REGINA LECTURES ON FAT POINTS

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1. Affine space and projective space

Let K be an algebraically closed field. For $n \geq 0$, let $\mathbb{A}^n$ denote K^n, and let $A = K[\mathbb{A}^n]$ denote $K[X_1, \ldots, X_n]$. We refer to $\mathbb{A}^n$ as *affine n-space*. For any subset $S \subseteq \mathbb{A}^n$, let $I(S) \subseteq A$ denote the ideal of all polynomials that vanish on S. (For those familiar with Spec, the *affine scheme* associated to S is $\mathrm{Spec}(A/I(S))$. Note that any ideal $I \subseteq A$ defines an affine subscheme of $\mathrm{Spec}(A)$, and ideals I and J define the same affine subscheme if and only if $I = J$.)

For $n \geq 0$, let $\mathbb{P}^n$ denote equivalence classes of nonzero $(n+1)$-tuples, where $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ are equivalent if there is a $0 \neq t \in K$ such that $(a_0, \ldots, a_n) = t(b_0, \ldots, b_n)$. Let $R = K[\mathbb{P}^n]$ denote $K[x_0, \ldots, x_n]$. We refer to $\mathbb{P}^n$ as *projective n-space*. For any subset $S \subseteq \mathbb{P}^n$, we obtain an associated homogeneous ideal (i.e., an ideal generated by homogeneous polynomials, also called forms) $I(S) \subseteq R$, the ideal generated by all homogeneous polynomials that vanish on S, where we regard R as being a graded ring with each variable having degree 1 and constants having degree 0. For those familiar with Proj, the *projective scheme* associated to S is $\mathrm{Proj}(R/I(S))$. If $M = (x_0, \ldots, x_n)$, any homogeneous ideal $I \subseteq M \subset R$ defines a subscheme $\mathrm{Proj}(R/I) \subseteq \mathrm{Proj}(R) = \mathbb{P}^n$, and homogeneous ideals $I \subseteq M$ and $J \subseteq M$ define the same subscheme if and only if $I_t = J_t$ for $t \gg 0$ (or equivalently, if and only if $I \cap M^t = J \cap M^t$ for $t \gg 0$), where I_t and J_t are the homogeneous components of the ideals of degree t. (Thus I_t is the vector space span of the elements of I of degree t. This applies in particular to R, so R_t is the K-vector space span of the homogeneous polynomials in R of degree t, and we have $I_t = R_t \cap I$.) Given a homogeneous ideal I, among all homogeneous ideals J such that $I_t = J_t$ for $t \gg 0$ there is a largest such ideal contained in M which contains all of the others, called the *saturation* of I, denoted $\mathrm{sat}(I)$. Thus given homogeneous ideals $I \subseteq M$ and $J \subseteq M$, we have $\mathrm{Proj}(R/I) = \mathrm{Proj}(R/J)$ if and only if $\mathrm{sat}(I) = \mathrm{sat}(J)$. We say an ideal is *saturated* if it is equal to its saturation. Thus geometrically we are most interested in homogeneous ideals which are saturated.

We can regard $\mathbb{A}^n \subset \mathbb{P}^n$ via the inclusion $(a_1, \ldots, a_n) \mapsto (1, a_1, \ldots, a_n)$. We have an isomorphism of function fields $K(X_1, \ldots, X_n) = K(\mathbb{A}^n) \cong K(\mathbb{P}^n) = K(x_1/x_0, \ldots, x_n/x_0)$ defined by $X_i \mapsto \frac{x_i}{x_0}$.

Given any ideal $0 \neq I \subseteq A$, define $\alpha(I)$ to be the degree of the nonzero element of I of least degree.

If $0 \neq J \subseteq R$ is a homogeneous ideal, then $\alpha(J)$ again is the degree of a nonzero element of J of least degree. (Such an element is necessarily homogeneous if J is homogeneous.)

Remark 1.1. Some authors use $\mathbb{A}^n$ to denote $\mathrm{Spec}(K[x_1, \ldots, x_n])$. Since we are assuming K is algebraically closed, our usage is (by the Nullstellensatz) equivalent to taking $\mathbb{A}^n$ to be the set of closed points (i.e., of points corresponding to maximal ideals) of $\mathrm{Spec}(K[x_1, \ldots, x_n])$. Likewise, some authors use $\mathbb{P}^n$ to denote $\mathrm{Proj}(K[x_0, \ldots, x_n])$. In our definition, $\mathbb{P}^n$ denotes the set of closed points of $\mathrm{Proj}(K[x_0, \ldots, x_n])$.

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We will denote the span of all polynomials of degree at most t by $A_{\leq t}$. Given an ideal $I \subseteq A$, let $I_{\leq t}$ denote $A_{\leq t} \cap I$, so $I_{\leq t}$ is the subspace of I spanned by all $f \in I$ of degree at most t. Given an ideal $I \subseteq A$, the *Hilbert function* of I is the function $H_I^{\leq}$ where $H_I^{\leq}(t) = \dim_K(I_{\leq t})$; i.e., $H_I^{\leq}(t)$ is the K-vector space dimension of the vector space spanned by all $f \in I$ with $\deg(f) \leq t$. The *Hilbert function* of A/I (or of the scheme $\mathrm{Spec}(A/I)$) is $H_{A/I}^{\leq}(t) = \dim_K(A_{\leq t}/I_{\leq t}) = \binom{n+t}{n} - H_I^{\leq}(t)$. Given a homogeneous ideal $I \subseteq R$, the *Hilbert function* H_I of I is the function $H_I(t) = \dim_K(I_t)$; i.e., $H_I(t)$ is the K-vector space dimension of the vector space spanned by all homogeneous $f \in I$ with $\deg(f) = t$. The *Hilbert function* of R/I (or of the scheme $\mathrm{Proj}(R/I)$) is $H_{R/I}(t) = \dim_K(R_t/I_t) = \binom{t+n}{n} - H_I(t)$.

It is known that $H_I^{\leq}$ and $H_{A/I}^{\leq}$ become polynomials for $t \gg 0$ (see Exercise 2.8 for an example). This polynomial is called the *Hilbert polynomial* of I or A/I respectively. (We will see in the next section that the Hilbert polynomial for the ideal I of the fat point subscheme $m_1p_1+\cdots+m_rp_r$ which we define there is $\binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$. Similarly, $\sum_i \binom{m_i+n-1}{n}$ is the Hilbert polynomial for A/I.) Likewise, if $I \subseteq R$ is a homogeneous ideal, H_I and $H_{R/I}$ become polynomials for $t \gg 0$, called the *Hilbert polynomial* of I or R/I as the case may be. Note that $H_I^{\leq}(t) = H_A^{\leq}(t) - H_{A/I}^{\leq}(t) = \binom{t+n}{n} - H_{A/I}^{\leq}(t)$ for all $t \geq 0$. Using Exercise 1.1 we also see that $H_I(t) = H_R(t) - H_{R/I}(t) = \binom{t+n}{n} - H_{R/I}(t)$ for all $t \geq 0$.

It is a significant and often difficult problem to determine the least value i such that the Hilbert polynomial and Hilbert function become equal for all $t \geq i$. (For an ideal of fat points, this value is sometimes called the *regularity index* of I, and $i+1$ in the case of an ideal of fat points is known as the *Castelnuovo-Mumford regularity* $\mathrm{reg}(I)$ of I.)

Exercises

Exercise 1.1. Show that there is a bijection between the set $\mathcal{M}_{\leq t}(A)$ of monomials of degree at most t in $A = K[x_1, \ldots, x_n]$ and the set $\mathcal{M}_t(R)$ of monomials of degree exactly t in $R = K[x_0, \ldots, x_n]$ for every $t \geq 0$. (This shows that $H_A^{\leq}(t) = H_R(t)$ for all $t \geq 0$.)

Exercise 1.2. If $0 \neq I \subseteq A$ is an ideal, show that $\alpha(I^m) \leq m\alpha(I)$, but if $0 \neq J \subseteq R$ is homogeneous, then $\alpha(J^m) = m\alpha(J)$. (See Exercise 2.2 for an example where equality in $\alpha(I^m) \leq m\alpha(I)$ fails.)

Exercise 1.3. Let $I \subseteq M \subset R$ be a homogeneous ideal. Let P be the ideal generated by all homogeneous $f \in R$ such that $fM^i \subseteq I$ for some $i > 0$. Show that $I \subseteq P$, that P contains every homogeneous ideal $J \subseteq M$ such that $I_t = J_t$ for $t \gg 0$, and that $I_t = P_t$ for $t \gg 0$. Conclude that P is the saturation of I and that $P = \mathrm{sat}(P)$. (In terms of colon ideals, $\mathrm{sat}(I) = \cup_{i \geq 1} I : M^i$.)

2. Fat points in affine space

A *fat point* subscheme of affine n-space is the scheme corresponding to an ideal of the form $I = \cap_{i=1}^r I(p_i)^{m_i} \subset A$ for a finite set of points $p_1, \ldots, p_r \in \mathbb{A}^n$ and positive integers m_i. We denote $\mathrm{Spec}(A/I)$ in this case by $m_1p_1 + \cdots + m_rp_r$, and we denote the ideal $\cap_{i=1}^r I(p_i)^{m_i}$ by $I(m_1p_1 + \cdots + m_rp_r)$.

Given distinct points $p_1, \ldots, p_r \in \mathbb{A}^n$, let $I = \cap_{i=1}^r I(p_i)$; following Waldschmidt [W1] we define a constant we denote by $\gamma(I)$ as the following limit

$$\gamma(I) = \lim_{m \to \infty} \frac{\alpha(\cap_{i=1}^r (I(p_i)^m))}{m}.$$

By Exercise 2.1, $\cap_{i=1}^r (I(p_i)^m) = I^m$, so

$$\gamma(I) = \lim_{m \to \infty} \frac{\alpha(I^m)}{m},$$

but for a unified treatment, whether the points p_i are in affine space or projective space, it is better to take

$$\gamma(I) = \lim_{m\to\infty} \frac{\alpha(\cap_{i=1}^{r}(I(p_i)^m))}{m}$$

as the definition of $\gamma(I)$.

We say the points $p_1, \ldots, p_r \in \mathbb{A}^n$ are *generic* points if the coordinates of the points are algebraically independent over the prime field Π_K of K. (This is possible only if the transcendence degree of K over Π_K is at least rn.) The following problem is open for $n > 1$ and $r \gg 0$.

Problem 2.1. *Let I be the ideal of r generic points of $\mathbb{A}^n$. Determine $\gamma(I)$.*

There is a conjectural solution to the problem above, when $r \gg 0$, due to Nagata [N1] for $n = 2$ and Iarrobino [I] for $n > 2$:

Conjecture 2.2 (Nagata/Iarrobino Conjecture)**.** *Let I be the ideal of $r \gg 0$ generic points of $\mathbb{A}^n$. Then $\gamma(I) = \sqrt[n]{r}$ for $r \gg 0$.*

Remark 2.3. The value of $\gamma(I)$ is known for r generic points of $\mathbb{A}^2$ for $1 \le r \le 9$ (see for example [Ch, Appendix 1] and [N2, Theorem 7]) or when r is a square [N1]. In particular, $\gamma(I) = 1$ if $r = 1, 2$, while $\gamma(I) = 3/2$ if $r = 3$, $\gamma(I) = 2$ if $r = 4, 5$, $\gamma(I) = 12/5$ if $r = 6$, $\gamma(I) = 21/8$ if $r = 7$, $\gamma(I) = 48/17$ if $r = 8$, and $\gamma(I) = \sqrt{r}$ if $r \ge 9$ is a square. Moreover, when $n > 2$ and $\sqrt[n]{r}$ is an integer, then again $\gamma(I) = \sqrt[n]{r}$ (see [E, Theorem 6]).

We will for now just verify that the values given in Remark 2.3 are upper bounds. By Exercise 2.4, the Hilbert polynomial of the ideal of a fat point subscheme $m_1p_1 + \cdots + m_rp_r \subset \mathbb{A}^n$ is $\binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$, and so $\sum_i \binom{m_i+n-1}{n}$ is the *Hilbert polynomial* for A/I or equivalently for the scheme $m_1p_1 + \cdots + m_rp_r$.

Proposition 2.4. *Consider the ideal I of r distinct points of $\mathbb{A}^n$. Then $\gamma(I) \le \sqrt[n]{r}$. Moreover, if $n = 2$, then $\gamma(I) = 1$ if $r = 1, 2$, $\gamma(I) \le 3/2$ if $r = 3$, $\gamma(I) \le 2$ if $r = 4, 5$, $\gamma(I) \le 12/5$ if $r = 6$, $\gamma(I) \le 21/8$ if $r = 7$, and $\gamma(I) \le 48/17$ if $r = 8$.*

Proof. For $\gamma(I) \le \sqrt[n]{r}$, see Exercise 2.9. Now let $n = 2$. Say $r = 1$. Then by Exercise 2.6, $H^{\le}_{I^m}(t) = 0$ for $t < m$ (so $\alpha(I^m) \ge m$) and clearly I^m has elements of degree m (so $\alpha(I^m) \le m$), hence $\alpha(I^m) = m$. Thus $\gamma(I) = 1$ by definition.

Now let $r = 2$; let p_1 and p_2 be the $r = 2$ points. Then $I^m \subseteq I(p_1)^m$, so $\alpha(I(p_1)^m) \le \alpha(I^m)$, hence $1 = \gamma(I(p_1)) \le \gamma(I^m)$, but again I^m clearly has elements of degree m (take the mth power of the linear polynomial defining the line through p_1 and p_2), so $\alpha(I^m) \le m$, hence $\gamma(I) \le 1$ so we have $\gamma(I) = 1$.

Now let $r = 3$. If the points are collinear, then as for two points we have $\gamma(I) = 1$. Otherwise, the cubic polynomial corresponding to the three lines through pairs of the $r = 3$ points is in I^2 and has degree 3, so Exercise 2.3(c) shows that $\gamma(I) \le \alpha(I^2)/2 \le 3/2$.

For $r = 4$, it's easy to see that $\alpha(I) \le 2$, so $\gamma(I) \le \alpha(I)/1 \le 2$.

For $r = 5$, $H^{\le}_I(2) \ge \binom{2+2}{2} - 5\binom{1+2-1}{2} = 1$, so $\alpha(I) \le 2$ and $\gamma(I) \le \alpha(I)/1 \le 2$.

For $r = 6$, through every subset of 5 of the 6 points there is (as we just saw) a conic, hence I^5 contains a nonzero polynomial of degree 12 (coming from the conics through the 6 subsets of 5 of the 6 points), so $\alpha(I^5) \le 12$ and $\gamma(I) \le \alpha(I^5)/5 \le 12/5$.

For $r = 7$, there is a cubic which has a point of multiplicity at least 2 at any one of the points and multiplicity at least 1 at the other 6 points, since $H^{\le}_I(3) \ge \binom{3+2}{2} - \binom{2+2-1}{2} - 6\binom{1+2-1}{2} = 1$. Multiplying together the seven cubics (one having a point of multiplicity at least 2 at the first point, the next having a point of multiplicity 2 at the second point, etc.) gives a polynomial of degree 21 having multiplicity at least 8 at each of the points, so $\gamma(I) \leq \alpha(I^8)/8 \leq 21/8$.

For $r = 8$, there is a sextic which has a point of multiplicity at least 3 at any one of the points and multiplicity at least 2 at the other 7 points, since $H_I^{\leq}(6) \geq \binom{6+2}{2} - \binom{3+2-1}{2} - 7\binom{2+2-1}{2} = 1$. Multiplying together the eight sextics gives a polynomial of degree 48 having multiplicity at least 17 at each of the points, so $\gamma(I) \leq \alpha(I^{17})/17 \leq 48/17$. $\square$

We will see in Section 6 and its exercises and Section 7 why equality holds above for $r < 9$ when $n = 2$ if the points are sufficiently general.

Exercises

Exercise 2.1. Let $p_1, \ldots, p_r$ be distinct points of $\mathbb{A}^n$. Show that $\cap_{i=1}^r I(p_i)^{m_i} = I(p_1)^{m_1} \cdots I(p_r)^{m_r}$.

Exercise 2.2. Let p_1, p_2, p_3 be distinct noncollinear points of $\mathbb{A}^2$. If $I = I(p_1) \cap I(p_2)$, show that $\alpha(I^m) = m\alpha(I)$. If $J = I(p_1) \cap I(p_2) \cap I(p_3)$ and $m > 1$, show that $\alpha(J^m) < m\alpha(J)$.

Exercise 2.3. [Waldschmidt's constant, [W1, W2]] Let $p_1, \ldots, p_r$ be distinct points of $\mathbb{A}^n$ and let $I = \cap_{i=1}^r I(p_i)$. Let b and c be positive integers.

(a) Show that
$$\frac{\alpha(I^{bc})}{bc} \leq \frac{\alpha(I^b)}{b}.$$
(b) Show that
$$\lim_{m\to\infty} \frac{\alpha(I^{m!})}{m!}$$
exists.
(c) Show that
$$\lim_{m\to\infty} \frac{\alpha(I^m)}{m}$$
exists, is equal to the limit given in (b) and satisfies
$$\lim_{m\to\infty} \frac{\alpha(I^m)}{m} \leq \frac{\alpha(I^t)}{t}$$
for all $t \geq 1$.

Exercise 2.4. Show that the K-vector space dimension of $A_{\leq t}$ is $\dim_K(A_{\leq t}) = \binom{t+n}{n}$.

Exercise 2.5. Show that there are $\binom{t+n}{n}$ monomials of degree t in $n+1$ variables.

Exercise 2.6. Let I be the ideal of the point $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$. Show that $H_{I^m}^{\leq}(t) \geq \binom{t+n}{n} - \binom{m+n-1}{n}$, with equality for $t \geq m-1$.

Exercise 2.7. Let $I \subseteq A$ be an ideal. Show that $H_{A/I}^{\leq}$ is nondecreasing.

The following exercise is a version of the Chinese Remainder Theorem.

Exercise 2.8. Let I be the ideal of $m_1p_1 + \cdots + m_rp_r$ for r distinct points $p_i \in \mathbb{A}^n$. Show that $H_I^{\leq}(t) \geq \binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$, with equality if $t \gg 0$.

Exercise 2.9. Let I be the ideal of r distinct points of $\mathbb{A}^n$. Show that $\gamma(I) \leq \sqrt[n]{r}$. If $1 \leq r \leq n$, show that $\gamma(I) = 1$.

Exercise 2.10. If $s \gg 9$ and $n = 2$, show that $\inf\{\frac{t}{m} : \binom{t+n}{n} - s\binom{m+n-1}{n} > 0\} = \sqrt[n]{s}$. (The same fact is true for $n > 2$ with $s \gg 0$ replacing $s \geq 9$. This is part of the motivation for the Conjecture 2.2.)

Here is a more explicit version of Exercise 2.8, one solution of which applies Exercises 2.12, 2.13 and 2.14.

Exercise 2.11. Let I be the ideal of $m_1p_1 + \cdots + m_rp_r$ for r distinct points $p_i \in \mathbb{A}^n$. Show that $H_I^{\leq}(t) = \binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$ if $t \geq m_1 + \cdots + m_r - 1$. If the points are collinear, show that $H_I^{\leq}(t) > \binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$ if $t < m_1 + \cdots + m_r - 1$.

Exercise 2.12. Let $p \in \mathbb{A}^n$ and let $m > 0$. Show that every element $\overline{f} \in A/(I(p))^m$ is the image of a polynomial $f \in A$ of degree at most $m-1$, and that $\overline{f}$ is a unit if and only if $f(p) \neq 0$.

Exercise 2.13. For any nonzero element $f \in K[\mathbb{A}^n]$, show there exists a point $p \in \mathbb{A}^n$ such that $f(p) \neq 0$.

Exercise 2.14. Let $n \geq 1$ and let $p_1, \ldots, p_r$ be distinct points of $\mathbb{A}^n$. Show that there is a linear form $f \in K[\mathbb{A}^n]$ such that $f(p_i) \neq f(p_j)$ whenever $p_i \neq p_j$.

3. Fat points in projective space

A *fat point* subscheme of projective n-space is the scheme corresponding to an ideal of the form $I = \cap_{i=1}^r I(p_i)^{m_i} \subset R$ for a finite set of distinct points $p_1, \ldots, p_r \in \mathbb{P}^n$ and positive integers m_i. We again denote the subscheme defined by I by $m_1p_1 + \cdots + m_rp_r$ (in this case the subscheme is $\mathrm{Proj}(R/I)$), and we denote the ideal $\cap_{i=1}^r I(p_i)^{m_i}$ by $I(m_1p_1 + \cdots + m_rp_r)$.

Remark 3.1. If $p_1, \ldots, p_r \subset \mathbb{A}^n \subset \mathbb{P}^n$, then there is no ambiguity in the notation $m_1p_1+\cdots+m_rp_r$, since there is a canonical isomorphism from $m_1p_1 + \cdots + m_rp_r$ regarded as a subscheme of $\mathbb{A}^n$ and $m_1p_1 + \cdots + m_rp_r$ regarded as a subscheme of $\mathbb{P}^n$. However, there is ambiguity in the notation $I(m_1p_1+\cdots+m_rp_r)$, so we will sometimes use $I_A(m_1p_1+\cdots+m_rp_r)$ to denote the ideal in A and $I_R(m_1p_1 + \cdots + m_rp_r)$ to denote the homogeneous ideal in R of $m_1p_1 + \cdots + m_rp_r$.

Remark 3.2. If $I_R = \cap_{i=1}^r I_R(p_i)$, it can sometimes happen that $I_R^m = \cap_{i=1}^r (I_R(p_i)^m)$, but $I_R(p_1)^{m_1} \cdots I_R(p_r)^{m_r} = \cap_{i=1}^r I_R(p_i)^{m_i}$ essentially never happens (see Exercise 3.1), and in general the most one can say about I_R^m is that $I_R^m \subseteq \cap_{i=1}^r (I_R(p_i)^m)$. Thus, we define the mth *symbolic* power $I_R^{(m)}$ of $I_R = \cap_{i=1}^r I_R(p_i)$ to be $I_R^{(m)} = \cap_{i=1}^r (I_R(p_i)^m)$. One can see the difference between $I_R(p_1)^{m_1} \cdots I_R(p_r)^{m_r}$ and $\cap_{i=1}^r I_R(p_i)^{m_i}$ and between I_R^m and $I_R^{(m)}$ by looking at primary decompositions. The intersection $\cap_{i=1}^r (I_R(p_i)^m)$ is the primary decomposition of $I_R^{(m)}$, but I_R^m has a primary decomposition of the form $I_R^{(m)} \cap J$ where J is M-primary (possibly $J = M$, in which case we have $I_R^m = I_R^{(m)} \cap M = I_R^{(m)}$), M being the irrelevant ideal (the ideal generated by the coordinate variables in $K[\mathbb{P}^n]$). Similarly, the primary decomposition of $I_R(p_1)^{m_1} \cdots I_R(p_r)^{m_r}$ also has the form $I_R^{(m)} \cap J$ where J is M-primary. In any case, we see that $I_R^m \subseteq I_R^{(m)}$ for all $m \geq 1$. We also see that $(I_R^m)_t = (I_R^{(m)})_t$ for $t \gg 0$, since for large t, any M-primary ideal J contains M^t and thus has $J_t = M_t$.

By Exercise 3.7, we have $I^r \subseteq I^{(m)}$ if and only if $r \geq m$. However, it is a hard problem to determine for which m and r we have $I^{(m)} \subseteq I^r$. See for example [ELS, HH1, CHT, HaHu] and the references therein.

Problem 3.3 (Open Problem)**.** *Let $p_1, \ldots, p_s \in \mathbb{P}^n$ be distinct points. Let $I = I_R(p_1 + \cdots + p_s)$. Is it true that $I^{(ns-n+1)} \subseteq I^s$ for all $s \geq 1$? In particular, is it true that $I^{(3)} \subseteq I^2$ always holds when $n = 2$?*

Let $\delta_t : R_t \to A_{\leq t}$ be the map defined for any $F \in R_t$ by $\delta_t(F) = F(1, X_1, \ldots, X_n)$ and let $\eta_t : A_{\leq t} \to R_t$ be the map $\eta_t(f) = x_0^t f(x_1/x_0, \ldots, x_n/x_0)$. Note that these are K-linear maps, each being the inverse of the other. In particular, $\dim(R_t) = \dim(A_{\leq t}) = \binom{t+n}{n}$.

If $p \in \mathbb{A}^n \subset \mathbb{P}^n$, so $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and can be represented in projective coordinates by $p = (1, a_1, \ldots, a_n) \in \mathbb{P}^n$, let $I = (X_1 - a_1, \ldots, X_n - a_n)$ be the ideal of p in A and let $J = (x_1 - a_1x_0, \ldots, x_n - a_nx_0)$ be the ideal of $p \in \mathbb{P}^n$ in R. Then $\eta_t((I^m)_{\leq t}) \subseteq (J^m)_t$ and $\delta_t((J^m)_t) \subseteq (I^m)_{\leq t}$, so we have K-linear isomorphisms $(I^m)_{\leq t} \to (J^m)_t$ given by η_t, hence $H^{\leq}_{I^m}(t) = H_{J^m}(t)$ and $H^{\leq}_{A/I^m}(t) = H_{R/J^m}(t)$ for all t. Similarly, if $p_1, \ldots, p_r \in \mathbb{A}^n \subset \mathbb{P}^n$, and if $I = I_A(m_1p_1 + \cdots + m_rp_r) \subset A$ and $J = I_R(m_1p_1 + \cdots + m_rp_r) \subset R$, then again we have K-linear isomorphisms $I_{\leq t} \to J_t$ given by η_t, hence $H^{\leq}_I(t) = H_J(t)$ and $H^{\leq}_{A/I}(t) = H_{R/J}(t)$ for all t. Hence the Hilbert functions and Hilbert polynomials for $m_1p_1 + \cdots + m_rp_r$ are the same whether we regard them as affine or projective subschemes. In particular, if $p_1, \ldots, p_r \subset \mathbb{A}^n \subset \mathbb{P}^n$ and if $I_A = I_A(p_1 + \cdots + p_r)$ and $I_R = I_R(p_1 + \cdots + p_r)$, then $\alpha(I_R^{(m)}) = \alpha(I_A^m)$ for all $m \geq 1$ and $\gamma(I_A) = \gamma(I_R)$. By Exercise 2.8, we also have $H_{I_R}(t) \geq \binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}$ and hence clearly

$$H_{I_R}(t) \geq \max\left\{\binom{t+n}{n} - \sum_i \binom{m_i+n-1}{n}, 0\right\}.$$

This is an equality for $t \gg 0$. There is a conjecture, known as the SHGH Conjecture, that gives a conjectural value for $H_{I_R}(t)$ when $n = 2$ and the points p_i are generic. Here is a simple to state special case of the SHGH Conjecture, named for various people who published what turns out to be equivalent conjectures: B. Segre [Se] in 1961, B. Harbourne [H2] in 1986, A. Gimigliano [G1] in 1987 (also see [G2]) and A. Hirschowitz [Hi] in 1989.

Conjecture 3.4 (SHGH Conjecture)**.** *Given $r \geq 9$ generic points $p_i \in \mathbb{P}^2$ and any non-negative integers m and t, let $I = I_R(m(p_1 + \cdots + p_r))$. Then*

$$H_I(t) = \max\left\{\binom{t+2}{2} - r\binom{m+1}{2}, 0\right\}.$$

There has been a lot of work done on this conjecture, see for example [AH, CM, HR], but there are many more papers than this. Note that, once one knows the Hilbert function, as one would for ideals of fat generic points in $\mathbb{P}^2$ if the SHGH Conjecture is true, one might want to know the graded Betti numbers for a minimal free resolution. There are conjectures and results here too, mostly for $\mathbb{P}^2$. See for example [H4] for some conjectures, and [BI, C, FHH, GHI, GI, H5, HHF, Id] for various results.

Most questions about fat points can be studied either from the point of view of subschemes of affine space or of subschemes of projective space. It can be more convenient to work with homogeneous ideals, so we will focus on the latter point of view.

We now mention some bounds on $\gamma(I)$ for an ideal $I = I_R(p_1 + \cdots + p_r)$ of distinct points $p_i \in \mathbb{P}^n$. Waldschmidt and Skoda [W1, W2, Sk] showed $\gamma(I) \geq \frac{\alpha(I^{(m)})}{m+n-1}$ over the complex numbers, and in particular that $\gamma(I) \geq \frac{\alpha(I)}{n}$. The proof involved some hard complex analysis. Easier and more general proofs which hold for any field K in any characteristic can be given using recent results on containments of powers of I in symbolic powers. In particular, we have $I^{(nm)} \subseteq I^m$ [ELS, HH1]. Thus $m\alpha(I) = \alpha(I^m) \leq \alpha(I^{(nm)})$, so dividing by mn and taking the limit as $m \to \infty$ gives

$$\frac{\alpha(I)}{n} \leq \gamma(I).$$

(See [Sc] for a different specifically characteristic $p > 0$ argument.)

Chudnovsky [Ch] showed $\frac{\alpha(I)+1}{2} \leq \gamma(I)$ in case $n = 2$ and conjectured $\frac{\alpha(I)+n-1}{n} \leq \gamma(I)$ in general; this conjecture is still open. By Exercise 3.6 we know

$$\frac{\alpha(I^{(m)})}{n+m-1} \leq \gamma(I).$$

Esnault and Viehweg [EV] obtained $\frac{\alpha(I^{(m)})+1}{m+n-1} \leq \gamma(I)$ in characteristic 0. It seems reasonable to extend Chudnovsky's conjecture [HaHu, Question 4.2.1]:

Conjecture 3.5. *For an ideal $I = I_R(p_1 + \cdots + p_r)$ of distinct points $p_i \in \mathbb{P}^n$ and for all $m \geq 1$,*

$$\frac{\alpha(I^{(m)}) + n - 1}{n + m - 1} \leq \gamma(I).$$

If this conjecture is correct, it is sharp, since there are configurations of points (so called star configurations) for which equality holds (apply [B. et al, Lemma 8.4.7] with $j = 1$).

Exercises

Exercise 3.1. Given $r > 1$ and distinct points $p_1, \ldots, p_r \in \mathbb{P}^n$ with $m_i > 0$ for all i, show that $I(p_1)^{m_1} \cdots I(p_r)^{m_r} \subsetneq \cap_{i=1}^r I(p_i)^{m_i}$.

Exercise 3.2. Let $p_1, \ldots, p_r \in \mathbb{P}^n$ be distinct points. Let $I = I_R = I(m_1p_1 + \cdots + m_rp_r) \subset R$. Show that multiplication by a linear form F that doesn't vanish at any of the points p_i induces injective vector space homomorphisms $R_t/I_t \to R_{t+1}/I_{t+1}$. Conclude that $H_{R/I}$ is a nondecreasing function of t.

Exercise 3.3. Let $p_1, \ldots, p_r \in \mathbb{P}^n$ be distinct points. Let $I = I_R = I(m_1p_1 + \cdots + m_rp_r) \subset R$. Show that $H_{R/I}(t)$ is strictly increasing until it becomes constant (i.e., if c is the least t such that $H_{R/I}(c) = H_{R/I}(c+1)$, show that $H_{R/I}(t)$ is a strictly increasing function for $0 \leq t \leq c$, and that $H_{R/I}(t) = H_{R/I}(c)$ for all $t \geq c$).

Exercise 3.4. Give an example of a monomial ideal $J \subset K[x, y]$ such that $H_{R/J}$ is eventually constant but is not nondecreasing.

Exercise 3.5. Show that Conjecture 3.4 implies the $n = 2$ case of Conjecture 2.2.

Exercise 3.6. If $I \subset R$ is the radical ideal of a finite set of points in $\mathbb{P}^n$, then $I^{((t+m-1)n)} \subseteq (I^{(m)})^t$ [ELS, HH1]. Use this to show

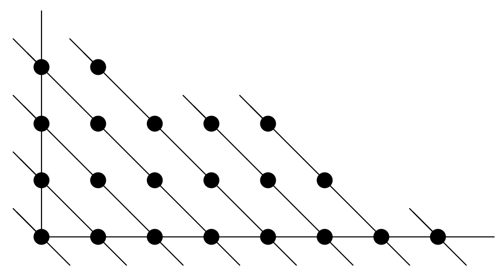
$$\frac{\alpha(I^{(m)})}{n + m - 1} \leq \gamma(I).$$

Exercise 3.7. Let $r, m \geq 1$. If $I = I(p_1 + \cdots + p_s) \subset R$ is the radical ideal of a finite set of distinct points $p_i \in \mathbb{P}^n$, show $I^r \subseteq I^{(m)}$ if and only if $r \geq m$.

4. Examples: bounds on the Hilbert function of fat point subschemes of $\mathbb{P}^2$

Let $p_1, \ldots, p_r \in \mathbb{P}^2$ be distinct points. Let $m_1, \ldots, m_r$ be positive integers. Let $L_0, \ldots, L_{s-1}$ be lines, repeats allowed, such that every point p_i is on at least m_i of the lines L_j. Let $Z = m_1p_1 + \cdots + m_rp_r$. For each $j \geq 0$, let l_{ij} be the number of times $p_i \in L_t$ for $1 \leq t \leq j$. Now define $Z_j = m_{1j}p_1 + \cdots + m_{rj}p_r$ where $m_{i0} = m_i$ for all i, and where $m_{ij} = \max\{m_i - l_{ij}, 0\}$. We get a sequence of fat point subschemes $Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_s = \varnothing$. Geometrically, Z_{j+1} is the fat point subscheme residual to Z_j with respect to the line L_{j+1}. Algebraically, $I(Z_{j+1}) = I(Z_j) : (F_{j+1})$, where F_{j+1} is the form defining the line L_{j+1}.

Define a reduction vector $\mathbf{d} = (d_0, \ldots, d_{s-1})$, where $d_j = \sum_{p_i \in L_j} m_{i\,j-1}$, so d_j is the sum of the multiplicities $m_{i\,j-1}$ for points $p_i \in L_j$. From the reduction vector we construct a new vector, $\mathrm{diag}(\mathbf{d})$. The entries of $\mathrm{diag}(\mathbf{d})$ are obtained as follows. Make an arrangements of dots in s rows, the first row at the bottom, the next row above it (aligned at the left), and so on, one row for each entry of $\mathbf{d}$, where the number of dots in each row is given by the corresponding entry of $\mathbf{d}$ and where the dots are placed at integer lattice points. The entries of $\mathrm{diag}(\mathbf{d})$ are obtained by counting the number of dots on each diagonal (of slope -1). Here is Example 2.5.5 of [CHT], where $\mathbf{d} = (8, 6, 5, 2)$ and $\operatorname{diag}(\mathbf{d}) = (1, 2, 3, 4, 4, 3, 3, 1, 0, 0, \ldots)$:



Theorem 4.1 ([CHT, Theorem 1.1]). *Let $\mathbf{d}$ be the reduction vector for a fat point scheme $Z \subset \mathbb{P}^2$ with respect to a given choice of lines L_i, and let v_{t+1} be the sum of the first $t+1$ entries of* $\operatorname{diag}(\mathbf{d})$. *Then $H_{R/I(Z)}(t) \geq v_{t+1}$, and equality holds for all t if the entries of $\mathbf{d}$ are strictly decreasing.*

Thus if we choose distinct lines L_0, L_1, L_2 and L_3, and 8 points on L_0, 6 on L_1, 5 on L_2 and 2 on L_3, then the reduction vector of the reduced scheme consisting of these 21 points is $\mathbf{d} = (8, 6, 5, 2)$, and (regarding a function of the nonnegative integers as a sequence) $H_{R/I(Z)}$ is $(1, 3, 6, 10, 14, 17, 20, 21, 21, 21, \ldots)$.

It is sometimes convenient to give not $H_{R/I(Z)}$ itself, but its first difference $\Delta H_{R/I(Z)}$, defined as $\Delta H_{R/I(Z)}(0) = 1$ and $\Delta H_{R/I(Z)}(t) = H_{R/I(Z)}(t) - H_{R/I(Z)}(t-1)$ for $t > 0$. In the preceding example, $\Delta H_{R/I(Z)}$ is $(1, 2, 3, 4, 4, 3, 3, 1, 0, 0, \ldots)$. In particular, when the entries of $\mathbf{d}$ are strictly decreasing, then $\Delta H_{R/I(Z)} = \operatorname{diag}(\mathbf{d})$.

Sketch of the proof of Theorem 4.1. We content ourselves here with merely obtaining an upper bound on $H_{R/I}(t)$. The fact that this bound agrees with the statement given in the theorem involves some combinatorial analysis, for which we refer you to the original paper.

Let $Z = Z_0$ be the original fat point scheme and let $Z_1, Z_2, \ldots, Z_s = \varnothing$ be the successive residuals with respect to the lines $L_0, L_1, \ldots, L_{s-1}$. Let $I = I(Z) \subset K[\mathbb{P}^2]$ be the ideal defining Z. Let $\mathbf{d} = (d_0, \ldots, d_{s-1})$. Let F_i be a linear form defining L_i. Given any fat point subscheme $X = a_1q_1 + \cdots + a_uq_u \subsetneq \mathbb{P}^2$, we have the ideal $I(X) \subset K[\mathbb{P}^2]$ as usual. Given a line $L \subset \mathbb{P}^2$ defined by a linear form F, the scheme theoretic intersection $X \cap L = \sum_{q_i \in L} a_iq_i$ is the fat point subscheme of $L \cong \mathbb{P}^1$ defined by the ideal $I_L(X \cap L) = \cap_{q_i \in L} I_L(q_i)^{a_i} \subset K[L] = K[\mathbb{P}^2]/(F) \cong K[\mathbb{P}^1]$, where for a point $q \in L \subset \mathbb{P}^2$, $I_L(q) \subset K[L]$ is the principal ideal defining q as a point of $L \cong \mathbb{P}^1$. Specifically, $I_L(q) = I(q)/(F) \subset K[L] = K[\mathbb{P}^2]/(F)$.

We have canonical inclusions $I(Z_{i+1}) \to I(Z_i)$ given by multiplying by F_i. The quotient $I(Z_i)/F_iI(Z_{i+1})$ is an ideal of $K[L_i]$ whose saturation is $I_L(Z_i \cap L)$. Thus we have an inclusion $I(Z_i)/F_iI(Z_{i+1}) \subseteq I_L(Z_i \cap L)$ which need not be an equality, but for $t \gg 0$ we do have $I(Z_i)_t/F_i(I(Z_{i+1}))_{t-1} = (I(Z_i)/F_iI(Z_{i+1}))_t \subseteq (I_L(Z_i \cap L))_t$.

Thus for each i and t we have an exact sequence

$$0 \to (I(Z_{i+1}))_{t-1} \to (I(Z_i))_t \to (I_{L_i}(Z_i \cap L_i))_t.$$

By definition of the reduction vector, $Z_i \cap L_i$ has degree d_i. Since $I_{L_i}(Z_i \cap L_i)$ is a principal ideal, we have $\dim_K((I_{L_i}(Z_i \cap L_i))_t) = \binom{t-d_i+1}{1} = \max\{t - d_i + 1, 0\}$, since there are $t - d_i + 1$ monomials in two variables of degree $t - d_i$ whenever $t - d_i \geq 0$. In particular,

$$\dim_K((I(Z_0))_t) \leq \dim_K((I(Z_1))_{t-1}) + \max\{t - d_0 + 1, 0\},$$

but likewise we have

$$\dim_K((I(Z_1))_{t-1}) \leq \dim_K((I(Z_2))_{t-2}) + \max\{t - 1 - d_1 + 1, 0\},$$

and this continues all the way to

$$\dim_K((I(Z_{s-1}))_{t-(s-1)}) \leq \dim_K((I(Z_s))_{t-s}) + \max\{t-(s-1)-d_{s-1}+1, 0\},$$

where we note that $(I(Z_s))_{t-s} = M_{t-s}$, M being the irrelevant ideal (so generated by the variables), hence $\dim_K((I(Z_s))_{t-s}) = \binom{t-s+2}{2}$.

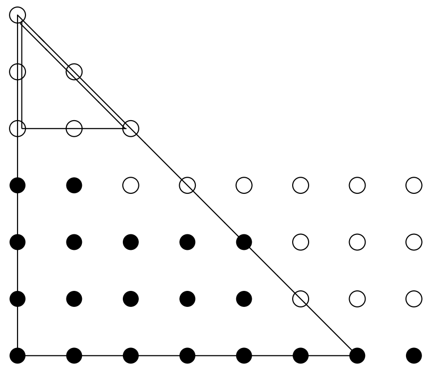
By back substitution, we get

$$\dim_K((I(Z_0))_t) \leq \binom{t-s+2}{2} + \sum_{0\leq i\leq s-1} \max\{t-i-d_i+1, 0\}.$$

Thus

$$H_{R/I}(t) = \binom{t+2}{2} - \dim_K((I(Z_0))_t) \geq \binom{t+2}{2} - \binom{t-s+2}{2} - \sum_{0\leq i\leq s-1} \max\{t-i-d_i+1, 0\}.$$

A combinatorial analysis shows this bound is what is claimed in the statement of the theorem. Basically, if you arrange the dots as specified by the reduction vector $\mathbf{d}$ (for the figure below $\mathbf{d} = (8, 5, 5, 2)$), then $\binom{t+2}{2} - \binom{t-s+2}{2} - \sum_{0\leq i\leq s-1} \max\{t-i-d_i+1, 0\}$ will for each t count the number of black dots in an isosceles right triangle with legs of length t; in the figure below this triangle is the big triangle, which has $t = 6$. The term $\binom{t+2}{2}$ counts the number of total number of dots in the big triangle, black and open (giving 28 in the figure below). To get the number of black dots, you must subtract the open dots in the little triangle; there are $\binom{t-s+2}{2}$ of these (where, in the figure, $t = 6$ and $s = 4$, giving 6 open dots). The remaining terms subtract off the number of open dots in the big triangle where each term accounts for each horizontal line on which there is a black dot (these terms would be $\max\{t-0-d_0+1, 0\} = \max\{6-8+1, 0\} = 0$ for the bottom row, $\max\{t-1-d_1+1, 0\} = \max\{6-1-5+1, 0\} = 1$ for the next row up, $\max\{t-2-d_2+1, 0\} = \max\{6-2-5+1, 0\} = 0$ for the row above that, and $\max\{t-3-d_3+1, 0\} = \max\{6-3-2+1, 0\} = 2$ for the top row below the little triangle.



The fact that the bound is an equality when the entries of the reduction vector are decreasing involves showing that the third map in the sequence

$$0 \to (I(Z_{i+1}))_{t-1} \to (I(Z_i))_t \to (I_{L_i}(Z_i \cap L_i))_t \qquad (*)$$

is surjective for every i and t. This is done using the long exact sequence in cohomology, where the terms in $(*)$ become modules of global sections of ideal sheaves, and where the lack of surjectivity on the right is controlled by an h^1 term. Working back from the last sequence, one shows for each i and t that either the controlling h^1 term is 0 (and hence we have surjectivity for that i and t) or $(I_{L_i}(Z_i \cap L_i))_t = 0$, hence again we have surjectivity for the given i and t. □

Exercises

Exercise 4.1. Let $r_1 > \cdots > r_s > 0$ be integers. Pick s distinct lines, and on line i pick any r_i points, such that none of the points chosen is a point of intersection of the ith line with another of the s lines. Let Z be the reduced scheme consisting of all of the chosen points. Show that $\Delta H_{R/I(Z)}$ is the sequence $(1, 2, \ldots, s, {}^{r_s-1}s, {}^{r_{s-1}-r_s-1}(s-1), {}^{r_{s-2}-r_{s-1}-1}(s-2), \ldots)$, where ${}^{i}j$ denotes a sequence consisting of i repetitions of j.

Exercise 4.2. Take any 4 distinct lines L_0, L_1, L_2, L_3, no three of which contain a point. There are 6 points, $p_1, \ldots, p_6$, where pairs of the lines intersect. Let $Z = 3p_1 + \cdots + 3p_6$. Determine the Hilbert function of $R/I(Z)$. (This generalizes to s lines, no 3 of which are coincident at a point; see [CHT].)

Exercise 4.3. Let $p_1, \ldots, p_r$ be distinct points of $\mathbb{P}^2$. Let $Z = m_1p_1 + \cdots + m_rp_r$. Pick lines $L_0, \ldots, L_{r-1}$ such that L_{i-1} contains p_i but does not contain p_j for $j \neq i$. Let $\mathbf{d}$ be the reduction vector obtained by choosing m_1 copies of L_0, then m_2 copies of L_1, etc. Show that $\mathbf{d} = (m_1, m_1 - 1, m_1 - 2, \ldots, m_1 - (m_1 - 1), m_2, m_2 - 1, \ldots, m_2 - (m_2 - 1), \ldots, m_r, m_r - 1, \ldots, m_r - (m_r - 1))$; conclude that $H_{R/I(Z)}(t) = \sum_i \binom{m_i+1}{2}$ for all $t \geq m_1 + \cdots + m_r - 1$.

5. Hilbert functions: some structural results

By Exercises 3.2 and 3.3, we know the Hilbert function of a fat point subscheme is nondecreasing in a strong way (it is strictly increasing until it is constant). It is possible to characterize the functions that are Hilbert functions of fat point subschemes: the Hilbert function of every fat point subscheme of projective space is a differentiable O-sequence, and for every differentiable O-sequence f there is an n and a finite set of points $p_1, \ldots, p_r \in \mathbb{P}^n$ such that $f = H_{R/I}$ where $R = K[\mathbb{P}^n]$ and $I = I_R(p_1 + \cdots + p_r)$.

It is worth noting that this leads to a characterization of Hilbert functions of reduced 0-dimensional subschemes of projective space: a function f is $H_{R/I}$ for some homogeneous radical ideal I of a finite set of points of projective space if and only if f is a 0-dimensional differentiable O-sequence. One can also say that a function f is $H_{R/I}$ for some homogeneous ideal $I = I(Z)$ for a fat point subscheme Z of projective space if and only if f is a 0-dimensional differentiable O-sequence. However, it is not known which 0-dimensional differentiable O-sequences occur as Hilbert functions $H_{R/I^{(2)}}$ for homogeneous radical ideals I defining finite sets of points in projective space.

Definition-Proposition 5.1. [GK] *Let h and d be positive integers. Then h can be expressed uniquely in the form*

$$\binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_j}{j}$$

where $m_d > m_{d-1} > \cdots > m_j \geq j \geq 1$. This expression for h is called the d-binomial expansion *of h. Given the d-binomial expansion of h, we also define*

$$h^{\langle d \rangle} = \binom{m_d + 1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_j + 1}{j+1}.$$

Example 5.2. The 3-binomial expansion of 16 is

$$15 = \binom{5}{3} + \binom{3}{2} + \binom{2}{1} = 10 + 3 + 2$$

and so

$$15^{\langle 3 \rangle} = \binom{6}{4} + \binom{4}{3} + \binom{3}{2} = 15 + 4 + 3 = 22.$$

Definition 5.3. A sequence of non-negative integers $\{h_d\}_{d \geq 0}$ is called an *O-sequence* if

- $h_0 = 1$
- $h_{d+1} \leq h_d^{\langle d \rangle}$ for all $d \geq 1$.

With these definitions we can state a well-known theorem of Macaulay (see [M] and [St] for full details):

Theorem 5.4 (Macaulay's Theorem). *The following are equivalent:*

(a) $\{h_d\}_{d\geq 0}$ *is an O-sequence;*
(b) $\{h_d\}_{d\geq 0}$ *is the Hilbert function $H_{R/I}$ for some homogeneous ideal $I \subseteq R$; and*
(c) $\{h_d\}_{d\geq 0}$ *is the Hilbert function $H_{R/J}$ for some monomial ideal $J \subseteq R$.*

Definition 5.5. Let $\mathcal{H} = \{h_d\}_{d\geq 0}$ be an O-sequence and $\Delta\mathcal{H} = \{e_d\}_{d\geq 0}$ be defined by $e_0 = h_0$ and $e_d = h_d - h_{d-1}$ for $d \geq 1$. We say that $\mathcal{H}$ is a *differentiable O-sequence* if $\Delta\mathcal{H}$ is also an O-sequence. We say $\mathcal{H}$ is *0-dimensional* if $\Delta\mathcal{H}$ is 0 for all $t \gg 0$.

Proposition 5.6. *Let $p_1, \ldots, p_s \in \mathbb{P}^n$ be distinct points, let $m_1, \ldots, m_s$ be positive integers, and let $I = I(m_1p_1 + \cdots + m_sp_s)$ be the ideal of the fat point subscheme $m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^n$. Then the Hilbert function $H_{R/I}$ is a differentiable 0-dimensional O-sequence.*

Proof. By Macaulay's Theorem, $H_{R/I}$ is an O-sequence. By Exercise 3.3, $H_{R/I}$ is 0-dimensional. But if $x \in R$ is a linear form that does not vanish at any of the points, and if $J = I + (x)$, then

$$\frac{R}{J} \cong \frac{R/I}{J/I} = \frac{R/I}{((x)+I)/I} \cong \frac{R/I}{x(R/I)}$$

so we have $H_{R/J} = H_{\frac{R/I}{x(R/I)}}$ and since x maps to a unit in R/I, we obtain $H_{\frac{R/I}{x(R/I)}} = \Delta H_{R/I}$. But by Macaulay's Theorem again, $H_{R/J}$ is an O-sequence, hence $H_{R/I}$ is a differentiable O-sequence. $\square$

There is also a converse:

Theorem 5.7. [GMR] *Let $\mathcal{H} = \{h_d\}_{d\geq 0}$ be a differentiable 0-dimensional O-sequence with $h_1 = n+1$. Then there is a finite set of points in $\mathbb{P}^n$ and the ideal $I \subseteq R$ of those points is a radical ideal such that $\mathcal{H} = H_{R/I}$. In case $n = 2$, those points can be chosen as in Exercise 4.1 and hence $\Delta\mathcal{H} = \mathrm{diag}(\mathbf{d})$ for some decreasing sequence $\mathbf{d}$ of positive integers.*

We give some idea how one can prove this, involving monomial ideals and their liftings. The original proof, given in [GMR], is somewhat different.

Definition 5.8. Let $J \subseteq K[x_1, x_2]$ be a homogeneous ideal and let $\phi : K[x_0, x_1, x_2] \to K[x_1, x_2]$ be defined by $\phi(x_0) = 0$ and $\phi(x_i) = x_i$ for $i > 0$. We say that J *lifts to* $I \subseteq K[x_0, x_1, x_2]$ if

- I is a radical ideal in $K[x_0, x_1, x_2]$;
- x_0 is not a zero-divisor on $K[x_0, x_1, x_2]/I$; and
- $\phi(I) = J$.

If $\mathcal{H} = \{h_d\}_{d\geq 0}$ is a differentiable 0-dimensional O-sequence (with $n = 2$), let $\Delta\mathcal{H} = \{e_d\}_{d\geq 0}$ be defined by $e_0 = 1, e_d = h_d - h_{d-1}$ for $d \geq 1$. By Macaulay's Theorem, there exists an ideal $J \subseteq K[x_1, x_2]$ generated by monomials $\{x_1^{m_{1i}} x_2^{m_{2i}} : i = 0, \ldots, r\}$ such that $H_{K[x_1,x_2]/J} = \Delta\mathcal{H}$. Since the O-sequence is 0-dimensional, we know that among the generators are pure powers of x_1 and x_2. In fact, Macaulay proved more than the statement we gave above of Macaulay's Theorem; he showed that J can be taken to be a lex ideal, which here means that we may assume that $m_{2i} = i$ and $m_{1i} - 1 > m_{1\,i+1}$ for all i, with $m_{1r} = 0$. Geramita–Gregory–Roberts [GGR] and Hartshorne [Ht] showed that J lifts to an ideal I which is the ideal of a finite set of points whose coordinates are given by the exponent vectors (m_{1i}, m_{2i}). To explain this in more detail we introduce some notation and bijections.

To an element $\alpha = (a_1, a_2) \in \mathbb{N}^2$ we associate the point $\overline{\alpha} = [1 : a_1 : a_2] \in \mathbb{P}^2$. Further, for each monomial $g = x^\alpha = x_1^{a_1} x_2^{a_2}$ we associate

$$\overline{g} = \prod_{j=1}^{2} \left(\prod_{i=0}^{a_j - 1} (x_j - i x_0) \right).$$

Observe that $\overline{g}$ is homogeneous.

Now, since J is a monomial ideal, the set $M \setminus N$, where M denotes the monomials in $K[x_1, x_2]$ (including 1) and N denotes the set of monomials in J, gives representatives for a K-basis of $K[x_1, x_2]/J$. Let $\overline{M}$ denote the set of all points $\overline{\alpha} = \overline{(a_1, a_2)} \in \mathbb{P}^2$ such that $x_1^{a_1} x_2^{a_2} \in M$. It can then be shown (see [GGR] for full details) that J lifts to $I = (\overline{g_i})$, where $\{g_i\}$ is the minimal generating set for J. The key step in the proof is to show that

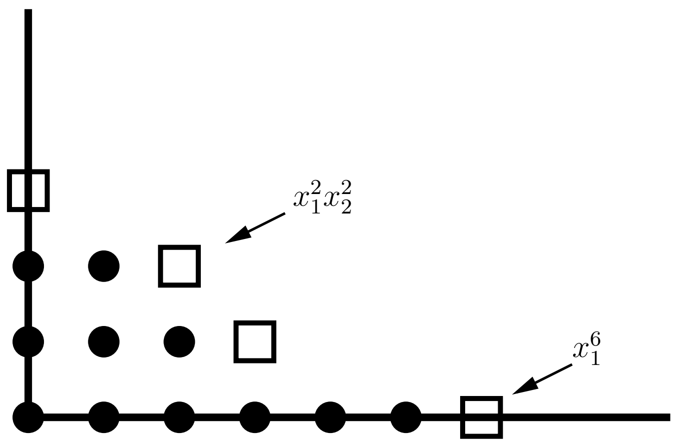
$$I = \{f \in K[x_0, x_1, x_2] : f(\overline{\alpha}) = 0 \text{ for all } \overline{\alpha} \in \overline{M}\}.$$

Note that I is the ideal of a finite set of points which can be chosen as in Exercise 4.1.

Example 5.9. Consider $\mathcal{H} = (1, 3, 6, 9, 10, 11, 11, 11, \ldots)$ which is a differentiable 0-dimensional O-sequence. Then $\Delta\mathcal{H} = (1, 2, 3, 3, 1, 1, 0, 0, \ldots)$. To find a finite set of points $\mathbb{X}$ where $H_{R/I(\mathbb{X})} = \mathcal{H}$ we consider the monomial ideal $J = (x_2^3, x_1^2 x_2^2, x_1^3 x_2, x_1^6)$. We can visualize the monomials in $M \setminus N$ as the circles in the following x_1x_2-plane, where the monomial $x_1^{a_1} x_2^{a_2}$ is represented by the pair (a_1, a_2). The squares represent the generators of J.

We see that the set $\mathbb{X}$ consisting of the points in $\mathbb{P}^2$ which are in $\overline{M}$ is:

$$\{[1:0:0], [1:1:0], [1:2:0], [1:3:0], [1:4:0], [1:5:0], [1:0:1], [1:1:1], [1:2:1], [1:0:2], [1:1:2]\}.$$



The ideal $I = I(\mathbb{X})$ is generated by:

$$\begin{aligned}
\overline{x_2^3} &= x_2(x_2 - x_0)(x_2 - 2x_0)\\
\overline{x_1^2x_2^2} &= x_1(x_1 - x_0)x_2(x_2 - x_0)\\
\overline{x_1^3x_2} &= x_1(x_1 - x_0)(x_1 - 2x_0)x_2\\
\overline{x_1^6} &= x_1(x_1 - x_0)(x_1 - 2x_0)(x_1 - 3x_0)(x_1 - 4x_0)(x_1 - 5x_0).
\end{aligned}$$

We have that J lifts to I. Observe that $\mathbb{X}$ is a configuration of points contained in a union of three "horizontal" lines in $\mathbb{P}^2$, with 6 points on the bottom line, 3 on the middle line and 2 on the top line.

The method used in the above example will work in general. Given a differentiable 0-dimensional O-sequence $\mathcal{H}$ where $\Delta\mathcal{H} = (h_0, h_1, h_2, \ldots)$, then one applies the steps above using the ideal J found by setting the degree t monomials of $M \setminus N$ to be the first h_t monomials in R using lexicographic ordering.

Exercises

Exercise 5.1. Let $I = I(3p)$ for a point $p \in \mathbb{P}^2$. Find a set of points $p_1, \ldots, p_r \in \mathbb{P}^2$ such that $H_{R/I} = H_{R/J}$ where $J = I(p_1 + \cdots + p_r)$.

Exercise 5.2. Show that **d** in the statement of Theorem 5.7 is unique.

6. Bézout's theorem and applications

Let $0 \neq F \in K[\mathbb{P}^2] = K[x_0, x_1, x_2]$ be homogeneous. The multiplicity $\mathrm{mult}_p(F)$ of F at a point $p \in \mathbb{P}^2$ is the largest m such that $F \in I(p)^m$, where we regard $I(p)^0$ as being R. If projective coordinates are chosen so that $p = (1, 0, 0)$, then $\mathrm{mult}_p(F)$ is the degree of a term of least degree in $F(1, x_1, x_2)$. The homogeneous component h of $F(1, x_1, x_2)$ of least degree factors as a product of powers of homogeneous linear factors l_i; i.e., $h = l_1^{m_1} \cdots l_s^{m_s}$. The factors l_i are the *tangents* to F at p, and the exponent m_i is the multiplicity of l_i.

We can regard F as defining a 1-dimensional subscheme $C_F \subset \mathbb{P}^2$. If F and G are homogeneous polynomials which do not have a common factor vanishing at p, we define the intersection multiplicity $I_p(F, G) = I_p(C_F, C_G)$ to be the K-vector space dimension of the tth homogeneous component of $R/((F, G) + I(p)^m)$ when m and t are large.

Assume that F, G and H are homogeneous polynomials which do not have a common factor vanishing at p. Then some facts about intersection multiplicities are (see [Hr] or [F]):

(a) $I_p(F,G) \geq \operatorname{mult}_p(F)\operatorname{mult}_p(G)$, where equality holds if and only if F and G have no tangent in common at p;
(b) $I_p(F,GH) = I_p(F,G) + I_p(F,H)$;
(c) intersection multiplicities are invariant under projective linear homogeneous changes of coordinates; and
(d) (Bézout's Theorem) if F and G have no common factor of positive degree, then

$$(\deg(F))(\deg(G)) = \sum_{p\in\mathbb{P}^2} I_p(F,G).$$

Example 6.1. Let $Z = m_1p_1 + \cdots + m_rp_r$, where $p_1,\ldots,p_r \in \mathbb{P}^2$ are distinct points and each m_i is a positive integer. Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d such that $\operatorname{mult}_{p_i}(C) = e_i$ for each i (i.e., $\operatorname{mult}_{p_i}(G) = e_i$ where G is the form defining C). Say $0 \neq F \in I(Z)_t$, so $\operatorname{mult}_{p_i}(F) \geq m_i$ for all i. If $\sum_i m_ie_i > td$, then $\sum_i I_{p_i}(F,G) \geq \sum_i \operatorname{mult}_{p_i}(F)\operatorname{mult}_{p_i}(G) \geq \sum_i m_ie_i > td$ so by Bézout's Theorem, G and F have a common factor, but G is irreducible, so $G|F$. Thus $H \in I((m_1-e_1)p_1 + \cdots + (m_r-e_r)p_r)$, where $H = F/G$.

We can apply this to get bounds on $\alpha(I(Z))$. For example, let $L_1, L_2, L_3, L_4 \subset \mathbb{P}^2$ be lines no three of which meet at a point. We will regard L_i as denoting either the line itself or the linear homogeneous form that defines the line, depending on context. Let $p_{ij} = L_i \cap L_j$ for $i \neq j$, so $\{p_{ij}\}$ are the six points of pair-wise intersections of the lines. Let $Z = \sum_{ij} 3p_{ij}$. It is easy to check that $(L_1L_2L_3)^2L_4$ is in $I(p_{ij})^3$ for each of the six points. Thus $(L_1L_2L_3)^2L_4 \in I(Z)_7$ so $\alpha(I(Z)) \leq 7$. On the other hand, assume we have $0 \neq F \in I(Z)_6$. There are three points where both F and L_i vanish, with F having multiplicity at least 3 at each and L_i having multiplicity 1. Since $3\cdot(3\cdot 1) > \deg(F)\deg(L_i) = 6$, then $L_i|F$. This is true for all i, so $L_1L_2L_3L_4|F$. Let $H = F/(L_1L_2L_3L_4)$. Then $\deg(H) = 2$ and $\operatorname{mult}_{p_{ij}}(H) \geq 1$. Now $3\cdot(1\cdot 1) > \deg(H)\deg(L_i) = 2$, so again $L_1\cdots L_4|H$, but this is impossible since $\deg(H) < \deg(L_1L_2L_3L_4)$. Thus H and therefore F must be 0, so $\alpha(I(Z)) > 6$ and hence $\alpha(I(Z)) = 7$. (Note that this is in agreement with the result of Exercise 4.2.)

Example 6.2. Let $I = I(p_1+p_2+p_3)$ for three noncolinear points of $\mathbb{P}^2$. We show that $\gamma(I) = 3/2$. Consider $I^{(m)} = I(m(p_1+p_2+p_3))$. Assume $m = 2s$ is even, and suppose $0 \neq F \in (I^{(m)})_{3s-1}$. Note that F vanishes to order at least m at each of two points for any line L_{ij} through two of the points p_i, p_j, $i \neq j$. Since $2m = 4s > 3s-1$, this means by Bézout that the linear forms (also denoted L_{ij}) defining the lines are factors of F. Dividing F by $L_{12}L_{13}L_{23}$ we obtain a form G of degree $3(s-1)-1$ in $I^{(m-2)}$. The same argument applies: $L_{12}L_{13}L_{23}$ must divide G. Eventually we obtain a form of degree 2 divisible by $L_{12}L_{13}L_{23}$, which is impossible. Thus $F = 0$, and $\alpha(I^{(m)}) > \frac{3m}{2} - 1$. Since $(L_{12}L_{13}L_{23})^s \in I^{(m)}$, we see that $\alpha(I^{(m)}) \leq \frac{3m}{2}$, thus $\alpha(I^{(m)}) = \frac{3m}{2}$, and hence $\gamma(I) = \lim_{m\to\infty} \alpha(I^{(m)})/m = 3/2$.

Exercises

Exercise 6.1. Show that $I_p(F,G) = 0$ if either F or G does not vanish at p.

Exercise 6.2. Let $p = (1,0,0)$, $F = x_1x_0 - x_2^2$ and $G = x_1x_0^2 - x_2^3$. Compute $I_p(F,G)$ and verify that $\sum_{p\in\mathbb{P}^2} I_p(F,G) = \deg(F)\deg(G)$ by explicit computation.

Exercise 6.3. Consider the $\binom{s}{2}$ points of pairwise intersection of s distinct lines in $\mathbb{P}^2$, no three of which meet at a point. Let I be the radical ideal of the points. Mimic Example 6.1 to show that $\alpha(I^{(m)}) = ms/2$ if m is even, and $\alpha(I^{(m)}) = (m+1)s/2 - 1$ if m is odd.

Exercise 6.4. Let $I = I(p_1+p_2+p_3+p_4)$ for four points of $\mathbb{P}^2$, no three of which are colinear. Show that $\gamma(I) = 2$.

Exercise 6.5. Let $I = I(p_1+p_2+p_3+p_4+p_5)$ for five points of $\mathbb{P}^2$, no three of which are colinear. Show that $\gamma(I) = 2$.

Exercise 6.6. Show that there exist 6 points of $\mathbb{P}^2$ which do not all lie on any conic, and no three of which are colinear.

Exercise 6.7. Let $I = I(p_1 + \cdots + p_6)$ for six points of $\mathbb{P}^2$, no three of which are colinear and which do not all lie on a conic (such point sets exist by Exercise 6.6). Show that $\gamma(I) = 12/5$.

Exercise 6.8. Show that there exist 7 points of $\mathbb{P}^2$ no three of which are colinear and no six of which lie on a conic.

Exercise 6.9. Let $I = I(p_1 + \cdots + p_7)$ for seven points of $\mathbb{P}^2$, no three of which are colinear and no six of which lie on a conic (such point sets exist by Exercise 6.8). Show that $\gamma(I) = 21/8$.

Exercise 6.10. Given 9 distinct points $p_i \in \mathbb{P}^2$ on an irreducible cubic C such that $\mathrm{mult}_{p_i}(C) = 1$ for all i, show that $\gamma(I) = 3$ for $I = I(p_1 + \cdots + p_9)$.

7. Divisors, global sections, the divisor class group and fat points

For this section, our references are [Hr], [N2], [D], [H3] and [H1]. Given any finite set of distinct points $p_1, \ldots, p_r \in \mathbb{P}^2$, there is a projective algebraic surface X, a projective morphism $\pi : X \to \mathbb{P}^2$ (obtained by blowing up the points p_i) such that each $\pi^{-1}(p_i) = E_i$ is a smooth rational curve and such that π induces an isomorphism $X \setminus \cup_i E_i \to \mathbb{P}^2 \setminus \{p_1, \ldots, p_r\}$.

The divisor class group $\mathrm{Cl}(X)$ (of divisors modulo linear equivalence, where a divisor is an element of the free abelian group on the irreducible curves on X) is the free group with basis $e_0, e_1, \ldots, e_r$, where e_0 is the class of the pullback E_0 to X of a line $L \subset \mathbb{P}^2$, and e_i for $i > 0$ is the class of the curve E_i. The group $\mathrm{Cl}(X)$ comes with a bilinear form, called the intersection form, defined as $-e_0^2 = e_i^2 = -1$ for all $i > 0$, and $e_i \cdot e_j = 0$ for $i \neq j$. An important element, known as the canonical class, is $K_X = -3e_0 + e_1 + \cdots + e_r$. If C and D are divisors, we define $C \cdot D = [C] \cdot [D]$. If C and D are prime divisors meeting transversely, then $C \cdot D$ is just the number of points of intersection of C with D.

If D is a divisor on X, its class can be written as $[D] = de_0 - \sum_i m_i e_i$ for some integers d and m_i. Associated to D is an invertible sheaf $\mathcal{O}_X(D)$. The space of global sections of this sheaf is a finite dimensional K-vector space, denoted $\Gamma(\mathcal{O}_X(D))$ and also $H^0(X, \mathcal{O}_X(D))$. The dimension of this vector space is denoted $h^0(X, \mathcal{O}_X(D))$; if $[D] = [D']$, then $h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D'))$.

In case $D = dE_0 - \sum_i m_i E_i$ such that each $m_i \geq 0$, then there is a canonical identification of $H^0(X, \mathcal{O}_X(D))$ with $I(m_1p_1+\cdots+m_rp_r)_d$ [H3, Proposition IV.1.1]. Thus techniques for computing $h^0(X, \mathcal{O}_X(D))$ can be applied to computing the Hilbert function of $m_1p_1+\cdots+m_rp_r$. One important tool is the theorem of Riemann-Roch for surfaces; see Exercise 7.2. Bézout's Theorem also has a natural interpretation in this context. If C and D are effective divisors such that $[C] = c_0e_0 - c_1e_1 - \cdots - c_re_r$ and $[d] = d_0e_0 - d_1e_1 - \cdots - d_re_r$, then $C \cdot D = c_0d_0 - c_1d_1 - \cdots - c_rd_r$; if this is negative then C and D have a common component. In particular, if C is a prime divisor, then C itself is the common component, hence $D - C$ is effective.

Another important technique involves a group action on $\mathrm{Cl}(X)$ related to the Cremona group of birational transformations of the plane. Given $\pi : X \to \mathbb{P}^2$ as above, there can exist morphisms $\pi' : X \to \mathbb{P}^2$ obtained by blowing up other points (possibly infinitely near) $p'_1, \ldots, p'_r \in \mathbb{P}^2$. The composition $\pi'\pi^{-1}$, defined away from the points p_i, is a birational transformation of $\mathbb{P}^2$, hence an element of the Cremona group (named for Luigi Cremona, after whom there is named a street in Rome near the Colosseum). We thus have a second basis $e'_0, e'_1, \ldots, e'_r$ of $\mathrm{Cl}(X)$ corresponding to curves E'_i. In particular, we can write $dE_0 - \sum_i m_iE_i$ as $d'E'_0 - \sum_i m'_iE'_i$. The change of basis transformation from the basis e_i to the basis e'_i is always an element of a particular group, now known as the Weyl group, W_r. For $r < 9$, W_r is finite, but it is infinite for all $r \geq 9$.

Example 7.1. Consider the quadratic Cremona transformation on $\mathbb{P}^2$, defined away from $x_0x_1x_2 = 0$ as $Q : (a, b, c) \mapsto (1/a, 1/b, 1/c)$. Alternatively, one can define it at all points of $\mathbb{P}^2$ except $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ as $(a, b, c) \mapsto (bc, ac, ab)$. It can also be obtained by as $\pi'\pi^{-1}$, where $\pi : X \to \mathbb{P}^2$ is the morphism given by blowing up the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and $\pi' : X \to \mathbb{P}^2$ contracts the proper transforms of the lines through pairs of those points. More generally one can define the quadratic transform at any three noncolinear points, by blowing them up and blowing down the proper transforms of the lines through pairs of the 3 points. An important theorem announced by M. Noether (but whose proof was felt to be incomplete), is that the Cremona group for $\mathbb{P}^2$ is generated by invertible linear transformations of the plane and quadratic transformations [A].

When the points p_i are sufficiently general (such as being generic, meaning, say, that the projective coordinates a_{ij} for each point $p_i = (a_{i0}, a_{i1}, a_{i2})$ are all nonzero, and the ratios $\frac{a_{11}}{a_{10}}$, $\frac{a_{12}}{a_{10}}$, $\frac{a_{21}}{a_{20}}$, $\frac{a_{22}}{a_{20}}, \dots, \frac{a_{r1}}{a_{r0}}$, $\frac{a_{r2}}{a_{r0}}$ are algebraically independent over the prime field of K) and given the surface $\pi : X \to \mathbb{P}^2$ obtained by blowing up the points p_i, the birational morphisms $X \to \mathbb{P}^2$ (up to projective equivalence) are in one-to-one correspondence with the elements of W_r. We denote by π_w the morphism corresponding to w. The identity element w corresponds to the basis $\{e_0, e_1, \dots, e_r\}$ obtained by blowing up the points p_i, and this gives π since for $i > 0$, E_i is the unique effective divisor whose class is e_i. Contracting $E_r, E_{r-1}, \dots, E_1$ in order gives π. Likewise, for any $w \in W_r$, the basis $e'_i = w(e_i)$ gives the sequence of curves E'_i which must be contracted to define π_w.

Let $n_0 = e_0 - e_1 - e_2 - e_3$ and let $n_i = e_1 - e_{i+1}$ for $i = 1, \dots, r - 1$. For any $x \in \mathrm{Cl}(X)$ and any $0 \leq i < r$, let $s_i(x) = x + (x \cdot n_i)n_i$. Then $s_i \in W_r$ and these generate W_r. When $i > 0$, the element s_i just transposes e_i and e_{i+1}, so $\{s_1, \dots, s_{r-1}\}$ generates the group of permutations on the set $\{e_1, \dots, e_r\}$. The element s_0 corresponds to the quadratic transformation $Q : (a, b, c) \mapsto (\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$. Note that $s_0(e_1) = e_0 - e_2 - e_3$, $s_0(e_2) = e_0 - e_1 - e_3$, and $s_0(e_3) = e_0 - e_1 - e_2$: blowing up p_1, p_2 and p_3, to get $E_1, E2, E_3$ and blowing down the proper transforms of the line through p_2 and p_3, the line through p_1 and p_3 and the line through p_1 and p_2 is precisely Q. (Note also that $s_0(e_0) = 2e_0 - e_1 - e_2 - e_3$ and a line $a_0x_0 + a_1x_1 + a_2x_2 = 0$ pulls back under Q to $a_0/x_0 + a_1/x_1 + a_2/x_2 = 0$ which, by multiplying through by $x_0x_1x_2$ to clear the denominators is the same as $a_0x_1x_2 + a_1x_0x_2 + a_2x_0x_1 = 0$; i.e., on the surface X obtained by blowing up the coordinate vertices we have $e'_0 = 2e_0 - e_1 - e_2 - e_3$.)

Given a divisor $F = dE_0 - \sum_i m_iE_i$, we denote by wF the divisor $d'E'_0 - \sum_i m'_iE'_i$ where $w(de_0 - \sum_i m_ie_i) = d'e'_0 - \sum_i m'_ie'_i$. Since w represents a change of basis, we have $H^0(X, \mathcal{O}_X(F)) = H^0(X, \mathcal{O}_X(wF))$ and thus $\dim I(\sum_i m_ip_i)_d = \dim I(\sum_i m'_ip'_i)_{d'}$. (The fact that $H^0(X, \mathcal{O}_X(F)) = H^0(X, \mathcal{O}_X(wF))$ also shows that $I(\sum_i m_ip_i)_d$ has an irreducible element if and only if $I(\sum_i m'_ip'_i)_{d'}$ does.) But if the points p_i are generic, so are the points p'_i (up to projective equivalence), so $\dim I(\sum_i m_ip_i)_d = \dim I(\sum_i m'_ip_i)_{d'}$. (There is an automorphism $\phi : K \to K$ such that the coordinates of the points p_i map to the coordinates of the points p'_i. This induces an invertible map $\Phi : I(\sum_i m'_ip_i)_{d'} \to I(\sum_i m'_ip'_i)_{d'}$ such that if $a_i \in K$ and $F_i \in I(\sum_i m'_ip_i)_{d'}$, then $\Phi(\sum_i a_iF_i) = \sum_i \phi(a_i)\Phi(F_i)$, from which it follows that $\dim I(\sum_i m'_ip_i)_{d'} = \dim I(\sum_i m'_ip_i)_{d'}$ and hence that $\dim I(\sum_i m_ip_i)_d = \dim I(\sum_i m'_ip_i)_{d'}$.)

Example 7.2. Let $p_1, \dots, p_8$ be generic points of $\mathbb{P}^2$. We show that $I(p_1 + \cdots + p_5)_2$, $I(2p_1 + p_2 + \cdots + p_7)_3$ and $I(3p_1 + 2p_2 + \cdots + 2p_8)_6$ each are 1-dimensional, with basis given by an irreducible form. In each case we have a homogeneous component of the form $I(\sum_i m_ip_i)_d$. It is enough to show that there is an element $w \in W_8$ such that $w[F] = e_0 - e_1 - e_2$, where $[F] = de_0 - \sum_i m_ie_i$. But $s_0(2e_0 - e_1 - \cdots - e_5 = e_0 - e_4 - e_5)$ and we apply a permutation σ to obtain $\sigma(e_0 - e_4 - e_5) = e_0 - e_1 - e_2$. Thus $\dim I(p_1 + \cdots + p_5)_2 = \dim I(p_1 + p_2)_1$ and since $I(p_1 + p_2)_1$ clearly has an irreducible element so does $I(p_1+\cdots+p_5)_2$. The other cases with $r<9$ are similar. The case that $r=9$ is also similar if we show that $I(p_1+\cdots+p_9)_3$ has an irreducible element.

Exercises

Exercise 7.1. Let X be the blow up of $\mathbb{P}^2$ at r distinct points. Show that $w(x)\cdot w(y)=x\cdot y$ for all $x,y\in \mathrm{Cl}(X)$ and all $w\in W_r$, and show that $w(K_X)=K_X$ for all $w\in W_r$, where $K_X=-3e_0+e_1+\cdots+e_r$.

Exercise 7.2. Let X be the blow up of $\mathbb{P}^2$ at s distinct points $p_i\in\mathbb{P}^2$. Let $F=tE_0-m_1E_1-\cdots-m_sE_s$. The theorem of Riemann-Roch for surfaces says that

$$h^0(X,\mathcal{O}_X(F))-h^1(X,\mathcal{O}_X(F))+h^2(X,\mathcal{O}_X(F))=\frac{F^2-K_X\cdot F}{2}+1.$$

Serre duality says that $h^2(X,\mathcal{O}_X(F))=h^0(X,\mathcal{O}_X(K_X-F))$, and hence that $h^2(X,\mathcal{O}_X(F))=0$ if $t\geq 0$. Thus for $t\geq 0$ and $m_i\geq 0$ for all i, taking $I=I(m_1p_1+\cdots+m_sp_s)$, we have $H_I(t)=h^0(X,\mathcal{O}_X(F))=\frac{F^2-K_X\cdot F}{2}+1+h^1(X,\mathcal{O}_X(F))$. Show that

$$\frac{F^2-K_X\cdot F}{2}+1=\binom{t+2}{2}-\sum_i\binom{m_i+1}{2}+h^1(X,\mathcal{O}_X(F)).$$

Conclude that $P_I(t)=\frac{F^2-K_X\cdot F}{2}+1$ where P_I is the Hilbert polynomial for I, and that $h^1(X,\mathcal{O}_X(F))=H_I(t)-P_I(t)$ is the difference between the Hilbert function and Hilbert polynomial for I.

Exercise 7.3. Let $I=I(p_1+\cdots+p_r)$ for generic points $p_i\in\mathbb{P}^2$. If $r=5$, show that $\gamma(I)=2$, if $r=6$ show that $\gamma(I)=12/5$, if $r=7$ show that $\gamma(I)=21/8$, if $r=8$ show that $\gamma(I)=48/17$ and if $r=9$, show $\gamma(I)=3$.

Exercise 7.4. Let $p_1,\ldots,p_8$ be generic points of $\mathbb{P}^2$. Show that $\alpha(I(6p_1+\cdots+6p_8))=17$.

Exercise 7.5. Let $p_1,\cdots,p_r\in\mathbb{P}^2$ be generic points of $\mathbb{P}^2$. Let X be the surface obtained by blowing up the points. Let $w\in W_r$ and let $[C]=w(e_1)$. Show that C is a smooth rational curve with $C^2=C\cdot K_X=1$. Conclude that $((mC)^2-K_X\cdot(mC))/2+1\leq 0$ for all $m>1$. Such a curve C is called an *exceptional curve*. (By [N2, Theorem 2b], when $r\geq 3$, the set of classes of exceptional curves is precisely the orbit $W_r(e_1)$.)

Exercise 7.6. Let $p_1,\cdots,p_r\in\mathbb{P}^2$ be distinct points of $\mathbb{P}^2$. Let X be the surface obtained by blowing up the points. Let C be an exceptional curve on X, let D be an effective divisor, let $m=-C\cdot D>0$ and let $F=D-mC$. If $m>1$, show that $h^0(X,\mathcal{O}_X(D)=h^0(X,\mathcal{O}_X(F))$ (hence C is a *fixed component* of $|D|$ of multiplicity m, where $|D|$ is the linear system of all curves corresponding to elements of $H^0(X,\mathcal{O}_X(F)))$, and that $(D^2-K_X\cdot D)/2<(F^2-K_X\cdot F)/2$; conclude that $h^0(X,\mathcal{O}_X(D)>(D^2-K_X\cdot D)/2+1$.

8. The SHGH Conjecture

The SHGH Conjecture [Se, H2, G1, Hi] gives an explicit conjectural value for the Hilbert function of the ideal of a fat point subscheme of $\mathbb{P}^2$ supported at generic (or even just sufficiently general) points.

Consider I_4 where I is the ideal of the fat point subscheme $3p_1+3p_2+p_3+p_4\subset\mathbb{P}^2$. Let $D=4E_0-3E_1-3E_2-E_3-E_4$ and let $C=E_0-E_1-E_2$. Note that $D\cdot C=-2$; let $F=D-2C=2E_0-E_1-\cdots-E_4$. We know $H_I(4)=h^0(X,\mathcal{O}_X(D))\geq(D^2-K_X\cdot D)/2+1=\binom{4+2}{2}-2\binom{3+1}{2}-2\binom{1+1}{2}=1$. But by Exercise 7.6 we also have

$$H_I(4)=h^0(X,\mathcal{O}_X(F))\geq(F^2-K_X\cdot F)/2+1=2.$$

The occurrence of C as a fixed component of $|D|$ of multiplicity more than 1 results in a strict inequality $h^0(X, \mathcal{O}_X(D)) > (D^2 - K_X \cdot D)/2 + 1$.

The SHGH Conjecture says that whenever we have a divisor $D = dE_0 - m_1E_1 - \cdots - m_rE_R$ with $d, m_1, \ldots, m_r \geq 0$, (assuming that the E_i were obtained by blowing up $r \geq 3$ generic points of $\mathbb{P}^2$) then either $h^0(X, \mathcal{O}_X(D)) = \max(0, (D^2 - K_X \cdot D)/2 + 1)$ or there is an exceptional curve C (i.e., an effective divisor whose class is an element of the W_r-orbit of E_1) such that $C \cdot D < -1$. If $h^0(X, \mathcal{O}_X(D)) > 0$, it is easy to find all such C and subtract them off, leaving one with F such that $h^0(X, \mathcal{O}_X(F)) = (F^2 - K_X \cdot F)/2 + 1$. (If $D \cdot C \geq 0$ for all C, one can show that $[D]$ can be reduced by W_r to a nonnegative linear combination of the classes e_0, $e_0 - e_1$, $2e_0 - e_1 - e_2$, $3e_0 - e_1 - e_2 - e_3, \cdots, 3e_0 - e_1 - \cdots - e_r$; see [H1].)

The SHGH Conjecture is known to hold for $r \leq 9$.

Example 8.1. Consider the fat point subscheme $Z = 13p_1 + 13p_2 + 10p_3 + \cdots + 10p_7$ for generic points $p_i \in \mathbb{P}^2$. We determine the Hilbert function of $I = I(Z)$. First $H_I(28) = 0$. We have $H_I(28) = h^0(X, \mathcal{O}_X(D))$ for the divisor $D = 28E_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7$. But $[D]$ reduces via W_7 to $-2e_0 + 2e_4 + 2e_5 + 5e_6 + 5e_7$, so $h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D'))$, where $D' = -2E_0 + 2E_4 + 2E_5 + 5E_6 + 5E_7$. The occurrence of a negative coefficient for e_0 means $h^0(X, \mathcal{O}_X(D')) = 0$, hence $H_I(t) = 0$ for $t < 29$. Now consider $D = 29E_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7$. Then via the action of W_7 we obtain $D' = 4E_0 - E_1 - \cdots - E_5 + 2E_6 + 2E_7$. As in Exercise 7.6, we can subtract off $2E_6 + 2E_7$ to get $F = D - (2E_6 + 2E_7) = 4E_0 - E_1 - \cdots - E_5 = (E_0) + (3E_0 - E_1 - \cdots - E_5)$. Thus $F \cdot C \geq 0$ for all exceptional C, so by the SHGH Conjecture $H_I(29) = h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D')) = h^0(X, \mathcal{O}_X(F)) = (F^2 - K_X \cdot F)/2 + 1 = 10$. Finally consider $D = 30E_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7$. Here we get $F = D' = 12E_0 - 4(E_1 + \cdots + E_5) - E_6 - E_7 = 3(3E_0 - E_1 - \cdots - E_5) + (3E_0 - E_1 - \cdots - E_7)$. Thus $D' \cdot C \geq 0$ for all exceptional C, so we get $H_I(30) = h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D')) = h^0(X, \mathcal{O}_X(F)) = (F^2 - K_X \cdot F)/2 + 1 = 39$. For $t \geq 30$ and $D = tE_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7$, we have $D = (t-30)E_0 + (30E_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7)$ so $D \cdot C = (t-30)E_0 \cdot C + C \cdot (30E_0 - 13E_1 - 13E_2 - 10E_3 - \cdots - 10E_7) \geq 0$ for all exceptional C, so $H_I(t) = h^0(X, \mathcal{O}_X(D)) = \max(0, (D^2 - K_X \cdot D)/2 + 1)$, but $(D^2 - K_X \cdot D)/2 + 1$ was positive for $t = 30$ and adding a nonnegative multiple of E_0 only makes it bigger so we have $H_I(t) = h^0(X, \mathcal{O}_X(D)) = (D^2 - K_X \cdot D)/2 + 1 = \binom{t+2}{2} - 2\binom{13+1}{2} - 5\binom{10+1}{2}$.

Exercises

Exercise 8.1. Find the Hilbert function of the ideal I of $Z = 12p_1 + 10p_2 + \cdots + 10p_8 \subset \mathbb{P}^2$, assuming the points are generic.

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