LECTURE THREE

ENRICO CARLINI

In the last lecture we showed the solution the big Waring problem, that is we showed how to determine the Waring rank $\mathrm{rk}(F)$ for F a generic form. We will now focus on the general question: given any form F what can we say on $\mathrm{rk}(F)$?

The main tool we will use is *Apolarity* and in order to do this we will need the following setting. Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[y_0, \ldots, y_n]$. We make T act on S via differentiation, i.e. we define

$$y_i \circ x_j = \frac{\partial}{\partial x_i} x_j,$$

i.e. $y_i \circ x_j = 1$ if $i = j$ and it is zero otherwise. We then extend the action to all T so that $\partial \in T$ is seen as a differential operator on element of S; from now on we will omit $\circ$.

Definition 0.1. Given $F \in S_d$ we define the *annihilator*, or *perp ideal*, of F as follows:

$$F^\perp = \{\partial \in T : \partial F = 0\}.$$

Exercise 0.2. Show that $F^\perp \subset T$ is an ideal and that it also is artinian, i.e. $(T/F^\perp)_i$ is zero for $\imath \geq d$.

Exercise 0.3. Show that the map

$$S_i \times T_i \longrightarrow \mathbb{C}$$
$$(F, \partial) \mapsto \partial F$$

is a perfect pairing, i.e.

$$(F, \partial_0) \mapsto 0, \forall F \in S_i \Longrightarrow \partial_0 = 0$$

and

$$(F_0, \partial) \mapsto 0, \forall \partial \in T_i \Longrightarrow F_0 = 0$$

Remark 0.4. *Actually even more is true, and $A = T/F^\perp$ is artinian and Gorenstein with socle degree d.* Using the perfect pairing $S_i \times T_i \longrightarrow \mathbb{C}$ we see that $\dim A_d = \dim A_0 = 1$ and that A_d is the socle of A.

In what follows we will make use of Hilbert functions, thus we define them here.

Definition 0.5. For an ideal $I \subset T$ we define the *Hilbert function* of T/I as

$$HF(T/I, t) = \dim(T/I)_t.$$

Example 0.6. Let $F \in S_d$. We see that $HF(T/F^\perp, t) = 0$ for all $t > d$, in fact all partials of degree $t > 0$ will annihilate the degree d form F and hence $(T/F^\perp)_t = 0$. From the remak above we also see that $HF(T/F^\perp, d) = 1$.

Exercise 0.7. Given $F \in S_d$ show that $HF(T/F^\perp, t)$ is a symmetric function of t.

An interesting property of the ideal $F^\perp$ is described by Macaulay's Theorem

Theorem 0.8. *If $F \in S_d$, then $T/F^\perp$ is an artinian Gorenstein ring with socle degree d. Conversely, if T/I is an artinian Gorenstein ring with socle degree d, then $I = F^\perp$ for some $F \in S_d$.*

Let's now see how apolarity relates to the Waring rank. Recall that $s = \mathrm{rk}(F)$ if and only if $F = \sum_1^s L_i^d$ and no shorter presentation exists.

Example 0.9. We now compute the possible Waring ranks for a binary cubic, i.e. for $F \in S_3$ where $S = \mathbb{C}[x_0, x_1]$. We begin by describing the Hilbert function of $F^\perp$. There are only two possibilities:

case 1

t	0	1	2	3	4
$HF(T/F^\perp, t)$	1	1	1	1	$0 \rightarrow$

case 2

t	0	1	2	3	4
$HF(T/F^\perp, t)$	1	2	2	1	$0 \rightarrow$

We want to show that in **case** 1 we have $F = L^3$. From the Hilbert function we see that $(F^\perp)_1 = \langle \partial_1 \rangle$. From the perfect pairing property we see that

$$\{L \in S_1 : \partial_1 L = 0\} = \langle L_1 \rangle.$$

Thus we can find $L_0 \in S_1$ such that $\partial_1 L_0 = 1$ and

$$S_1 = \langle x_0, x_1 \rangle = \langle L_0, L_1 \rangle.$$

We now perform a liner change of variables and we obtain a polynomial $G(L_0, L_1) = aL_0^3 + bL_0^2 L_1 + cL_0 L_1^2 + dL_1^3$ such that

$$G(L_0, L_1) = F(x_0, x_1).$$

As $\partial_1 L_0 \neq 0$ and $\partial_1 L_1 = 0$ we get

$$0 = \partial_1 G = 2bL_0 L_1 + cL_1^2 + 3dL_1^2$$

and hence $G = F = aL_0^3$ thus $\mathrm{rk}(F) = 1$.

We want now to show that in **case** 2 we have $\mathrm{rk}(F) = 2$ or $\mathrm{rk}(F) = 3$. We note that $\mathrm{rk}(F) \neq 1$, otherwise $(F^\perp)_1 \neq 0$. As in this case $(F^\perp)_1 = 0$, we consider the degree two piece, $(F^\perp)_2 = \langle Q \rangle$. We have to possibilities

$$Q = \partial\partial' \text{ or } Q = \partial^2.$$

If $Q = \partial\partial'$ we can construct a basis for $S_1 = \langle L, L' \rangle$ in such a way that

$$\partial L = \partial' L' = 1$$

and

$$\partial' L = \partial L' = 0.$$

Then we perform a change of variables and we get

$$F(x_0, x_1) = G(L_0, L_1) = aL_0^3 + bL_0^2 L_1 + cL_0 L_1^2 + dL_1^3.$$

We want to show that $F(x_0, x_1) = aL_0^3 + dL_1^3$. To do this we define

$$H(x_0, x_1) = G(L_0, L_1) - aL_0^3 - dL_1^3$$

and we show that the degree 3 polynomial H is the zero polynomial. To do this, it is enough to show that $(H^\perp)_3 = T_3$. We now compute

$$\partial^3 H = 6aL - 6aL = 0,$$

$$\partial'^3 H = 6dL' - 6dL' = 0,$$

then we notice that $\partial^2\partial' = \partial Q \in F^\perp$ and then $\partial^2\partial' H = 0$, similarly for $\partial\partial'^2$. Thus $H = 0$ and $F(x_0, x_1) = aL_0^3 + dL_1^3$. As $(F^\perp)_1 = 0$ this means that $\text{rk}(F) = 2$.

Finally, if $Q = \partial^2$ we assume by contradiction that $\text{rk}(F) = 2$, thus $F = N^3 + M^3$ for some linear forms N and M. There exist partial linearly independent$\partial_N, \partial_M \in S_1$ such that

$$\partial_N N = \partial_M M = 1$$

and

$$\partial_N M = \partial_M N = 0.$$

And then $\partial_N \partial_M \in F^\perp$ and this is a contradiction as Q is the only element in $(F^\perp)_2$ and it is a square.

Remark 0.10. We consider again the case of binary cubic forms. We want to make a connection between the Waring rank of F and ceratin ideals contained in $F^\perp$. If $\text{rk}(F) = 1$ we saw that $F^\perp \subset (\partial_1)$ and this is the ideal of one point in $\mathbb{P}^1$. If $\text{rk}(F) = 2$ then $F^\perp \subset (\partial\partial')$ and this the ideal of two distinct points in $\mathbb{P}^1$; as $(F^\perp)_1 = 0$ there is no ideal of one point contained in the annihilator. Finally, if $\text{rk}(F) = 3$, then $F^\perp \subset (\partial^2)$ and there is no ideal of two points, or one point, contained in the annihilator. However, $(F^\perp)_3 = T_3$ and we can find many ideal of three points.

There is connection between $\text{rk}(F)$ and set of points whose ideal I is such that $I \subset F^\perp$. This connection is the content of the *Apolarity Lemma*.

Lemma 0.11. *Let $F \in S_d$ be a degree d form in $n+1$ variables. Then the following facts are equivalent:*

- *$F = L_1^d + \ldots + L_s^d$;*
- *$F^\perp \supset I$ such that I is the ideal of a set of s distinct points in $\mathbb{P}^n$.*

Example 0.12. We use the Apolarity Lemma to explain the Alexander-Hirschowitz defective case $n = 2$ and $d = 4$. Given a generic $F \in S_4$ we want to show that $\text{rk}(F) = 6$ and not 5 as expected. To do this we use Hilbert functions. Clearly, if $I \subset F^\perp$ then $HF(T/I, t) \geq HF(T/F^\perp, t)$ for all t. Thus by computing $HF(T/F^\perp, t)$ we get information on the Hilbert function of any ideal contained in the annihilator, and in particular for ideal of sets of points.

t	0	1	2	3	4
$HF(T/F^\perp, t)$	1	3	6	3	$\rightarrow$

In particular, $HF(T/F^\perp, 2) = 6$ means that for no set of 5 points its defining ideal I could be such that $I \subset F^\perp$.

Exercise 0.13. Use the Apolarity Lemma to compute $\text{rk}(x_0 x_1^2)$. Then try the binary forms $x_0 x_1^d$.

Exercise 0.14. Use the Apolarity Lemma to explain Alexander-Hirschowitz exceptional cases.

It is in general very difficult to compute the Waring rank of a given form and no algorithm exists which can compute it for you in all cases. However, we know $\text{rk}(F)$ when F is a quadratic form, and we do have an efficient algorithm when F is a binary form.

Remark 0.15. *There is an algorithm, attributed to Sylvester, to compute* $\mathrm{rk}(F)$ *for a binary form and it uses the Apolarity Lemma.* The idea is to notice that $F^{\perp} = (\partial_1, \partial_2)$, i.e. the annihilator is a complete intersection ideal, say, with generators in degree $d_1 = \deg \partial_1 \leq d_2 = \deg \partial_2$. If ∂_1 is square free, we are done, and $\mathrm{rk}(F) = d_1$. If not, as ∂_1 and ∂_2 do not have common factors, there is a square free degree d_2 element in $F^{\perp}$. Hence, $\mathrm{rk}(F) = d_2$.

Exercise 0.16. Compute $\mathrm{rk}(F)$ when F is a quadratic form.

Remark 0.17. The Waring rank was recently for a monomials in 2011 paper by Carlini, Catalisano e Geramita. In particular, it shown that

$$\mathrm{rk}(x_0^{a_0} \ldots x_n^{a_n}) = \frac{1}{(a_0+1)} \Pi_{i=0}^{n}(a_i+1),$$

where $1 \leq a_0 \leq a_1 \leq \ldots \leq a_n$.

We conclude by studying the Waring rank of degree d forms of the kind $L_1^d + \ldots L_s^d$. Clearly, $\mathrm{rk}(L_1^d) = 1$ and $\mathrm{rk}(L_1^d + L_2^d) = 2$, if L_1 and L_2 are linearly independent. If the linear forms L_i are not linearly independent, then the situation is more interesting.

Example 0.18. Consider the binary cubic form $F = ax_0^3 + bx_1^3 + (x_0 + x_1)^3$. We want to know $\mathrm{rk}(F)$. For a generic choice of a and b, we have $\mathrm{rk}(F) = 2$, but for special values of a and b $\mathrm{rk}(F) = 3$. The idea is that the rank three element of $\mathbb{P}S_3$ lie on the tangent developable of the twisted cubic curve, which is an irreducible surface. Hence, the general element of the plane

$$\langle [x_0^3], [x_1^3], [(x_0 + x_1)^3] \rangle$$

has rank two, but there are rank three elements.

Exercise 0.19. Prove that $\mathrm{rk}(L^d + M^d + N^d) = 3$ whenever L, M and N are linearly independent linear forms.

(E. Carlini) Dipartimento di Scienze Matematiche, Politecnico di Torino, Turin, Italy
E-mail address: enrico.carlini@polito.it