LECTURE ONE

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In what follows, $X \subset \mathbb{P}^N$ will denote an irreducible, reduced algebraic variety; we work over an algebraically closed field of characteristic zero, which we assume to be $\mathbb{C}$.

The topic of these lecture are *higher secant varieties* of X.

Definition 0.1. The s-th higher secant variety of X is

$$\sigma_s(X) = \overline{\bigcup_{P_1,\ldots,P_s \in X} \langle P_1, \ldots, P_s \rangle},$$

where the over bar denotes the Zariski closure.

In words, $\sigma_s(X)$ is the closure of the union of s-secant spaces to X.

Example 0.2. If $X \subset \mathbb{P}^2$ is a curve (not a line) then $\sigma_2(X) = \mathbb{P}^2$, the same is true for hypersurfaces which are not hyperplanes. But, if $X \subset \mathbb{P}^3$ is a non-degenerate curve (i.e. not contained in a hyperplane), then $\sigma_2(X)$ can be, in principle, either a surface or a threefold.

We note that the closure operation is in general necessary, but there are cases in which it is not.

Exercise 0.3. Show that the union of chords (secant lines) to a plane conic is closed. However, the union of the chords of the twisted cubic curve in $\mathbb{P}^3$ is not.

In general, we have a sequence of inclusions

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \ldots \subseteq \sigma_r(X) \subseteq \ldots \subseteq \mathbb{P}^N.$$

If X is a projective space, then $\sigma_i(X) = X$ for all i and all of the elements of the sequence are equal.

Remark 0.4. If $X = \sigma_2(X)$ then X is a projective space. To see this consider a point $P \in X$ and the projection map $\pi_P : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$. Let $X_1 = \pi_P(X)$ and notice that $\dim X_1 = \dim X - 1$ and that $\sigma_2(X_1) = X_1$. If X_1 is a projective space also X is and we are done. Otherwise iterate the process constructing a sequence of varieties $X_2, \ldots, X_m$ of decreasing dimension. The process will end with X_m equal to a point and then X_{m-1} a projective space. Thus X_{m-2} is a projective space and so on up to the original variety X.

Exercise 0.5. For $X \subset \mathbb{P}^N$, show that, if $\sigma_i(X) = \sigma_{i+1}(X) \neq \mathbb{P}^N$, then $\sigma_i(X)$ is a projective space and hence $\sigma_j(X) = \sigma_i(X)$ for all $j \geq i$.

Using this remark and the exercise, we can refine our chain of inclusions for X a non degenerate variety (not contained in a hyperplane)

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \ldots \subset \sigma_r(X) = \mathbb{P}^N.$$

In particular, all inclusions are strict and there is a higher secant variety which coincides with the ambient space.

It is natural to ask: what is the smallest r such that $\sigma_r(X) = \mathbb{P}^N$? Or more generally: what is the value of $\sigma_i(X)$ for all i?

As a preliminary move in this direction, we notice that there is an expected value for the dimension of any higher secant variety of X.

Definition 0.6. For $X \subset \mathbb{P}^n$, set $n = \dim X$. The *expected dimension* of $\sigma_s(X)$ is

$$\mathrm{expdim}(\sigma_s(X)) = \min\{sn + s - 1, N\}.$$

Example 0.7. If X is a curve, then $\mathrm{expdim}(\sigma_2(X)) = 2$, if the curve is a plane curve, and $\mathrm{expdim}(\sigma_2(X)) = 3$ otherwise. This is why any curve is isomorphic to a curve in $\mathbb{P}^3$, but only birational to a plane curve.

There are cases in which $\mathrm{expdim}(\sigma_i(X)) \neq \dim(\sigma_i(X))$ and these motivate the following

Definition 0.8. If $\mathrm{expdim}(\sigma_i(X)) \neq \dim(\sigma_i(X))$ then X is said to be *i-defective.*

Remark 0.9. Notice that $\dim(\sigma_{i+1}(X)) \leq \dim(\sigma_i(X)) + n + 1$, where $n = \dim X$. This means that if $\sigma_i(X) \neq \mathbb{P}^N$ and X is i-defective, then X is j-defective for $j \leq i$.

Let's now see the most celebrated example of a defective variety, the Veronese surface in $\mathbb{P}^5$.

Example 0.10. Consider the polynomial ring $S = \mathbb{C}[x, y, z]$ and its homogeneous pieces S_d. The Veronese map ν_2 is defined as follows

$$\mathbb{P}(S_1) \longrightarrow \mathbb{P}(S_2)$$
$$[L] \mapsto [L^2]$$

and it can be described in coordinates by fixing the standard monomial basis in S_1 and the following basis in S_2

$$\langle x^2, 2xy, 2xz, y^2, 2yz, z^2\rangle.$$

Thus the Veronese map can be written as

$$\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$
$$[a : b : c] \mapsto [a^2 : ab : ac : b^2 : bc : c^2].$$

The Veronese surface in $\mathbb{P}^2$ is then defined as the image of this map, i.e. the Veronese surface is $X = \nu_2(\mathbb{P}^2)$.

We now want to study higher secant varieties of X, and in particular we ask: is $\dim \sigma_2(X) = \mathrm{expdim}\sigma_2(X) = 5$? In other words, is $\sigma_2(X) = \mathbb{P}^5$?

It is useful to notice that elements in S_2 are quadratic forms, and hence they are uniquely determined by 3×3 symmetric matrices. In particular, $P \in \mathbb{P}^5$ can be seen as $P = [Q]$ where Q is a 3×3 symmetric matrix. If $P \in X$ then Q also has rank equal one. Thus we have,

$$\sigma_2(X) = \overline{\bigcup_{P_1, P_2} \langle P_1, P_2\rangle}$$
$$= \overline{\{[Q_1 + Q_2] : Q_i \text{ is a } 3 \times 3 \text{ symmetric matrix and } \mathrm{rk}(Q_i) = 1\}}$$
$$\subseteq H = \{3 \times 3 \text{ symmetric matrices of rank at most two}\}.$$

Clearly H is the hypersurface defined by the vanishing of the determinant of the general 3×3 symmetric matrix and hence X is 2-defective.

Exercise 0.11. Show that $H = \sigma_2(X)$.

Exercise 0.12. Repeat the same argument for $X = \nu_2(\mathbb{P}^3)$. Is X 2-defective?

In order to deal with the problem of studying the dimension of the higher secant varieties of X we need to introduce a celebrated tool, namely Terracini's Lemma.

Lemma 0.13 (Terracini's Lemma)**.** *Let $P_1, \ldots, P_s \in X$ be general points and $P \in \langle P_1, \ldots, P_s \rangle \subset \sigma_s(X)$ be a general point. Then the tangent space to $\sigma_s(X)$ in P is*

$$T_P(\sigma_s(X)) = \langle T_{P_1}(\sigma_s(X)), \ldots, T_{P_s}(\sigma_s(X)) \rangle.$$

Remark 0.14. To get a (affine) geometric heuristic idea of why Terracini's Lemma holds, we consider an affine curve $\gamma(t)$. A general point on $P \in \sigma_2(\gamma)$ is described as $\gamma(s_0) + \lambda_0[\gamma(t_0) - \gamma(s_0)]$. A neighborhood of P is then described as $\gamma(s) + \lambda[\gamma(t) - \gamma(s)]$. Hence the tangent space $T_P(\sigma_s(\gamma))$ is spanned by

$$\gamma'(s_0) - \lambda_0\gamma'(s_0), \lambda_0\gamma'(t_0), \gamma(t_0) - \gamma(s_0)$$

As a first application of Terracini's Lemma, we consider the twisted cubic curve.

Example 0.15. Let X be the twisted cubic curve in $\mathbb{P}^3$, i.e. $X = \nu(\mathbb{P}^1)$ where ν is the map

$$\nu : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

We want to compute $\dim \sigma_2(X) = \dim T_P(\sigma_2(X))$ at a generic point P. Using Terracini's Lemma it is enough to choose generic points $P_1, P_2 \in X$ and to study the linear span

$$\langle T_{P_1}(X), T_{P_1}(X) \rangle.$$

In particular, $\sigma_2(X) = \mathbb{P}^3$ if and only if the lines $T_{P_1}(X)$ and $T_{P_2}(X)$ do not intersect, that is, if and only if there does not exist a hyperplane containing both lines.

If $H \subset \mathbb{P}^3$ is a hyperplane the points of $H \cap X$ are determined by finding the roots of the degree three homogeneous polynomial $g(s,t)$ defining $\nu^{-1}(H) \subset \mathbb{P}^1$. If $H \supset T_{P_1}(X)$ then g has a double root. Thus, no hyperplane exists containing both tangent lines.

In conclusion, $\sigma_2(X) = \mathbb{P}^3$.

Exercise 0.16. Prove that, if $H \supset T_P(X)$ then the polynomial defining $\nu^{-1}(H)$ has a double root.

We now introduce the Veronese variety in general.

Definition 0.17. Consider the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ and its homogeneous pieces S_d. The *d-th Veronese map* ν_d is defined as follows

$$\mathbb{P}(S_1) \longrightarrow \mathbb{P}(S_d)$$
$$[L] \mapsto [L^d]$$

and it can be described in coordinates by using a monomial basis of S_d

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$
$$[x_0 : \ldots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : x_0^{d-1}x_2 : \ldots : x_n^d],$$

where $N = \binom{n+d}{d} - 1$.

We call $\nu_d(\mathbb{P}^n)$ a *Veronese variety.*

Example 0.18. A relevant family of Veronese varieties are the *rational normal curves* which are Veronese varieties of dimension one, i.e $n = 1$. In this situation $S = \mathbb{C}[x_0, x_1]$ and S_d is the vector space of degree d binary forms. The rational normal curve $\nu_d(\mathbb{P}S_1) \subseteq \mathbb{P}S_d$ is represented by d-th powers of binary linear forms.

Example 0.19. The rational normal curve $X = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$ is an irreducible conic. It is easy to see that $\sigma_2(X) = \mathbb{P}^2 = \mathbb{P}S_2$. This equality can also be explained by saying that any binary quadratic form Q is the sum of two squares of linear forms, i.e. $Q = L^2 + M^2$.

Exercise 0.20. Consider the rational normal curve in $\mathbb{P}^3$, i.e. the twisted cubic curve $X = \nu_3(\mathbb{P}S_1) \subset \mathbb{P}S_3$. We know that $\sigma_2(X)$ fills up all the space. Can we write any binary cubic as the sum of two cubes of linear forms? Try $x_0x_1^2$.

Exercise 0.21. We described the veronese variety $X = \nu_d(\mathbb{P}^n)$ in parametric form by means of the relation: $[F] \in X$ if and only if $F = L^d$. Use this description and standard differential geometry to compute $T_{[L^d]}(X)$ (describe this as vector space of homogeneous polynomials). This can be used to apply Terracini's Lemma, for example, to the twisted cubic curve.

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