrates our constructions. And even it has some non-trivial surprises.

Example 5.14. We consider the trivial Koszul complex $U = K^R(0)$:

$$U = \quad 0 \to Re \xrightarrow{0} R1 \to 0.$$

The notation indicates that we are using the basis $e \in U_1$ and $1 \in U_0$.

Exercise 5.12 shows that R is a DG U-module. Another example is the following, again with specified basis in each degree:

$$G = \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0.$$

The notation for the bases is chosen to help remember the DG U-module structure:

$$1 \cdot 1_{2n} = 1_{2n} \qquad\qquad 1 \cdot e_{2n+1} = e_{2n+1}$$
$$e \cdot 1_{2n} = e_{2n+1} \qquad\qquad e \cdot e_{2n+1} = 0.$$

One checks directly that G satisfies the axioms to be a DG U-module. It is worth noting that $\mathrm{H}_0(G) \cong R$ and $\mathrm{H}_i(G) = 0$ for all $i \neq 0$. (For perspective, G is modeled on the free resolution $\cdots \xrightarrow{e} R[e]/(e^2) \xrightarrow{e} R[e]/(e^2) \to 0$ of R over $R[e]/(e^2)$.)

We continue with Example 5.14, but working over a field F instead of R.

Example 5.15. We consider the trivial Koszul complex $U = K^F(0)$:

$$U = \quad 0 \to Fe \xrightarrow{0} F1 \to 0.$$

For W, we use $W_0 = F\eta_0 \cong F$ with basis η_0 and $W_i = 0$ for $i \neq 0$:

$$W = \quad 0 \oplus F\nu_0 \oplus 0.$$

For this vector space, we have no choice for the differential since it maps $W_i \to W_{i-1}$ and at least one of these modules is 0: $\partial_i = 0$ for all i. Also, we have no choice for the scalar multiplication: multiplication by 1 must be the identity, and multiplication by e maps $W_i \to W_{i+1}$ and at least one of these modules is 0. (this example is trivial, but it will be helpful later.)

Similarly, we consider

$$W' = \quad 0 \oplus F\eta_1 \oplus F\eta_0 \oplus 0.$$

This vector space allows for one non-trivial differential

$$\partial_1' \in \operatorname{Hom}_F(F\eta_1, F\eta_0) \cong F.$$

For the scalar multiplication, again multiplication by 1 must be the identity, but multiplication by e has one nontrivial option which we write as

$$\mu_0' \in \operatorname{Hom}_F(F\eta_0, F\eta_1) \cong F.$$

In other words, we have elements $x_1, x_0 \in F$ such that

$$\partial_1'(\eta_1) = x_1\eta_0 \qquad \text{and} \qquad e\eta_0 = x_0\eta_1.$$

For the Leibniz Rule to be satisfied, we must have

$$\partial_{i+1}'(e \cdot \eta_i) = \partial_1^U(e) \cdot \eta_i + (-1)^{|e|} e \cdot \partial_i'(\eta_i)$$

for $i = 0, 1$. We begin with $i = 0$:

$$\begin{aligned}
\partial_1'(e \cdot \eta_0) &= \partial_1^U(e) \cdot \eta_0 + (-1)^{|e|} e \cdot \partial_0'(\eta_0) \\
\partial_1'(x_0\eta_1) &= 0 \cdot \eta_0 - e \cdot 0 \\
x_0\partial_1'(\eta_1) &= 0 \\
x_0x_1\eta_0 &= 0
\end{aligned}$$

so we have $x_0x_1 = 0$, that is, either $x_0 = 0$ or $x_1 = 0$. For $i = 1$, we have

$$\begin{aligned}
\partial_2'(e \cdot \eta_1) &= \partial_1^U(e) \cdot \eta_1 + (-1)^{|e|} e \cdot \partial_1'(\eta_1) \\
0 &= 0 \cdot \eta_1 - e \cdot (x_1\eta_0) \\
0 &= -x_1 e \cdot \eta_0 \\
0 &= -x_1x_0\eta_1
\end{aligned}$$

so we again conclude that $x_0 = 0$ or $x_1 = 0$.

Definition 5.16. Let A be a DG R-algebra, and let i be an integer. The ith *suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$. The notation ΣM is short for $\Sigma^1 M$.

Definition 5.17. Let A be a DG R-algebra. A DG A-module M is *bounded below* if $M_n = 0$ for all $n \ll 0$; and it is *homologically finite* if each $\mathrm{H}_0(A)$-module $\mathrm{H}_n(M)$ is finitely generated and $\mathrm{H}_n(M) = 0$ for $|n| \gg 0$.

Example 5.18. In Exercise 5.12, the DG K-module $R/(\mathbf{x})$ is bounded below and homologically finite. In Example 5.14, the DG U-module G is bounded below and homologically finite.

Morphisms of DG Modules.

Definition 5.19. A *morphism* of DG A-modules is a chain map $f\colon M \to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. A *quasiisomorphism* is a morphism $M \to N$ such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism; these are identified by the symbol $\simeq$.

Exercise 5.20. Prove that a morphism of DG R-modules is simply a chain map. Prove that a quasiisomorphism of DG R-modules is simply a quasiisomorphism in the sense of Definition 3.7.

Exercise 5.21. Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$. Given an R-module homomorphism $f\colon M \to N$, prove that the chain map $K \otimes_R f\colon K \otimes_R M \to K \otimes_R N$ is a morphism of DG K-modules. More generally, given a chain map of R-complexes $g\colon X \to Y$ and a DG R-algebra A, prove that the chain map $A \otimes_R g\colon A \otimes_R X \to A \otimes_R Y$ is a morphism of DG A-modules.

Give an example showing that if g is a quasiisomorphism, then $A \otimes_R g$ need not be a quasiisomorphism. However, note that if A_i is is R-projective for each i (e.g., if A is a Koszul complex over R), then g being a quasiisomorphism implies that $A \otimes_R g$ is a quasiisomorphism by Fact 4.23.

Prove that the natural map $K \to R/(\mathbf{x})$ is a morphism of DG K-modules.

Example 5.22. We continue with the notation of Example 5.14.

Let $f\colon G \to \Sigma R$ be a morphism of DG U-modules:

$$\begin{array}{cccccccccccc}
G = & & \cdots \xrightarrow{1} & Re_3 & \xrightarrow{0} & R1_2 & \xrightarrow{1} & Re_1 & \xrightarrow{0} & R1_0 & \longrightarrow & 0 \\
{\scriptstyle f}\downarrow & & & & & \downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow & & \\
\Sigma R & & & & & 0 & \longrightarrow & R & \longrightarrow & 0. & &
\end{array}$$

Commutativity of the first square shows that $f = 0$. One can also see this from the following computation:

$$f_1(e_1) = f_1(e \cdot 1_0) = ef_0(1_0) = 0.$$

The same conclusion holds for any morphism $f\colon G \to \Sigma^{2n+1} R$.

On the other hand, every element $r \in R$ determines a morphism $g^r\colon G \to \Sigma^{2n} R$, via multiplication. For instance in the case $n = 1$:

$$\begin{array}{cccccccccccc}
G = & & \cdots \xrightarrow{1} & Re_3 & \xrightarrow{0} & R1_2 & \xrightarrow{1} & Re_1 & \xrightarrow{0} & R1_0 & \longrightarrow & 0 \\
{\scriptstyle g^r}\downarrow & & & \downarrow & & {\scriptstyle g_2^r}\downarrow & & \downarrow & & & & \\
\Sigma^2 R & & & 0 & \longrightarrow & R & \longrightarrow & 0. & & & &
\end{array}$$

Each square commutes, and the linearity condition is from the next computations:

$$\begin{aligned}
g_2^r(1 \cdot 1_2) &= g_2^r(1_2) = r = 1 \cdot r = 1 \cdot g_2^r(1_2) \\
g_2^r(e \cdot e_1) &= g_2^r(0) = 0 = e \cdot 0 = e \cdot g_1^r(e_1) \\
g_3^r(e \cdot 1_2) &= g_3^r(e_3) = 0 = e \cdot r = e \cdot g_2^r(1_2).
\end{aligned}$$

One checks readily that the natural map $G \xrightarrow{g^1} R$ is a quasiisomorphism.

Fact 5.23. Let $Q \to R$ be a ring epimorphism. Then there is a quasiisomorphism $A \xrightarrow{\simeq} R$ of DG Q-algebras such that each A_i is finitely generated and projective over R and $A_i = 0$ for $i > \operatorname{pd}_Q(R)$. See, e.g., [4, Proposition 2.2.8].

Definition 5.24. In Fact 5.23, the quasiisomorphism $A \xrightarrow{\simeq} R$ is a *DG algebra resolution* of R over Q.

Remark 5.25. When $\mathbf{y} \in Q$ is a Q-regular sequence, the Koszul complex $K^Q(\mathbf{y})$ is a DG algebra resolution of $Q/(\mathbf{y})$ over Q. Section 6 contains other famous examples.

5.26 (First part of the Proof of Theorem 1.3)**.** As in the proof of Theorem 2.9, there is a flat local ring homomorphism $R \to R'$ such that R' is complete with algebraically closed residue field. Since $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$ by Fact 2.6, we can replace R with R' and assume without loss of generality that R is complete with algebraically closed residue field.

Since R is complete and local, Cohen's structure theorem provides a ring epimorphism $\tau\colon (Q, \mathfrak{n}, k) \to (R, \mathfrak{m}, k)$ where Q is a complete regular local ring such that $\mathfrak{m}$ and $\mathfrak{n}$ have the same minimal number of generators. Let $\mathbf{y} = y_1, \ldots, y_n \in \mathfrak{n}$ be a minimal generating sequence for $\mathfrak{n}$, and set $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ where $x_i := \tau(y_i)$. It follows that we have $K^R(\mathbf{x}) \cong R \otimes_Q K^Q(\mathbf{y})$. Since Q is regular and $\mathbf{y}$ is a minimal generating sequence for $\mathfrak{n}$, the Koszul complex $K^Q(\mathbf{y})$ is a minimal Q-free resolution of k.

Fact 5.23 provides a quasiisomorphism $A \xrightarrow{\simeq} R$ of DG Q-algebras such that each A_i is finitely generated and projective over R and $A_i = 0$ for $i > \operatorname{pd}_Q(R)$. Note that $\operatorname{pd}_Q(R) < \infty$ since Q is regular. We consider the following diagram of morphisms of DG Q-algebras:

$$R \to K^R(\mathbf{x}) \cong K^Q(\mathbf{y}) \otimes_Q R \xleftarrow{\simeq} K^Q(\mathbf{y}) \otimes_Q A \xrightarrow{\simeq} k \otimes_Q A. \tag{5.26.1}$$

The first map is from Exercise 5.6. The isomorphism is from the previous paragraph. The first quasiisomorphism comes from an application of $K^Q(\mathbf{y}) \otimes_Q -$ to the quasiisomorphism $R \xleftarrow{\simeq} A$, using Fact 4.23. The second quasiisomorphism comes from an application of $- \otimes_Q A$ to the quasiisomorphism $K^Q(\mathbf{y}) \xrightarrow{\simeq} k$. Note that $k \otimes_Q A$ is a finite dimensional DG k-algebra because of the assumptions on A. (This is the finite dimensional DG algebra B described in Remark 2.11.)

We show in 7.21 below how this provides a diagram

$$\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(K^R(\mathbf{x})) \xleftarrow{\cong} \mathfrak{S}(K^Q(\mathbf{y}) \otimes_Q A) \xrightarrow{\cong} \mathfrak{S}(k \otimes_Q A) \tag{5.26.2}$$

where $\cong$ identifies bijections of sets. We then show in 8.16 that $\mathfrak{S}(k \otimes_Q A)$ is finite, and it follows that $\mathfrak{S}_0(R)$ is finite, as desired. (In the notation of Remark 2.11, we have $U = k \otimes_Q A$.)

6. Examples of Algebra Resolutions

In this section we provide two classes of examples which illustrate, at least partially, the existence of DG algebra resolutions defined in Fact 5.23. It should be noted that Fact 5.23 does assume anything about the minimality of the resolution. Indeed, we shall later see by way of counter-example that this condition is, in fact, too strong.

The simplest examples of (non-trivial) DG algebra resolutions are those of length one. The next fact explains that such resolutions occur only in the presence of a non-zerodivisor.

Fact 6.1. Let $x \in R$ be a non-zerodivisor. Then the deleted free resolution

$$0 \to R \xrightarrow{x} R \to 0$$

of $R/(x)$ is precisely the Koszul complex on x, and it therefore possesses the structure of a DG R-algebra. Indeed, this structure is unique, and is guaranteed by the condition $e^2 = 0$, where

$$0 \to Re \xrightarrow{x} R1 \to 0.$$

More generally, if I is generated by a regular sequence $\mathbf{x}$ in R, then the Koszul complex on $\mathbf{x}$ is a free resolution of R/I, and clearly possesses a DG algebra structure. However, what can be said in the case that I is not necessarily generated by a regular sequence? We consider this question for resolutions of lengths two and three next.

Notation 6.2. Let A be a matrix and $J, K \subset \mathbb{N}$. The submatrix of A obtained by deleting columns indexed by J and rows indexed by K is denoted A_K^J.

The next result is well-known as the Hilbert-Burch Theorem. It was first proven by Hilbert in 1890 in the case that R is a polynomial ring [24]; the more general statement (the one given below) was proven by Burch in 1968 [12].

Theorem 6.3. [12, 24] *Let I be an ideal of R, and suppose that a deleted free resolution of R/I is given by*

$$0 \to R^n \xrightarrow{\beta} R^{n+1} \xrightarrow{\alpha} R \to 0.$$

If, upon fixing bases, A and B are matrix representations of α and β, respectively, then there exists some non-zerodivisor $a \in R$ such that the ith column of A is given by $(-1)^{i-1} a \det(B_i)$. In particular, I is a multiple of the ideal generated by the $n \times n$ minors of B.

Fact 6.4. The Hilbert-Burch Theorem actually provides a structure theorem for perfect ideals of grade two over a local ring. Indeed, if I is such an ideal over some local ring R, then the minimal free resolution of R/I has the form prescribed in Theorem 6.3.

Exercise 6.5. A converse statement to Theorem 6.3 provides a method for "cooking up" such resolutions: *Given $n \in \mathbb{N}$, suppose that B is an $(n+1) \times n$ matrix with entries in R such that the ideal $\mathcal{I}(B)$ of $n \times n$ minors of B has grade at least two on R. If $a \in R$ is a non-zerodivisor, then the $1 \times (n+1)$ matrix A defined in Theorem 6.3 makes*

$$0 \to R^n \xrightarrow{B} R^{n+1} \xrightarrow{A} R \to 0$$

into a deleted free resolution of $R/a\mathcal{I}(B)$.

Using this fact, construct an ideal of $R = k[x, y]$ (for some field k) with a free resolution characterized by Theorem 6.3.

In [23], Herzog showed that a resolution characterized by Theorem 6.3 admits the structure of a DG algebra.

Theorem 6.6. [23] *Given an $(n+1) \times n$ matrix B with entries in R, suppose that A is a $1 \times (n+1)$ matrix whose ith column is given by $(-1)^{i-1} \det(B_i)$. Then the graded R-complex*

$$F: \quad 0 \to \bigoplus_{\ell=1}^{n} Rf_\ell \xrightarrow{B} \bigoplus_{\ell=1}^{n+1} Re_\ell \xrightarrow{A} R1 \to 0$$

has the structure of a DG R-algebra whenever the following conditions are satisfied:

(1) $e_i^2 = 0$ *for all* $1 \leqslant i \leqslant n+1$, *and*

(2) $e_i e_j = -e_j e_i = -a \sum_{k=1}^{n} (-1)^{i+j+k} \det(B_{ij}^k) f_k$ *for all* $1 \leqslant i \neq j \leqslant n+1$.

Exercise 6.7. Check that the conditions specified in Theorem 6.6 give F the structure of a DG R-algebra.

In order to consider the case for resolutions of length three, we must first consider the following background material.

Definition 6.8. Let $\varphi : F^* \to F$ be a homomorphism of finitely generated free R-modules. Then φ is said to be *alternating* if $\varphi^* = -\varphi$ (ie. φ is *skew-symmetric*) and $x\varphi(x) = 0$ for all $x \in F^*$.

Exercise 6.9. [17] Show that there is a one-to-one correspondence between elements of $\bigwedge^2 F$ and alternating homomorphisms $F^* \to F$ given by the mapping $\alpha \mapsto (\varphi_\alpha : x \mapsto x(\alpha))$ for all $\alpha \in \bigwedge^2 F$.

Fact 6.10. Suppose that $\varphi : F^* \to F$ is an alternating map of free R-modules. If $\mathcal{E} = \{e_1, \ldots, e_n\}$ is a basis for F, and $\mathcal{E}^* = \{e_1^*, \ldots, e_n^*\}$ is the corresponding dual basis for F^*, then the matrix representing φ in the bases $\mathcal{E}$ and $\mathcal{E}^*$ is alternating; that is, it is skew-symmetric with zeros along its diagonal.

Definition 6.11. The *Pfaffian* of an alternating matrix A, denoted $\operatorname{Pf}(A)$, is defined by $\operatorname{Pf}(A)^2 = \det(A)$. If A is an $n \times n$ matrix, then the $(n-1)$*-st order Pfaffians* of A are given by the set of all $\operatorname{Pf}(A_i^i)$, where $1 \leqslant i \leqslant n$.

Definition 6.12. Let $\varphi : F^* \to F$ be an alternating map of free R-modules. If A is the matrix representation of φ with respect to some bases $\mathcal{E}$ and $\mathcal{E}^*$ (the corresponding dual basis) of F and F^*, respectively, then we define the *Pfaffian* of φ by $\operatorname{Pf}(\varphi) := \operatorname{Pf}(A)$. If $\operatorname{rank} F = n$, then the ideal generated by the $(n-1)$-st order Pfaffians of A is denoted $\operatorname{Pf}_{n-1}(\varphi)$.

Fact 6.13. The Pfaffian of an alternating map of finitely generated free modules is well-defined.

Exercise 6.14. Let $R = k[x_1, \ldots, x_{2n+1}]$ for some field k and some $n \geqslant 1$. Suppose that $A_n = (a_{ij})$ is an alternating matrix representation of $\varphi : R^{n+2} \to R^{n+2}$ defined by $a_{ij} = x_{i+j-2}$ for all $2 \leqslant i < j \leqslant n+2$. Find a minimal generating set for $\operatorname{Pf}_2(\varphi)$.

Exercise 6.15. Show that the Pfaffian of an alternating map $\varphi : F^* \to F$ is only non-vanishing whenever $\operatorname{rank} F$ is even.

In [10], Buchsbaum and Eisenbud study the structure of resolutions of length three. Here, the authors both give a characterization of the resolution and exhibit its DG structure.

Theorem 6.16. [10] *Let* $(R, \mathfrak{m})$ *be a local ring, and suppose that* I *is a grade three ideal of* R. *Then* I *is Gorenstein if and only if, for some odd* $n \geqslant 3$, *there exists an alternating map* $\alpha : R^n \to R^n$, *whose image is contained in* $\mathfrak{m}R^n$, *such that* $I = \operatorname{Pf}_{n-1}(\alpha)$. *In this case, the deleted minimal free resolution of* R/I *will take the form*

$$0 \to R \xrightarrow{\beta^*} R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} R \to 0$$

where, if A is an alternating matrix representation of α with respect to the basis $\mathcal{E} = \{e_1, \dots, e_n\}$ of R^n, then $\beta(e_i) = (-1)^{i-1}\operatorname{Pf}(A_i^i)$ for each $1 \leqslant i \leqslant n$.

Remark 6.17. Since the Pfaffian of an $(n-1)\times(n-1)$ matrix is non-vanishing only when n is odd, Theorem 6.16 implies that the minimal number of generators of a grade three Gorenstein ideal must be odd. The next two examples of Buchsbaum and Eisenbud [11] illustrate that there exists, for any odd $n \geqslant 3$, a grade three Gorenstein ideal which is n-generated.

Example 6.18. [11] Let k be a commutative ring and define $G_n(k)$ to be a generic $(2n+1)\times(2n+1)$ alternating matrix with entries belonging to the ring $R_n(k) := k\left[\{x_{i,j}\}_{1\leqslant i<j\leqslant 2n+1}\right]$. That is:

$$G_n(k) = \begin{bmatrix} 0 & x_{1,2} & x_{1,3} & \cdots & x_{1,2n+1} \\ -x_{1,2} & 0 & x_{2,3} & \cdots & x_{2,2n+1} \\ -x_{1,3} & -x_{2,3} & 0 & & \\ \vdots & \vdots & & \ddots & \\ -x_{1,2n+1} & -x_{2,2n+1} & & & 0 \end{bmatrix}$$

Then, for every $n \geqslant 1$, $\operatorname{Pf}_{2n}(G_n(k))$ is a Gorenstein ideal of height three over $R_n(k)$.

Example 6.19. [11] Let $R = k[\![x,y,z]\!]$ for some field k. For each $n \geqslant 3$ which is odd, define H_n to be an $n \times n$ alternating matrix whose entries above the diagonal are given by the following.

$$(H_n)_{ij} = \begin{cases} x & \text{if } i \text{ is odd and } j = i+1 \\ y & \text{if } i \text{ is even and } j = i+1 \\ z & \text{if } j = n-i+1 \end{cases}$$

Then $\operatorname{Pf}_{n-1}(H_n)$ is an n-generated Gorenstein ideal of height three over R.

Theorem 6.20. [10] *Let A be an $n \times n$ alternating matrix with entries in R, and also suppose that B is an $1 \times n$ matrix whose ith entry is given by $(-1)^{i-1}\operatorname{Pf}(A_i^i)$. Then the graded R-complex*

$$F: \quad 0 \to Rg \xrightarrow{B^{\intercal}} \bigoplus_{\ell=1}^{n} Rf_\ell \xrightarrow{A} \bigoplus_{\ell=1}^{n} Re_\ell \xrightarrow{B} R1 \to 0$$

admits the structure of a DG R-algebra, given that the following conditions are satisfied:

(1) $e_ie_i = 0$ and $e_if_j = f_je_i = \delta_{ij}g$ for all $1 \leqslant i,j \leqslant n$, and

(2) $e_ie_j = -e_je_i = \sum_{k=1}^{n}(-1)^{i+j+k}\rho_{ijk}\operatorname{Pf}(A_{ijk}^{ijk})f_k$ for all $1 \leqslant i \neq j \leqslant n$, where $\rho_{ijk} = -1$ whenever $i < k < j$, and $\rho_{ijk} = 1$ otherwise.

Exercise 6.21. Show that the conditions specified in Theorem 6.20 give F the structure of a DG R-algebra.

Buchsbaum and Eisenbud actually showed a bit more than this in [10]. They proved that a projective resolution of length at most three can always be given the structure of a DG algebra.

Proposition 6.22. [10] *A resolution of R-modules of the form*

$$0 \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \to R \to 0$$

where each P_i is projective, has the structure of a DG R-algebra.

Resolutions of length greater than three are not guaranteed to possess such a structure. However, in [26] Kustin and Miller demonstrate that this restriction can be slightly loosened in the presence of a Gorenstein ideal.

Example 6.23. Suppose that I is a Gorenstein ideal of a local ring R, and set $Q = R/I$. If $\operatorname{pd}_R(Q) = 4$, then the minimal R-free resolution of Q has the structure of a DG R-algebra.

Still, Avramov illustrates a Cohen-Macaulay residue ring of projective dimension four whose minimal free resolution does not possess the desired structure.

Example 6.24. [4] Let k be a field, and consider the local ring $R = k[\![w, x, y, z]\!]$. There exists no DG R-algebra structure on the minimal R-free resolution of the Cohen-Macaulay residue ring $Q = R/(wy^6, x^7, x^6z, y^7)$.

7. DG Algebras and DG Modules II

Here we continue to introduce the framework for the proof of Theorem 1.3.

Convention. Throughout this section, A is a DG R-algebra, and L, M, and N are DG A-modules.

Hom and Tensor Product for DG Modules.

Definition 7.1. Given an integer i, a *DG A-module homomorphism of degree i* is an element $f \in \operatorname{Hom}_R(M, N)_i$ such that $f(am) = (-1)^{i|a|}af(m)$ for all $a \in A$ and $m \in M$. The graded submodule of $\operatorname{Hom}_R(M, N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\operatorname{Hom}_A(M, N)$.

Exercise 7.2. Prove that the complex $\operatorname{Hom}_A(M, N)$ is a DG A-module via the action

$$(af)(m) := a(f(m)) = (-1)^{|a||f|}f(am).$$

Prove that for each $a \in A$ the multiplication map $\mu^{M,a} \colon M \to M$ given by $m \mapsto am$ is a homomorphism of degree $|a|$.

Example 7.3. We continue with the notation of Example 5.14. From the computations from Example 5.22, it follows that $\operatorname{Hom}_U(G, R)$ has the form

$$\operatorname{Hom}_U(G, R) = \quad 0 \to R \to 0 \to R \to 0 \to \cdots$$

where the copies of R are in even non-positive degrees. Due to degree considerations, multiplication by e is 0 on $\operatorname{Hom}_U(G, R)$, and multiplication by 1 is the identity because it must be.

Definition 7.4. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is the subcomplex generated by all elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Exercise 7.5. Prove that the tensor product $M \otimes_A N$ is a DG A-module via the scalar multiplication

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an).$$

Exercise 7.6. Let $A \to B$ be a morphism of DG R-algebras. The "base changed" complex $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG A-module structure on $B \otimes_A M$ via restriction of scalars.

Given $f \in \operatorname{Hom}_A(M, N)_i$, define $B \otimes_A f \in \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)_i$ by the formula $(B \otimes_A f)(b \otimes m) := (-1)^{i|b|} b \otimes f(m)$. This yields a morphism of DG A-modules $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$ given by $f \mapsto B \otimes_A f$.

Exercise 7.7. Verify the following isomorphisms of DG A-modules:

$$\begin{aligned} \operatorname{Hom}_A(A, L) &\cong L && \text{Hom cancellation} \\ A \otimes_A L &\cong L && \text{tensor cancellation} \\ L \otimes_A M &\cong M \otimes_A L && \text{tensor comutativity} \\ \operatorname{Hom}_A(L \otimes_A M, N) &\cong \operatorname{Hom}_A(M, \operatorname{Hom}_A(L, N)) && \text{Hom tensor adjointness.} \end{aligned}$$

In particular, $\operatorname{Hom}_A(A, A) \cong A$. (Note: Adjointness is a bookkeeping nightmare.)

Semifree Resolutions. We need resolutions in order to do homological algebra in the DG setting.

Definition 7.8. A subset E of L is called a *semibasis* if it is a basis of the underlying $A^\natural$-module $L^\natural$. If L is bounded below, then L is called *semi-free* if it has a semibasis.[4] A *semi-free resolution* of a DG A-module M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of DG A-modules such that F is semi-free.

Exercise 7.9. Prove that a semi-free DG R-module is simply a bounded below complex of free R-modules. Prove that a free resolution F of an R-module M is equivalent to a semi-free resolution $F \xrightarrow{\simeq} M$; see Exercise 3.9.

Exercise 7.10. Prove that M is exact (as an R-complex) if and only if $0 \xrightarrow{\simeq} M$ is a semi-free resolution. Prove that the DG A-module A is semi-free, as is $\bigoplus_{n \geqslant n_0} A^{\beta_n}$ for all $n_0 \in \mathbb{Z}$ and $\beta_n \in \mathbb{N}$.

Exercise 7.11. Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$. Given a bounded below complex F of free R-modules, prove that the complex $K \otimes_R M$ is a semi-free DG K-module. If $F \xrightarrow{\simeq} M$ is a free resolution of an R-module M, prove that $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a semi-free resolution of the DG K-module $K \otimes_R M$. More generally, if $F \xrightarrow{\simeq} M$ is a semi-free resolution of a DG R-module M, prove that $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a semi-free resolution of the DG K-module $K \otimes_R M$.

Example 7.12. In the notation of Example 5.14, the natural map $G \to R$ is a semi-free resolution of R over U; see Example 5.22. The following diagrams indicate why G is semi-free over U, that is, why $G^\natural$ is free over $U^\natural$:

$$\begin{aligned} U &= \quad 0 \to Re \xrightarrow{0} R1 \to 0 \\ U^\natural &= \quad Re \oplus R1 \\ G &= \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0 \\ G^\natural &= \cdots (Re_3 \oplus R1_2) \oplus (Re_1 \oplus R1_0). \end{aligned}$$

[4] As is noted in [6], when L is not bounded below, the definition of "semi-free" is significantly more technical. However, our results do not require this level of generality, so we focus only on this case.

Fact 7.13. The DG A-module M has a semi-free resolution if and only if $\mathrm{H}_i(M) = 0$ for $i \ll 0$, by [6, Theorem 2.7.4.2].

Assume that A is noetherian, and let j be an integer. Assume that each module $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ and that $\mathrm{H}_i(M) = 0$ for $i < j$. Then M has a semi-free resolution $F \xrightarrow{\simeq} M$ such that $F^\natural \cong \oplus_{i=j}^{\infty} \Sigma^i (A^\natural)^{\beta_i}$ for some integers β_i, and so $F_i = 0$ for all $i < j$; see [1, Proposition 1]. In particular, homologically finite DG A-modules admit such "degree-wise finite, bounded below" semi-free resolutions.

Fact 7.14. Assume that L and M are semi-free. If there is a quasiisomorphism $L \xrightarrow{\simeq} M$, then there is also a quasiisomorphism $M \xrightarrow{\simeq} L$. [4]

Definition 7.15. Two semifree DG A-modules L and M are *quasiisomorphic* if there is a quasiisomorphism $L \xrightarrow{\simeq} M$; this equivalence relation is denoted by the symbol $\simeq$. Two semifree DG A-modules L and M are *shift-quasiisomorphic* if there is an integer m such that $L \simeq \Sigma^m M$; this equivalence relation is denoted by the symbol $\sim$.

Semidualizing DG Modules. We use Christensen and Sather-Wagstaff's notion of semidualizing DG U-modules from [15], defined next.

Definition 7.16. The *homothety morphism* $\chi_M^A \colon A \to \mathrm{Hom}_A(M, M)$ is given by $\chi_M^A(a) := \mu^{M,a}$, i.e., $\chi_M^A(a)(m) = am$.

Assume that A is noetherian. Then M is a *semidualizing* DG A-module if M is homologically finite and semi-free such that $\chi_M^A \colon A \to \mathrm{Hom}_A(M, M)$ is a quasiisomorphism. Let $\mathfrak{S}(A)$ denote the set of shift-quasiisomorphism classes of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules under the relation $\sim$ from Definition 7.15.

Fact 7.17. Let M be an R-module with projective resolution P. Then Fact 3.13 shows that M is a semidualizing R-module if and only if P is a semidualizing DG R-module. Thus, we have $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$.

Example 7.18. Let B and C be semi-free DG A-modules such that $B \simeq C$. Then B is semidualizing over A if and only if C is semidualizing over A. The point here is the following. The condition $B \simeq C$ tells us that B is homologically finite if and only if C is homologically finite. Fact 7.14 provides a quasiisomorphism $B \xrightarrow[\simeq]{f} C$. Thus, there is a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\chi_B^A} & \mathrm{Hom}_A(B, B) \\
{\scriptstyle \chi_C^A}\downarrow & & \downarrow{\scriptstyle \simeq\ \mathrm{Hom}_A(B,f)} \\
\mathrm{Hom}_A(C, C) & \xrightarrow[\simeq]{\mathrm{Hom}_A(f,C)} & \mathrm{Hom}_A(B, C).
\end{array}$$

The morphisms $\mathrm{Hom}_A(f, C)$ and $\mathrm{Hom}_A(B, f)$ are quasiisomorphisms because B and C are semi-free and f is a quasiisomorphism. It follows that χ_B^A is a quasiisomorphism if and only if χ_C^A is a quasiisomorphism.

The next lemmas are blerf [15, 29] and [25, 30, 31]. See Exercise 7.6.

Fact 7.19. Assume that $(R, \mathfrak{m})$ is local. Fix a list of elements $\mathbf{x} \in \mathfrak{m}$ and set $K = K^R(\mathbf{x})$. The base change functor $K \otimes_R -$ induces an injective map $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(K)$; if R is complete, then this map is a bijection.

Fact 7.20. Let $\varphi\colon A \xrightarrow{\simeq} B$ be a quasiisomorphism of noetherian DG R-algebras. The base change functor $B \otimes_A -$ induces a bijection from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$.

7.21 (Second part of the proof of Theorem 1.3)**.** We continue with the notation established in 5.26. Diagram (5.26.2) follows from (5.26.1) because of Facts 7.19 and 7.20. Thus, it remains to show that $\mathfrak{S}(k \otimes_Q A)$ is finite.

Ext for DG Modules. The proof of Fact 7.19 is not trivial. Essentially, the bijective part depends on a version of a famous result of Auslander, Ding, and Solberg [3] for the map $R \to K$. This result is in [29]. Its proof is quite technical. One other subtlety of the proof is found in the behavior of Ext for DG modules, which we describe next. Here there be dragons.

Definition 7.22. Given a semi-free resolution $F \xrightarrow{\simeq} M$, set and $\operatorname{Ext}^i_A(M,N) := \operatorname{H}_{-i}(\operatorname{Hom}_A(F,N))$ for each integer i.[5]

Exercise 7.23. Given R-modules M and N, prove that the module $\operatorname{Ext}^i_R(M,N)$ defined in 7.22 is the usual $\operatorname{Ext}^i_R(M,N)$; see Exercise 7.9.

Example 7.24. In the notation of Example 5.14, we have

$$\operatorname{Ext}^i_U(R,R) = \operatorname{H}_{-i}(\operatorname{Hom}_U(G,R)) = \begin{cases} R & \text{if } i \geqslant 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

This follows from Examples 7.3 and 7.12.

This example is a bit strange to us because it shows that U is fundamentally different from $U^\natural \cong R[X]/(X^2)$, even though U is obtained using a trivial differential on $R[X]/(X^2)$ with the natural grading. The fundamental difference comes from the fact that $\operatorname{Ext}^i_{U^\natural}(R,R) = R$ for all $i \geqslant 0$; contrast this with the previous display.

Fact 7.25. For each index i, the module $\operatorname{Ext}^i_A(M,N)$ is independent of the choice of semi-free resolution of M. [4]

Remark 7.26. An important fact about $\operatorname{Ext}^1_R(M,N)$ for R-modules M and N is the following: the elements of $\operatorname{Ext}^1_R(M,N)$ are in bijection with the equivalence classes of short exact sequences (i.e., "extensions") of the form $0 \to N \to X \to M \to 0$. For DG modules over a DG R-algebra A, things are a bit more subtle.

Given DG A-modules M and N, one defines the notion of a short exact sequence of the form $0 \to N \to X \to M \to 0$ in the naive way: the arrows are morphisms of DG A-modules such that for each $i \in \mathbb{Z}$ the sequence $0 \to N_i \to X_i \to M_i \to 0$ is exact. One defines an equivalence relation on the set of short exact sequences of this form (i.e., "extensions") in the natural way: two extensions $0 \to N \xrightarrow{f} X \xrightarrow{g} M \to 0$ and $0 \to N \xrightarrow{f'} X' \xrightarrow{g'} M \to 0$ are equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & X & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{h} & & \Big\downarrow{=} & & \\ 0 & \longrightarrow & N & \xrightarrow{f'} & X' & \xrightarrow{g'} & M & \longrightarrow & 0 \end{array}$$

[5] One can also define $\operatorname{Tor}_i^R(M,N) := \operatorname{H}_i(F \otimes_A N)$, but we do not need this here.

of morphisms of DG A-modules. Let the set of equivalence classes of such extensions be denoted $\mathrm{YExt}^1_A(M,N)$. The "Y" is for "Yoneda". Analogous to the case of R-modules, one can define an abelian group structure on $\mathrm{YExt}^1_A(M,N)$. However, in general one has $\mathrm{YExt}^1_A(M,N) \not\cong \mathrm{Ext}^1_A(M,N)$, even when $A = R$.

Example 7.27. Let $R = k[\![X]\!]$, and consider the following exact sequence of DG R-modules, i.e., exact sequence of R-complexes:

$$0 \longrightarrow \underline{R} \longrightarrow \underline{R} \longrightarrow \underline{k} \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

This sequence does not split over R (it is not even degree-wise split) so it gives a non-trivial class in $\mathrm{YExt}^1_R(\underline{k},\underline{R})$, and we conclude that $\mathrm{YExt}^1_R(\underline{k},\underline{R}) \neq 0$. On the other hand, $\underline{k}$ is homologically trivial, so we have $\mathrm{Ext}^1_R(\underline{k},\underline{R}) = 0$ since 0 is a semi-free resolution of $\underline{k}$.

For our proof of Theorem 1.3, the connection between Ext and YExt is quite important. The following result is proved in [30, Section 3]. We only need the second part for our work. The other part is used in the proof of the first par, and provides some perspective on the situation.

Fact 7.28. If L is semi-free, then $\mathrm{YExt}^1_A(L,M) \cong \mathrm{Ext}^1_A(L,M)$; if furthermore $\mathrm{Ext}^1_R(L,L) = 0$, then for each $n \geqslant \sup(L)$, one has

$$\mathrm{YExt}^1_A(L,L) = 0 = \mathrm{YExt}^1_A(\tau(L)_{(\leqslant n)}, \tau(L)_{(\leqslant n)}).$$

The notions $\sup(C)$ and $\tau(C)_{(\leqslant n)}$ are described next.

Truncations of DG Modules.

Definition 7.29. The *supremum* of M is

$$\sup(M) := \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(M) \neq 0\}.$$

Given an integer n, the *nth soft left truncation of M* is the complex

$$\tau(M)_{(\leqslant n)} := \quad 0 \to M_n/\operatorname{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M.

Exercise 7.30. Fix an integer n. Then the truncation $\tau(M)_{(\leqslant n)}$ is a DG A-module with the obvious scalar multiplication, and the natural chain map $M \to \tau(M)_{(\leqslant n)}$ is a morphism of DG A-modules. This morphism is a quasiisomorphism if and only if $n \geqslant \sup(M)$. See [6, (4.1)].

Exercise 7.31. We continue with the notation of Example 5.14. For each $n \geqslant 1$, prove that

$$\tau(G)_{(\leqslant n)} = \quad 0 \to R1_{2i} \xrightarrow{1} Re_{2i-1} \xrightarrow{0} \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0$$

where $i = \lfloor n/2 \rfloor$.

8. A Version of Happel's Result for DG Modules

The ideas for this section are from [2, 19, 22, 34]. The idea is bold one: study all possible module structures on a reasonable set. We have already seen a simple case of this in Example 5.15.

Notation 8.1. Let F be an algebraically closed field, and let

$$U := (0 \to U_q \xrightarrow{\partial_q^U} U_{q-1} \xrightarrow{\partial_{q-1}^U} \cdots \xrightarrow{\partial_1^U} U_0 \to 0)$$

be a finite-dimensional DG F-algebra. Let $\dim_F(U_i) = n_i$ for $i = 0, \ldots, q$. Let

$$W := \bigoplus_{i=0}^{s} W_i$$

be a graded F-vector space with $r_i := \dim_F(W_i)$ for $i = 0, \ldots, s$.

A DG U-module structure on W consists of two pieces of data. First, we need a differential ∂. Second, once the differential ∂ has been chosen, we need a scalar multiplication μ. Let $\operatorname{Mod}^U(W)$ denote the set of all ordered pairs (∂, μ) making W into a DG U-module. Let $\operatorname{End}_F(W)_0$ denote the set of F-linear endomorphisms of W that are homogeneous of degree 0. Let $\operatorname{GL}(W)_0$ denote the set of F-linear automorphisms of W that are homogeneous of degree 0, that is, the invertible elements of $\operatorname{End}_F(W)_0$.

Example 8.2. We continue with the notation of Example 5.15. In this example, we have $\operatorname{Mod}^U(W) = \{(0,0)\}$ and $\operatorname{Mod}^U(W') = \{(x_1, x_0) \in F^2 \mid x_1 x_0 = 0\}$. Rewriting F^2 as $\mathbb{A}_F^2$, we see that $\operatorname{Mod}^U(W)$ is a single point (the origin) in $\mathbb{A}_F^2$ and $\operatorname{Mod}^U(W')$ is the union of the two coordinate axes $V(x_1x_0) = V(x_0) \cup V(x_1)$.

It is straightforward to show that

$$\begin{aligned}
\operatorname{End}_F(W)_0 &= \operatorname{Hom}_F(F\nu_0, F\nu_0) \cong F = \mathbb{A}_F^1 \\
\operatorname{GL}_F(W)_0 &= \operatorname{Aut}_F(F\nu_0) \cong F^\times = U_x \subset \mathbb{A}_F^1 \\
\operatorname{End}_F(W')_0 &= \operatorname{Hom}_F(F\eta_1, F\eta_1) \oplus \operatorname{Hom}_F(F\eta_0, F\eta_0) \cong F \times F = \mathbb{A}_F^2 \\
\operatorname{GL}_F(W')_0 &= \operatorname{Aut}_F(F\eta_1) \oplus \operatorname{Aut}_F(F\eta_0) \cong F^\times \times F^\times = U_{x_1x_0} \subset \mathbb{A}_F^2.
\end{aligned}$$

Here U_x is the subset $\mathbb{A}_F^1 \smallsetminus V(x)$, and $U_{x_1x_0} = \mathbb{A}_F^2 \smallsetminus V(x_1x_0)$.

We next describe geometric structures on the sets $\operatorname{Mod}^U(W)$ and $\operatorname{GL}(W)_0$, as suggested by Example 8.2.

Remark 8.3. We work in the setting of Notation 8.1.

A differential ∂ on W is an element of the graded vector space $\operatorname{Hom}_F(W,W)_{-1} = \bigoplus_{i=0}^{s} \operatorname{Hom}_F(W_i, W_{i-1})$ such that $\partial\partial = 0$. The vector space $\operatorname{Hom}_F(W_i, W_{i-1})$ has dimension $r_i r_{i-1}$, so the map ∂ corresponds to an element of the affine space $\mathbb{A}_F^d$ where $d := \sum_i r_i r_{i-1}$. The vanishing condition $\partial\partial = 0$ is equivalent to the entries of the matrices representing ∂ satisfying certain fixed homogeneous quadratic polynomial equations over F. Hence, the set of all differentials on W is a Zariski-closed subset of $\mathbb{A}_F^d$.

Once the differential ∂ has been chosen, a scalar multiplication μ is in particular a cycle in $\operatorname{Hom}_F(U \otimes_F W, W)_0 = \bigoplus_{i,j} \operatorname{Hom}_F(U_i \otimes_F W_j, W_{i+j})$. For all i, j, the vector space $\operatorname{Hom}_F(U_i \otimes_F W_j, W_{i+j})$ has dimension $n_i r_j r_{i+j}$, so the map μ corresponds to an element of the affine space $\mathbb{A}_F^{d'}$ where $d' := \sum_{i,j} n_i r_j r_{i+j}$. The condition that μ be an associative, unital cycle is equivalent to the entries of the matrices representing ∂ and μ satisfying certain fixed polynomials over F. Thus, the set $\operatorname{Mod}^U(W)$ is a Zariski-closed subset of $\mathbb{A}_F^d \times \mathbb{A}_F^{d'} \cong \mathbb{A}_F^{d+d'}$.

Exercise 8.4. Continue with the notation of Example 5.15. Write out the coordinates and equations describing $\operatorname{Mod}^U(W'')$ and $\operatorname{Mod}^U(W''')$ where

$$\begin{aligned} W'' &= \quad 0 \oplus Fw_2 \oplus Fw_1 \oplus Fw_0 \oplus 0 \\ W''' &= \quad 0 \oplus Fz_2 \oplus (Fz_{1,1} \oplus Fz_{1,2}) \oplus Fz_0 \oplus 0. \end{aligned}$$

For scalar multiplication, note that since multiplication by 1 is already determined by the F-vector space structure, we only need to worry about multiplication by e which maps $W''_i \to W''_{i+1}$ and $W'''_i \to W'''_{i+1}$ for $i = 0, 1, 2$.

Remark 8.5. We work in the setting of Notation 8.1.

A map $\alpha \in \operatorname{GL}(W)_0$ is an element of the graded vector space $\operatorname{Hom}_F(W, W)_0 = \bigoplus_{i=0}^s \operatorname{Hom}_F(W_i, W_i)$ with a multiplicative inverse. The vector space $\operatorname{Hom}_F(W_i, W_i)$ has dimension r_i^2, so the map α corresponds to an element of the affine space $\mathbb{A}_F^e$ where $e := \sum_i r_i^2$. The invertibility of α is equivalent to the invertibility of each "block" $\alpha_i \in \operatorname{Hom}_F(W_i, W_i)$, which is an open condition defined by the non-vanishing of the determinant polynomial. Thus, the set $\operatorname{GL}(W)_0$ is a Zariski-open subset of $\mathbb{A}_F^e$, so it is smooth over F.

Alternately, one can view $\operatorname{GL}(W)_0$ as the product $\operatorname{GL}(W_0) \times \cdots \times \operatorname{GL}(W_s)$. Since each $\operatorname{GL}(W_i)$ is an algebraic group smooth over F, it follows that $\operatorname{GL}(W)_0$ is also an algebraic group that is smooth over F.

Exercise 8.6. Continue with the notation of Example 5.15. Write out the coordinates and equations describing $\operatorname{GL}^U(W'')_0$ and $\operatorname{GL}^U(W''')_0$ where W'' and W''' are from Exercise 8.4

Next, we describe an action of $\operatorname{GL}(W)_0$ on $\operatorname{Mod}^U(W)$.

Remark 8.7. We work in the setting of Notation 8.1.

Let $\alpha \in \operatorname{GL}(W)_0$. For every $(\partial, \mu) \in \operatorname{Mod}^U(W)$, we define $\alpha \cdot (\partial, \mu) := (\widetilde{\partial}, \widetilde{\mu})$, where $\widetilde{\partial}_i := \alpha_{i-1} \circ \partial_i \circ \alpha_i^{-1}$ and $\widetilde{\mu}_{i+j} := \alpha_{i+j} \circ \mu_{i+j} \circ (U \otimes_F \alpha_j^{-1})$. For the multiplication, this defines a new multiplication

$$u_i \cdot_\alpha w_j := \alpha_{i+j}(u_i \cdot \alpha_j^{-1}(w_j))$$

where $\cdot$ is the multiplication given by μ: $u_i \cdot w_j := \mu_{i+j}(u_i \otimes w_j)$. Note that this leave multiplication by 1 unaffected:

$$1 \cdot_\alpha w_j = \alpha_j(1 \cdot \alpha_j^{-1}(w_j)) = \alpha_j(\alpha_j^{-1}(w_j)) = w_j.$$

It is straightforward to show that the ordered pair $(\widetilde{\partial}, \widetilde{\mu})$ describes a DG U-module structure for W, that is, we have $\alpha \cdot (\partial, \mu) := (\widetilde{\partial}, \widetilde{\mu}) \in \operatorname{Mod}^U(W)$. From the definition of $\alpha\cdot(\partial,\mu)$, it follows readily that this describes a $\mathrm{GL}(W)_0$-action on $\mathrm{Mod}^U(W)$.

Example 8.8. Continue with the notation of Example 5.15.

In this case, the only DG U-module structure on W is the trivial one $(\partial,\mu)=(0,0)$, so we have $\alpha\cdot(\partial,\mu)=(\partial,\mu)$ for all $\alpha\in\mathrm{GL}(W)_0$.

The action on $\mathrm{Mod}^U(W')$ is a bit more interesting. Let $x_0,x_1\in F$ such that $x_0x_1=0$, as in Example 5.15. Identify $\mathrm{GL}_F(W')_0$ with $F^\times\times F^\times$, as in Example 8.2, and let $\alpha\in\mathrm{GL}_F(W')_0$ be given by the ordered pair $(y_1,y_0)\in F^\times\times F^\times$. The differential $\widetilde{\partial}$ is defined so that the following diagram commutes.

$$\begin{array}{rccccccccc}
\partial: & 0 & \longrightarrow & F\eta_1 & \xrightarrow{x_1} & F\eta_0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle y_1} & & \downarrow{\scriptstyle y_0} & & \\
\widetilde{\partial}: & 0 & \longrightarrow & F\widetilde{\eta}_1 & \xrightarrow{\widetilde{x}_1} & F\widetilde{\eta}_0 & \longrightarrow & 0
\end{array}$$

so we have $\widetilde{\partial}_1(\widetilde{\eta}_1)=y_0x_1y_1^{-1}\widetilde{\eta}_0$.

Since multiplication by 1 is already determined, and we have $e\cdot_\alpha\widetilde{\eta}_1=0$ because of degree considerations, we only need to understand $e\cdot_\alpha\widetilde{\eta}_0$. From Remark 8.7, this is given by

$$e\cdot_\alpha\widetilde{\eta}_0=\alpha_1(e\cdot\alpha_0^{-1}(\widetilde{\eta}_0))=\alpha_1(e\cdot y_0^{-1}\eta_0)=y_0^{-1}\alpha_1(e\cdot\eta_0)=y_0^{-1}\alpha_1(x_0\eta_1)=y_0^{-1}y_1x_0\widetilde{\eta}_1.$$

Exercise 8.9. Continue with the notation of Example 5.15. Using the solutions to Exercises 8.4 and 8.6 describe the action on $\mathrm{Mod}^U(W'')$ and $\mathrm{Mod}^U(W''')$ as in the previous example.

Remark 8.10. We work in the setting of Notation 8.1.

Let $\alpha\in\mathrm{GL}(W)_0$. For every $(\partial,\mu)\in\mathrm{Mod}^U(W)$, let $\alpha\cdot(\partial,\mu):=(\widetilde{\partial},\widetilde{\mu})$ be as in Remark 8.7. It is straightforward to show that a map α gives a DG U-module isomorphism $(W,\partial,\mu)\xrightarrow{\cong}(W,\widetilde{\partial},\widetilde{\mu})$. Conversely, given another element $(\partial',\mu')\in\mathrm{Mod}^U(W)$, if there is a DG U-module isomorphism $\beta\colon(W,\partial,\mu)\xrightarrow{\cong}(W,\partial',\mu')$, then $\beta\in\mathrm{GL}(W)_0$ and $(\partial',\mu')=\beta\cdot(\partial,\mu)$. In other words, the orbits in $\mathrm{Mod}^U(W)$ under the action of $\mathrm{GL}(W)_0$ are the isomorphism classes of DG U-module structures on W. Let $M\in\mathrm{Mod}^U(W)$. The orbit $\mathrm{GL}(W)_0\cdot M$ is locally closed in $\mathrm{Mod}^U(W)$; see [16, II, §5, 3].

Note that the maps defining the action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$ are regular, that is, determined by polynomial functions. This is because the inversion map $\alpha\mapsto\alpha^{-1}$ on $\mathrm{GL}(W)_0$ is regular, as is the multiplication of matrices corresponding to the compositions defining $\widetilde{\partial}$ and $\widetilde{\mu}$.

Notation 8.11. We work in the setting of Notation 8.1. Let $F[\epsilon]:=F\epsilon\oplus F$ be the algebra of dual numbers, where $\epsilon^2=0$. For our convenience, we write elements of $F[\epsilon]$ as column vectors: $a\epsilon+b=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$. We identify $U[\epsilon]:=F[\epsilon]\otimes_F U$ with $U\epsilon\oplus U\cong U\oplus U$, and $W[\epsilon]:=F[\epsilon]\otimes_F W$ with $W\epsilon\oplus W\cong W\oplus W$. Using this protocol, we have $\partial_i^{U[\epsilon]}=\left[\begin{smallmatrix}\partial_i^U&0\\0&\partial_i^U\end{smallmatrix}\right]$.

Let $\mathrm{Mod}^{U[\epsilon]}(W[\epsilon])$ denote the set of all ordered pairs (∂,μ) making $W[\epsilon]$ into a DG $U[\epsilon]$-module. Let $\mathrm{End}_{F[\epsilon]}(W[\epsilon])_0$ denote the set of $F[\epsilon]$-linear endomorphisms of $W[\epsilon]$ that are homogeneous of degree 0. Let $\mathrm{GL}(W[\epsilon])_0$ denote the set of $F[\epsilon]$-linear automorphisms of $W[\epsilon]$ that are homogeneous of degree 0, that is, the invertible elements of $\operatorname{End}_{F[\epsilon]}(W[\epsilon])_0$.

Given an element $M = (\partial, \mu) \in \operatorname{Mod}^U(W)$, the tangent space $\mathsf{T}_M^{\operatorname{Mod}^U(W)}$ is the set of all ordered pairs $(\overline{\partial}, \overline{\mu}) \in \operatorname{Mod}^{U[\epsilon]}(W[\epsilon])$ that give rise to M modulo ϵ. The tangent space $\mathsf{T}_{\operatorname{id}_W}^{\operatorname{GL}(W)_0}$ is the set of all elements of $\operatorname{GL}(W[\epsilon])_0$ that give rise to id_W modulo ϵ. There are alternate descriptions in [30, Lemmas 4.8 and 4.10]. Because of smoothness considerations, the map $\operatorname{GL}(W)_0 \xrightarrow{\cdot M} \operatorname{Mod}^U(W)$ induces a linear transformation $\mathsf{T}_{\operatorname{id}_W}^{\operatorname{GL}(W)_0} \to \mathsf{T}_M^{\operatorname{Mod}^U(W)}$ whose image is $\mathsf{T}_M^{\operatorname{GL}(W)_0 \cdot M}$.

Example 8.12. Continue with the notation of Example 5.15.

Theorem 8.13. *We work in the setting of Notation 8.1. Given an element* $M = (\partial, \mu) \in \operatorname{Mod}^U(W)$*, there is an isomorphism of abelian groups*

$$\mathsf{T}_M^{\operatorname{Mod}^U(W)} \big/ \mathsf{T}_M^{\operatorname{GL}(W)_0 \cdot M} \cong \operatorname{YExt}^1_U(M, M).$$

Sketch of proof. Using Notation 8.11, let $N = (\overline{\partial}, \overline{\mu})$ be an element of $\mathsf{T}_M^{\operatorname{Mod}^U(W)}$. Since N is a DG $U[\epsilon]$-module, restriction of scalars along the natural inclusion $U \to U[\epsilon]$ makes N a DG U-module.

Define $\rho \colon M \to N$ and $\pi \colon N \to M$ by the formulas $\rho(w) := \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$ and $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) := w$. With [30, Lemmas 4.8 and 4.10], one shows that ρ and π are chain maps and that ρ and π are U-linear. In other words, we have an exact sequence

$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$

of DG U-module morphisms. So, we obtain a map $\tau \colon \mathsf{T}_M^{\operatorname{Mod}^U(W)} \to \operatorname{YExt}^1_U(M, M)$ where $\tau(N)$ is the equivalence class of the displayed sequence in $\operatorname{YExt}^1_U(M, M)$. One shows that τ is a surjective abelian group homomorphism with $\operatorname{Ker}(\tau) = \mathsf{T}_M^{\operatorname{GL}(W)_0 \cdot M}$, and the result follows from the first Isomorphism Theorem.

To show that τ is onto, fix an arbitrary element $\zeta \in \operatorname{YExt}^1_U(M, M)$, represented by the sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. In particular, this is an exact sequence of F-complexes, so it is degree-wise split. This implies that we have a commutative diagram of graded vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & Z & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \vartheta} & & \big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & M & \xrightarrow{\rho} & W[\epsilon] & \xrightarrow{\pi} & M & \longrightarrow & 0 \end{array}$$

where $\rho(w) = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$, $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = w$, and ϑ is an isomorphism of graded F-vector spaces. The map ϑ allows us to endow $W[\epsilon]$ with a DG $U[\epsilon]$-module structure $(\overline{\partial}, \overline{\mu})$ that gives rise to M modulo ϵ, so $N = (\overline{\partial}, \overline{\mu}) \in \mathsf{T}_M^{\operatorname{Mod}^U(W)}$. Furthermore, we have $\tau(N) = \zeta$, so τ is surjective.

See [30, 4.11. Proof of Theorem B] for more details. □

The next result follows the ideas of Gabriel [19, 1.2 Corollary]. Or Voigt?

Corollary 8.14. *We work in the setting of Notations 8.1 and 8.11. Let C be a semidualizing DG U-module, and let $s \geqslant \sup(C)$. Set $M = \tau(C)_{(\leqslant s)}$ and $W = M^\natural$. Then the orbit* $\operatorname{GL}(W)_0 \cdot M$ *is open in* $\operatorname{Mod}^U(W)$.

Proof. Fact 7.28 implies that $\mathrm{YExt}^1_U(M,M) = 0$, so by Theorem 8.13 we have $\mathsf{T}_M^{\mathrm{Mod}^U(W)} = \mathsf{T}_M^{\mathrm{GL}(W)_0\cdot M}$. Since the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is smooth and locally closed, this implies that $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$. See [30, Corollary 4.12] for more details. $\square$

Lemma 8.15. *We work in the setting of Notations 8.1 and 8.11. Let $\mathfrak{S}_W(U)$ denote the set of quasiisomorphism classes of semi-free semidualizing DG U-modules C such that $s \geqslant \sup(C)$, $C_i = 0$ for all $i < 0$, and $(\tau(C)_{(\leqslant s)})^\natural \cong W$. Then $\mathfrak{S}_W(U)$ is a finite set.*

Proof. Fix a representative C for each quasiisomorphism class in $\mathfrak{S}_W(U)$, and write $[C] \in \mathfrak{S}_W(U)$ and $M_C = \tau(C)_{(\leqslant s)}$.

Let $[C], [C'] \in \mathfrak{S}_W(U)$. If $\mathrm{GL}(W)_0 \cdot M_C = \mathrm{GL}(W)_0 \cdot M_{C'}$, then $[C] = [C']$: indeed, Remark 8.7 explains the second step in the next display

$$C \simeq M_C \cong M_{C'} \simeq C'$$

and the remaining steps follow from the assumptions $s \geqslant \sup(C)$ and $s \geqslant \sup(C')$, by Exercise 7.30.

Now, each orbit $\mathrm{GL}(W)_0 \cdot M_C$ is open in $\mathrm{Mod}^U(W)$ by Corollary 8.14. Since $\mathrm{Mod}^U(W)$ is a subset of an affine space over F, it is quasi-compact, so it can only have finitely many open orbits. By the previous paragraph, this implies that there are only finitely many distinct elements $[C] \in \mathfrak{S}_W(U)$. $\square$

8.16 (Final part of the proof of Theorem 1.3)**.** We need to prove that $\mathfrak{S}(U)$ is finite where $U = k \otimes_Q A$. Set $s = \dim(R) - \mathrm{depth}(R) + n$. One uses various accounting principles to prove that every semidualizing DG U-module is equivalent to a semidualizing DG U-module C' such that $\mathrm{H}_i(C') = 0$ for all $i < 0$ and for all $i > s$. Let $L \xrightarrow{\simeq} C'$ be a minimal semi-free resolution of C' over U. The conditions $\sup(L) = \sup(C') \leqslant s$ imply that L (and hence C') is quasiisomorphic to the truncation $\widetilde{L} := \tau(L)_{\leqslant s}$. We set $W := \widetilde{L}^\natural$ and work in the setting of Notations 8.1 and 8.11.

One then uses further accounting principles to prove that there is an integer $\lambda \geqslant 0$, depending only on R and U, such that $\sum_{i=0}^s r_i \leqslant \lambda$. Compare this with Lemma 2.8. (Recall that r_i and other quantities are fixed in Notation 8.1.) Then, because there are only finitely many $(r_0, \ldots, r_s) \in \mathbb{N}^{s+1}$ with $\sum_{i=0}^s r_i \leqslant \lambda$, there are only finitely many W that occur from this construction, say $W^{(1)}, \ldots, W^{(b)}$. Lemma 8.15 implies that $\mathfrak{S}(U) = \mathfrak{S}_{W^{(1)}}(U) \cup \cdots \cup \mathfrak{S}_{W^{(b)}}(U) \cup \{[U]\}$ is finite. $\square$

Appendix A. Applications of Semidualizing Modules

This section includes some applications of semidualizing modules, to indicate why Theorem 1.3 might be interesting.

Application I. Asymptotic Behavior of Bass Numbers.

Definition A.1. Assume that $(R, \mathfrak{m}, k)$ is local. The *ith Bass number* of R is $\mu^i_R := \mathrm{rank}_k(\mathrm{Ext}^i_R(k,R))$. The *Bass series* of R is the formal power series $I^R(t) = \sum_{i=1}^\infty \mu^i_R t^i$.

Remark A.2. Assume that $(R, \mathfrak{m}, k)$ is local. The Bass numbers of R contain important structural information about the minimal injective resolution J of R. They also keep track of the depth and injective dimension of R:

$$\operatorname{depth}(R) = \min\{i \geqslant 0 \mid \mu_R^i \neq 0\}$$
$$\operatorname{id}_R(R) = \sup\{i \geqslant 0 \mid \mu_R^i \neq 0\}.$$

In particular, R is Gorenstein if and only if the sequence $\{\mu_R^i\}$ is eventually 0. If R has a dualizing module D, then the Bass numbers of R are related to the Betti numbers of D by the formula

$$\mu_R^{i+\operatorname{depth}(R)} = \beta_i^R(D) := \operatorname{rank}_k(\operatorname{Ext}_R^i(D,k)).$$

Viewed in the context of the characterization of Gorenstein rings in Remark A.2, the next question is natural, even if it is a bit bold.

Question A.3 (Huneke)**.** Assume that R is local. If the sequence $\{\mu_R^i\}$ is bounded, must R be Gorenstein? Equivalently, if R is not Gorenstein, must the sequence $\{\mu_R^i\}$ be unbounded?

The connection between semidualizing modules and Huneke's question is found in the following result. It shows that Huneke's question reduces to the case where R has only trivial semidualizing modules.

Theorem A.4 ([32])**.** *Assume that R is local. If R has a semidualizing module that is neither free nor dualizing, then the sequence $\{\mu_R^i\}$ is unbounded.*

Application II. Structure of Quasi-deformations. Semidualizing modules provide for extra structure in places where you might not expect it.

Definition A.5 ([7])**.** Assume that R is local. A *quasi-deformation* of R is a diagram $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ of local ring homomorphisms such that φ is flat and τ is surjective with kernel generated by a Q-regular sequence. A finitely generated R-module M has *finite CI-dimension* if there is a quasideformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M) < \infty$.

Remark A.6. Assume that R is local. A straightforward localization argument [7] shows that, if M is an R-module of finite CI-dimension, then there is a quasideformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M) < \infty$ and such that $R'/\mathfrak{m}R'$ is artinian, hence Cohen-Macaulay.

The next result is a souped-up version of the previous remark. In contrast to the previous application of semidualizing modules, this one does not refer to any semidualizing modules in the statement.

Theorem A.7. *Assume that R is local. If M is an R-module of finite CI-dimension, then there is a quasideformation $R \to R' \leftarrow Q$ such that* $\operatorname{pd}_Q(R' \otimes_R M) < \infty$ *and such that $R'/\mathfrak{m}R'$ is artinian and Gorenstein.*

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