LECTURE TWO

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In the last lecture we spoke about higher secant varieties in general. Now we focus on the special case of Veronese varieties. Throughout this lecture we will consider the polynomial ring $S = \mathbb{C}[x_0, \dots, x_n]$.

An explicit description of the tangent space to a Veronese variety will be useful, so we give it here.

Remark 0.1. Let $X = \nu_d(\mathbb{P}^n)$ and consider $P = [L^d] \in X$ where $L \in S_1$ is a linear form. Then

$$T_P(X) = \langle [L^{d-1}M] : M \in S_1 \rangle.$$

We can use this to revisit the Veronese surface example.

Example 0.2. Consider the Veronese surface $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. To compute $\dim \sigma_2(X)$ we use Terracini's Lemma. Hence we choose two general points $P = [L^2], Q = [N^2] \in X$ and we consider the linear span of their tangent spaces

$$T = \langle T_P(X), T_Q(X) \rangle.$$

By applying Grassmann's formula, and noticing that $T_P(X) \cap T_Q(X) = [LN]$ we get $\dim T = 3 + 3 - 1 - 1 = 4$ and hence $\sigma_2(X)$ is a hypersurface.

The study of higher secant varieties of Veronese varieties is strictly connected with a problem in polynomial algebra: the *Waring problem for forms*, i.e. for homogeneous polynomials. We begin by introducing the notion of Waring rank.

Definition 0.3. Let $F \in S$ be a degree d form. The *Waring rank of F* is denoted $\mathrm{rk}(F)$ and it is the **minimum** s such that we have

$$F = L_1^d + \dots + L_s^d$$

for linear forms $L_i \in S_1$.

Remark 0.4. It is clear that $\mathrm{rk}(L^d) = 1$ if L is a linear form, however $\mathrm{rk}(L^d + N^d) \leq 2$: it is 1 if L and N are proportional and 2 otherwise. For more than two factors the computation of the Waring rank for a sum of powers of linear form is not trivial.

We can now state the *Waring problem for forms*, which actually comes in two fashions. The **big** Waring problem asks for the computation of

$$g(n, d)$$

the minimal integer such that

$$\mathrm{rk}(F) \leq g(n, d)$$

for the **generic** element of S_d, i.e. for the generic degree d form in $n + 1$ variables. The **little** Waring problem is more ambitious and asks us to determine the smallest integer

$$G(n, d)$$

such that

$$\mathrm{rk}(F) \leq G(n, d)$$

for **any** element of S_d.

Remark 0.5. To understand the difference between the big and the little Waring problem we can refer to a probabilistic description. Pick a random element $F \in S_d$, then with *probability one* $\mathrm{rk}(F) \leq g(n, d)$ (actually they will be equal). However, if the choice of F is unlucky, it could be that $\mathrm{rk}(F) > g(n, d)$.

Remark 0.6. To make precise the notion of generic element, we use topology. The big Waring problem asks us to bound the Waring rank for all elements belonging to a non-empty Zariski open subset of $\mathbb{P}S_d$; as this subset would also be dense, this explains the probabilistic interpretation.

The big Waring problem has a nice geometric interpretation using Veronese varieties, and this interpretation allows for a complete solution of the problem. Also the little Waring problem has a geometric aspect, but this problem, in its full generality, is still unsolved.

Remark 0.7. As the Veronese variety $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ parameterizes pure powers in S_d, it is clear that $g(n, d)$ is the smallest s such that $\sigma_s(X) = \mathbb{P}^N$. Thus solving the big Waring problem is equivalent to finding the smallest higher secant variety of X filling up all the space. As the Zariski closure is involved in defining X, this is not the same as solving the little Waring problem.

Remark 0.8. To solve the little Waring problem one has to find the smallest s such that every single element $[F] \in \mathbb{P}S_d$ lies on the span of s points on X.

Let's consider two examples to better understand the difference between the two problems.

Example 0.9. Let $X = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$ be the rational normal curve. We know that $\sigma_2(X) = \mathbb{P}^2$ and hence $g(n = 1, d = 2) = 2$. But we also know that each point of $\mathbb{P}^2$ lies on the span of two distinct points of X, thus $G(n = 1, d = 2) = 2$. In particular this means that the Waring rank of a binary quadratic form is always at most two.

Example 0.10. Let $X = \nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ be the rational normal curve. Again, we know that $\sigma_2(X) = \mathbb{P}^3$ and hence $g(n = 1, d = 3) = 2$. However, there are degree three binary forms F such that $\mathrm{rk}(F) = 3$, and actually $G(n = 1, d = 3) = 3$. To understand which are the bad forms, consider the projection map π_P from any point $P = [F] \in \mathbb{P}^3$. Clearly, if $P \notin X$, $\pi_p(X)$ is a degree 3 rational plane curve. Hence, it is singular, and being irreducible, only two possibilities arise. If the singularity is a node, then $P = [F]$ lies on a chord of X, and thus $F = L^3 + N^3$. But, if the singularity is a cusp, this is no longer true as P lies on a tangent line to X and not on a chord. Thus, the bad binary cubics are of the form L^2N.

Exercise 0.11. For binary forms, we can stratify $\mathbb{P}S_2$ using the Waring rank: rank one elements correspond to points of the rational normal curve, while all the points outside the curve have rank two. Do the same for binary cubics and stratify $\mathbb{P}S_3 = \mathbb{P}^3$.

We can produce a useful interpretation of Terracini's Lemma in the case of Veronese varieties. We consider the Veronese variety $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$.

Remark 0.12. *If $H \subset \mathbb{P}^N$ is a hyperplane, then $\nu_d^{-1}(H)$ is a degree d hypersurface.* To see this, notice that H has an equation of the kind $a_0z_0 + \ldots + a_Nz_N$ where z_i are the coordinates of $\mathbb{P}^N$. To determine an equation for $\nu_d^{-1}(H)$ it is enough to substitute each z_i with the corresponding degree d monomial in the $x_0, \ldots, x_n$.

Remark 0.13. *If $H \subset \mathbb{P}^N$ is a hyperplane and $[L^d] \in H$, then $\nu_d^{-1}(H)$ is a degree d hypersurface passing through the point $[L] \in \mathbb{P}^n$.* This is clearly true as $\nu_d^{-1}([L^d]) = [L]$.

Remark 0.14. *If $H \subset \mathbb{P}^N$ is a hyperplane such that $T_{[L^d]}(X) \subset H$, then $\nu_d^{-1}(H)$ is a degree d hypersurface singular at the point $[L] \in \mathbb{P}^n$.* This can be seen using *apolarity* or by direct computation choosing $L^d = x_0^d$.

We illustrate the last remark in an example.

Example 0.15. Consider X the Veronese surface of $\mathbb{P}^5$, $P = [1 : 0 : 0 : 0 : 0 : 0] = [x^2] \in X$ and let $\mathbb{C}[z_0, z_1, \ldots, z_5]$ be the coordinate ring of $\mathbb{P}^5$. If H is a hyperplane containing P then H has equation

$$0z_0 + a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 + a_5z_5 = 0.$$

and hence $\nu_2^{-1}(H)$ is the plane conic of equation

$$a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 = 0,$$

which passes through the point $\nu^{-1}(P) = [1 : 0 : 0]$. The tangent space $T_P(X)$ is the linear span of the forms

$$x^2, xy, xz$$

and hence it is the linear span of the points

$$[1 : 0 : 0 : 0 : 0 : 0], [0 : 1 : 0 : 0 : 0 : 0], [0 : 0 : 1 : 0 : 0 : 0].$$

Thus, if $H \supset T_P(X)$ then $a_1 = a_2 = 0$ and the corresponding conic has equation

$$a_3y^2 + a_4yz + a_5z^2 = 0,$$

which is singular at the point $[1 : 0 : 0]$.

Exercise 0.16. Repeat the argument above to prove the general statement: if $T_{[L^d]}(\nu_d^{-1}(\mathbb{P}^n)) \subset H$, then $\nu_d^{-1}(H)$ is a degree d hypersurface singular at the point $[L] \in \mathbb{P}^n$.

We will now elaborate on the connection between double points and higher secant varieties to Veronese varieties.

Definition 0.17. Let $P_1, \ldots, P_s \in \mathbb{P}^n$ be points with defining ideals $\wp_1, \ldots, \wp_s$ respectively. The scheme defined by the ideal $\wp_1^2 \cap \ldots \cap \wp_s^2$ is called a 2-*fat point* scheme or a *double point* scheme.

Remark 0.18. Let $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$. There is a bijection between

$$\{H \subset \mathbb{P}^N \text{ a hyperplane } : H \supset T_{P_1}(X), \ldots, T_{P_s}(X)\}$$

and

$$\{\text{degree } d \text{ hypersurface of } \mathbb{P}^n \text{ singular at } P_1, \ldots, P_s\} = (\wp_1^2 \cap \ldots \cap \wp_s^2)_d$$

Using the double point interpretation of Terracini's Lemma we get the following criterion to study the dimension of higher secant varieties to Veronese varieties.

Lemma 0.19. *Let $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ and choose generic points $P_1, \ldots, P_s \in \mathbb{P}^n$ with defining ideals $\wp_1, \ldots, \wp_s$ respectively. Then*

$$\dim \sigma_s(X) = N - \dim(\wp_1^2 \cap \ldots \cap \wp_s^2)_d$$

Example 0.20. We consider, again, X the Veronese surface in $\mathbb{P}^5$. To determine $\dim \sigma_2(X)$ we choose generic points $P_1, P_2 \in \mathbb{P}^2$ and we look for conics singular at both points, i.e. elements in $(\wp_1^2 \cap \wp_s^2)_2$. Exactly one such conic exists (the line through P_1 and P_2 doubled) and hence $\sigma_2(X)$ is a hypersurface.

Exercise 0.21. Solve the big Waring problem for $n = 1$ using the double points interpretation.

We now go back to the big Waring problem. Notice that there is an expected value for $g(n, d)$ coming from the secant variety interpretation:

$$g(n,d) = \left\lceil \frac{\binom{d+n}{n}}{n+1} \right\rceil .$$

A complete solution for the big Waring problem is given by a celebrated result by Alexander and Hirschowitz.

Theorem 0.22. *Let F be a generic degree d form in $n + 1$ variables. Then*

$$\mathrm{rk}(F) = \left\lceil \frac{\binom{d+n}{n}}{n+1} \right\rceil$$

unless

- $d = 2$*, any* n *where* $\mathrm{rk}(F) = n + 1$.
- $d = 4, n = 2$ *where* $\mathrm{rk}(F) = 6$ *and not* 5 *as expected.*
- $d = 4, n = 3$ *where* $\mathrm{rk}(F) = 10$ *and not* 9 *as expected.*
- $d = 3, n = 4$ *where* $\mathrm{rk}(F) = 8$ *and not* 7 *as expected.*
- $d = 4, n = 4$ *where* $\mathrm{rk}(F) = 15$ *and not* 14 *as expected.*

Remark 0.23. A straightforward interpretation of the Alexander and Hirschowitz result in terms of higher secants is the following: $s = g(n, d)$ is the smallest s such that

$$\sigma_s(\nu_d(\mathbb{P}^)) = \mathbb{P}^N,$$

unless n and d are one of the exceptional cases above.

Remark 0.24. Actually the Alexander and Hirschowitz result gives even more for higher secant varieties of the Veronese varieties, namely that $\nu_d(\mathbb{P}^n)$ is not defective for all s, unless in the exceptional cases.

Let's now try to explain some of the defective cases of the Alexander-Hirschowitz result.

Example 0.25. For $n = 2, d = 4$ we consider $X = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^14$. In particular, we are looking for the smallest s such that $\sigma_s(X) = \mathbb{P}^14$. We expect $s = 5$ to work and we want to check whether this is the case or not. To use the double points interpretation, we choose 5 generic points $P_1, \ldots, P_5\mathbb{P}^2$ and we want to determine

$$(\wp_1^2 \cap \ldots \cap \wp_5^2)_4,$$

i.e. we want to know how many quartic curves exist which are singular at each P_i. Counting conditions we expect $15 - 5 \times 3 = 0$ such curves to exist. However, there exists a conic passing through the points P_i and this conic doubled is a quartic with the required properties. Thus,

$$\dim(\wp_1^2 \cap \ldots \cap \wp_5^2)_4 \geq 1$$

and $\dim \sigma_5(X) \leq 14 - 1 = 13$.

Exercise 0.26. Show that $\sigma_5(\nu_4(\mathbb{P}^2))$ is a hypersurface, i.e. it has dimension exactly 13.

Exercise 0.27. Explain the exceptional cases $d = 2$ any n.

Exercise 0.28. Explain the exceptional cases $d = 4$ and $n = 3, 4$.

Exercise 0.29. Explain the exceptional case $d = 3$ and $n = 4$. (Hint: use Castelnuovo's Theorem stating that there exists a (unique) rational normal curve passing through $n + 3$ generic points in $\mathbb{P}^n$).

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