4 Some Applications of Fourier Series

> Fourier series and analogous expansions intervene very naturally in the general theory of curves and surfaces. In effect, this theory, conceived from the point of view of analysis, deals obviously with the study of arbitrary functions. I was thus led to use Fourier series in several questions of geometry, and I have obtained in this direction a number of results which will be presented in this work. One notes that my considerations form only a beginning of a principal series of researches, which would without doubt give many new results.
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> *A. Hurwitz,* 1902

In the previous chapters we introduced some basic facts about Fourier analysis, motivated by problems that arose in physics. The motion of a string and the diffusion of heat were two instances that led naturally to the expansion of a function in terms of a Fourier series. We propose next to give the reader a flavor of the broader impact of Fourier analysis, and illustrate how these ideas reach out to other areas of mathematics. In particular, consider the following three problems:

I. Among all simple closed curves of length ℓ in the plane $\mathbb{R}^2$, which one encloses the largest area?

II. Given an irrational number γ, what can be said about the distribution of the fractional parts of the sequence of numbers $n\gamma$, for $n = 1, 2, 3, \ldots$?

III. Does there exist a continuous function that is nowhere differentiable?

The first problem is clearly geometric in nature, and at first sight, would seem to have little to do with Fourier series. The second question lies on the border between number theory and the study of dynamical systems, and gives us the simplest example of the idea of "ergodicity." The third problem, while analytic in nature, resisted many attempts before the solution was finally discovered. It is remarkable that all three questions can be resolved quite simply and directly by the use of Fourier series.

In the last section of this chapter, we return to a problem that provided our initial motivation. We consider the time-dependent heat equation on the circle. Here our investigation will lead us to the important but enigmatic heat kernel for the circle. However, the mysteries surrounding its basic properties will not be fully understood until we can apply the Poisson summation formula, which we will do in the next chapter.

1 The isoperimetric inequality

Let Γ denote a closed curve in the plane which does not intersect itself. Also, let ℓ denote the length of Γ, and $\mathcal{A}$ the area of the bounded region in $\mathbb{R}^2$ enclosed by Γ. The problem now is to determine for a given ℓ the curve Γ which maximizes $\mathcal{A}$ (if any such curve exists).

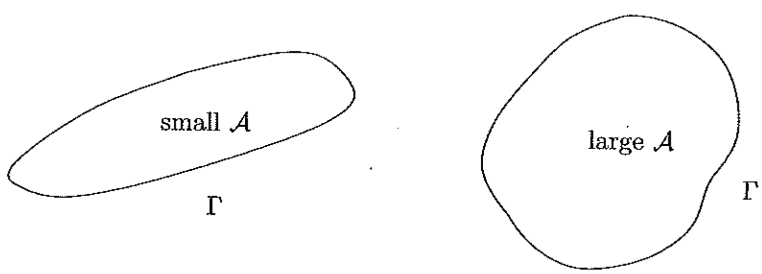

Figure 1. The isoperimetric problem

A little experimentation and reflection suggests that the solution should be a circle. This conclusion can be reached by the following heuristic considerations. The curve can be thought of as a closed piece of string lying flat on a table. If the region enclosed by the string is not convex (for example), one can deform part of the string and increase the area enclosed by it. Also, playing with some simple examples, one can convince oneself that the "flatter" the curve is in some portion, the less efficient it is in enclosing area. Therefore we want to maximize the "roundness" of the curve at each point.

Although the circle is the correct guess, making the above ideas precise is a difficult matter.

The key idea in the solution we give to the isoperimetric problem consists of an application of Parseval's identity for Fourier series. However, before we can attempt a solution to this problem, we must define the notion of a simple closed curve, its length, and what we mean by the area of the region enclosed by it.

Curves, length and area

A parametrized curve γ is a mapping

$$\gamma : [a, b] \to \mathbb{R}^2.$$

The image of γ is a set of points in the plane which we call a **curve** and denote by Γ. The curve Γ is **simple** if it does not intersect itself, and **closed** if its two end-points coincide. In terms of the parametrization above, these two conditions translate into $\gamma(s_1) \neq \gamma(s_2)$ unless $s_1 = a$ and $s_2 = b$, in which case $\gamma(a) = \gamma(b)$. We may extend γ to a periodic function on $\mathbb{R}$ of period $b - a$, and think of γ as a function on the circle. We also always impose some smoothness on our curves by assuming that γ is of class C^1, and that its derivative γ' satisfies $\gamma'(s) \neq 0$. Altogether, these conditions guarantee that Γ has a well-defined tangent at each point, which varies continuously as the point on the curve varies. Moreover, the parametrization γ induces an orientation on Γ as the parameter s travels from a to b.

Any C^1 bijective mapping $s : [c, d] \to [a, b]$ gives rise to another parametrization of Γ by the formula

$$\eta(t) = \gamma(s(t)).$$

Clearly, the conditions that Γ be closed and simple are independent of the chosen parametrization. Also, we say that the two parametrizations γ and η are equivalent if $s'(t) > 0$ for all t; this means that η and γ induce the same orientation on the curve Γ. If, however, $s'(t) < 0$, then η reverses the orientation.

If Γ is parametrized by $\gamma(s) = (x(s), y(s))$, then the **length** of the curve Γ is defined by

$$\ell = \int_a^b |\gamma'(s)|\, ds = \int_a^b \left(x'(s)^2 + y'(s)^2\right)^{1/2} ds.$$

The length of Γ is a notion intrinsic to the curve, and does not depend on its parametrization. To see that this is indeed the case, suppose that $\gamma(s(t)) = \eta(t)$. Then, the change of variables formula and the chain rule imply that

$$\int_a^b |\gamma'(s)|\, ds = \int_c^d |\gamma'(s(t))|\, |s'(t)|\, dt = \int_c^d |\eta'(t)|\, dt,$$

as desired.

In the proof of the theorem below, we shall use a special type of parametrization for Γ. We say that γ is a **parametrization by arc-length** if $|\gamma'(s)| = 1$ for all s. This means that $\gamma(s)$ travels at a constant speed, and as a consequence, the length of Γ is precisely $b - a$. Therefore, after a possible additional translation, a parametrization by arc-length will be defined on $[0, \ell]$. Any curve admits a parametrization by arc-length (Exercise 1).

We now turn to the isoperimetric problem.

The attempt to give a precise formulation of the area $\mathcal{A}$ of the region enclosed by a simple closed curve Γ raises a number of tricky questions. In a variety of simple situations, it is evident that the area is given by the following familiar formula of the calculus:

$$(1) \qquad \mathcal{A} = \frac{1}{2}\left|\int_\Gamma (x\,dy - y\,dx)\right| = \frac{1}{2}\left|\int_a^b x(s)y'(s) - y(s)x'(s)\,ds\right|;$$

see, for example, Exercise 3. Thus in formulating our result we shall adopt the easy expedient of taking (1) as our definition of area. This device allows us to give a quick and neat proof of the isoperimetric inequality. A listing of issues this simplification leaves unresolved can be found after the proof of the theorem.

Statement and proof of the isoperimetric inequality

Theorem 1.1 *Suppose that Γ is a simple closed curve in $\mathbb{R}^2$ of length ℓ, and let $\mathcal{A}$ denote the area of the region enclosed by this curve. Then*

$$\mathcal{A} \leq \frac{\ell^2}{4\pi},$$

with equality if and only if Γ is a circle.

The first observation is that we can rescale the problem. This means that we can change the units of measurement by a factor of $\delta > 0$ as follows. Consider the mapping of the plane $\mathbb{R}^2$ to itself, which sends the point (x, y) to $(\delta x, \delta y)$. A look at the formula defining the length of a curve shows that if Γ is of length ℓ, then its image under this mapping has length $\delta\ell$. So this operation magnifies or contracts lengths by a factor of δ depending on whether $\delta \geq 1$ or $\delta \leq 1$. Similarly, we see that the mapping magnifies (or contracts) areas by a factor of δ^2. By taking $\delta = 2\pi/\ell$, we see that it suffices to prove that if $\ell = 2\pi$ then $\mathcal{A} \leq \pi$, with equality only if Γ is a circle.

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ with $\gamma(s) = (x(s), y(s))$ be a parametrization by arc-length of the curve Γ, that is, $x'(s)^2 + y'(s)^2 = 1$ for all $s \in [0, 2\pi]$. This implies that

$$\frac{1}{2\pi}\int_0^{2\pi} (x'(s)^2 + y'(s)^2)\, ds = 1. \tag{2}$$

Since the curve is closed, the functions $x(s)$ and $y(s)$ are 2π-periodic, so we may consider their Fourier series

$$x(s) \sim \sum a_n e^{ins} \quad \text{and} \quad y(s) \sim \sum b_n e^{ins}.$$

Then, as we remarked in the later part of Section 2 of Chapter 2, we have

$$x'(s) \sim \sum a_n in e^{ins} \quad \text{and} \quad y'(s) \sim \sum b_n in e^{ins}.$$

Parseval's identity applied to (2) gives

$$\sum_{n=-\infty}^{\infty} |n|^2 \left(|a_n|^2 + |b_n|^2\right) = 1. \tag{3}$$

We now apply the bilinear form of Parseval's identity (Lemma 1.5, Chapter 3) to the integral defining $\mathcal{A}$. Since $x(s)$ and $y(s)$ are real-valued, we have $a_n = \overline{a_{-n}}$ and $b_n = \overline{b_{-n}}$, so we find that

$$\mathcal{A} = \frac{1}{2}\left|\int_0^{2\pi} x(s)y'(s) - y(s)x'(s)\, ds\right| = \pi\left|\sum_{n=-\infty}^{\infty} n\left(a_n\overline{b_n} - b_n\overline{a_n}\right)\right|.$$

We observe next that

$$|a_n\overline{b_n} - b_n\overline{a_n}| \leq 2\,|a_n|\,|b_n| \leq |a_n|^2 + |b_n|^2, \tag{4}$$

and since $|n| \leq |n|^2$, we may use (3) to get

$$\begin{aligned}\mathcal{A} &\leq \pi \sum_{n=-\infty}^{\infty} |n|^2 \left(|a_n|^2 + |b_n|^2\right)\\ &\leq \pi,\end{aligned}$$

as desired.

When $\mathcal{A} = \pi$, we see from the above argument that

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is} \quad \text{and} \quad y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

because $|n| < |n|^2$ as soon as $|n| \geq 2$. We know that $x(s)$ and $y(s)$ are real-valued, so $a_{-1} = \overline{a_1}$ and $b_{-1} = \overline{b_1}$. The identity (3) implies that $2\left(|a_1|^2 + |b_1|^2\right) = 1$, and since we have equality in (4) we must have $|a_1| = |b_1| = 1/2$. We write

$$a_1 = \frac{1}{2}e^{i\alpha} \quad \text{and} \quad b_1 = \frac{1}{2}e^{i\beta}.$$

The fact that $1 = 2|a_1\overline{b_1} - \overline{a_1}b_1|$ implies that $|\sin(\alpha - \beta)| = 1$, hence $\alpha - \beta = k\pi/2$ where k is an odd integer. From this we find that

$$x(s) = a_0 + \cos(\alpha + s) \quad \text{and} \quad y(s) = b_0 \pm \sin(\alpha + s),$$

where the sign in $y(s)$ depends on the parity of $(k-1)/2$. In any case, we see that Γ is a circle, for which the case of equality obviously holds, and the proof of the theorem is complete.

The solution given above (due to Hurwitz in 1901) is indeed very elegant, but clearly leaves some important issues unanswered. We list these as follows. Suppose Γ is a simple closed curve.

(i) How is the "region enclosed by Γ" defined?

(ii) What is the geometric definition of the "area" of this region? Does this definition accord with (1)?

(iii) Can these results be extended to the most general class of simple closed curves relevant to the problem—those curves which are "rectifiable"—that is, those to which we can ascribe a finite length?

It turns out that the clarifications of the problems raised are connected to a number of other significant ideas in analysis. We shall return to these questions in succeeding books of this series.

2 Weyl's equidistribution theorem

We now apply ideas coming from Fourier series to a problem dealing with properties of irrational numbers. We begin with a brief discussion of congruences, a concept needed to understand our main theorem.

where H_t is the **heat kernel for the circle**, given by

$$(9) \qquad H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x},$$

and where the convolution for functions with period 1 is defined by

$$(f * g)(x) = \int_0^1 f(x-y)g(y)\, dy.$$

An analogy between the heat kernel and the Poisson kernel (of Chapter 2) is given in Exercise 12. However, unlike in the case of the Poisson kernel, there is no elementary formula for the heat kernel. Nevertheless, it turns out that it is a good kernel (in the sense of Chapter 2). The proof is not obvious and requires the use of the celebrated Poisson summation formula, which will be taken up in Chapter 5. As a corollary, we will also find that H_t is everywhere positive, a fact that is also not obvious from its defining expression (9). We can, however, give the following heuristic argument for the positivity of H_t. Suppose that we begin with an initial temperature distribution f which is everywhere ≤ 0. Then it is physically reasonable to expect $u(x,t) \leq 0$ for all t since heat travels from hot to cold. Now

$$u(x,t) = \int_0^1 f(x-y)H_t(y)\, dy.$$

If H_t is negative for some x_0, then we may choose $f \leq 0$ supported near x_0, and this would imply $u(x_0, t) > 0$, which is a contradiction.

5 Exercises

1. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a parametrization for the closed curve Γ.

(a) Prove that γ is a parametrization by arc-length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ is precisely $s - a$, that is,

$$\int_a^s |\gamma'(t)|\, dt = s - a.$$

(b) Prove that any curve Γ admits a parametrization by arc-length. [Hint: If η is any parametrization, let $h(s) = \int_a^s |\eta'(t)|\, dt$ and consider $\gamma = \eta \circ h^{-1}$.]

2. Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a parametrization for a closed curve Γ, with $\gamma(t) = (x(t), y(t))$.

(a) Show that

$$\frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,ds = \int_a^b x(s)y'(s)\,ds = -\int_a^b y(s)x'(s)\,ds.$$

(b) Define the **reverse parametrization** of γ by $\gamma^- : [a,b] \to \mathbb{R}^2$ with $\gamma^-(t) = \gamma(b+a-t)$. The image of γ^- is precisely Γ, except that the points $\gamma^-(t)$ and $\gamma(t)$ travel in opposite directions. Thus γ^- "reverses" the orientation of the curve. Prove that

$$\int_\gamma (x\,dy - y\,dx) = -\int_{\gamma^-} (x\,dy - y\,dx).$$

In particular, we may assume (after a possible change in orientation) that

$$\mathcal{A} = \frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,ds = \int_a^b x(s)y'(s)\,ds.$$

3. Suppose Γ is a curve in the plane, and that there exists a set of coordinates x and y so that the x-axis divides the curve into the union of the graph of two continuous functions $y = f(x)$ and $y = g(x)$ for $0 \le x \le 1$, and with $f(x) \ge g(x)$ (see Figure 6). Let Ω denote the region between the graphs of these two functions:

$$\Omega = \{(x,y) : \ 0 \le x \le 1 \ \text{ and } \ g(x) \le y \le f(x)\}.$$

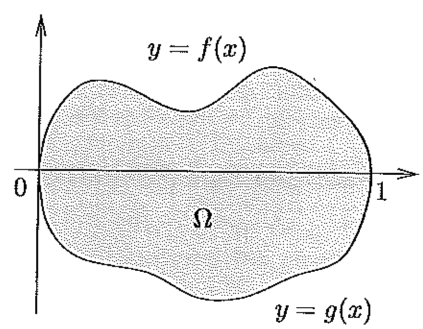

Figure 6. Simple version of the area formula

With the familiar interpretation that the integral $\int h(x)\,dx$ gives the area under the graph of the function h, we see that the area of Ω is $\int_0^1 f(x)\,dx -$

$\int_0^1 g(x)\,dx$. Show that this definition coincides with the area formula $\mathcal{A}$ given in the text, that is,

$$\int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx = \left| -\int_\Gamma y\,dx \right| = \mathcal{A}.$$

Also, note that if the orientation of the curve is chosen so that Ω "lies to the left" of Γ, then the above formula holds without the absolute value signs.

This formula generalizes to any set that can be written as a finite union of domains like Ω above.

4. Observe that with the definition of ℓ and $\mathcal{A}$ given in the text, the isoperimetric inequality continues to hold (with the same proof) even when Γ is not simple.

Show that this stronger version of the isoperimetric inequality is equivalent to Wirtinger's inequality, which says that if f is 2π-periodic, of class C^1, and satisfies $\int_0^{2\pi} f(t)\,dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2\,dt \le \int_0^{2\pi} |f'(t)|^2\,dt$$

with equality if and only if $f(t) = A\sin t + B\cos t$ (Exercise 11, Chapter 3).

[Hint: In one direction, note that if the length of the curve is 2π and γ is an appropriate arc-length parametrization, then

$$2(\pi - \mathcal{A}) = \int_0^{2\pi} [x'(s) + y(s)]^2\,ds + \int_0^{2\pi} (y'(s)^2 - y(s)^2)\,ds.$$

A change of coordinates will guarantee $\int_0^{2\pi} y(s)\,ds = 0$. For the other direction, start with a real-valued f satisfying all the hypotheses of Wirtinger's inequality, and construct g, 2π-periodic and so that the term in brackets above vanishes.]

5. Prove that the sequence $\{\gamma_n\}_{n=1}^\infty$, where γ_n is the fractional part of

$$\left(\frac{1+\sqrt{5}}{2}\right)^n,$$

is not equidistributed in $[0,1]$.

[Hint: Show that $U_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ is the solution of the difference equation $U_{r+1} = U_r + U_{r-1}$ with $U_0 = 2$ and $U_1 = 1$. The U_n satisfy the same difference equation as the Fibonacci numbers.]

6. Let $\theta = p/q$ be a rational number where p and q are relatively prime integers (that is, θ is in lowest form). We assume without loss of generality that $q > 0$. Define a sequence of numbers in $[0,1)$ by $\xi_n = \langle n\theta \rangle$ where $\langle \cdot \rangle$ denotes the

[Hint: For (a) compare the sum $\sum_{-\infty}^{\infty} e^{-cn^2t}$ with the integral $\int_{-\infty}^{\infty} e^{-cx^2t}\,dx$ where $c > 0$. For (b) use $x^2 \leq C(\sin \pi x)^2$ for $-1/2 \leq x \leq 1/2$, and apply the mean value theorem to e^{-cx^2t}.]

6 Problems

1.* This problem explores another relationship between the geometry of a curve and Fourier series. The diameter of a closed curve Γ parametrized by $\gamma(t) = (x(t), y(t))$ on $[-\pi, \pi]$ is defined by

$$d = \sup_{P,\,Q\in\Gamma} |P - Q| = \sup_{t_1,\,t_2\in[-\pi,\pi]} |\gamma(t_1) - \gamma(t_2)|.$$

If a_n is the n^{th} Fourier coefficient of $\gamma(t) = x(t) + iy(t)$ and ℓ denotes the length of Γ, then

(a) $2|a_n| \leq d$ for all $n \neq 0$.

(b) $\ell \leq \pi d$, whenever Γ is convex.

Property (a) follows from the fact that $2a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\gamma(t) - \gamma(t + \pi/n)]e^{-int}\,dt$.

The equality $\ell = \pi d$ is satisfied when Γ is a circle, but surprisingly, this is not the only case. In fact, one finds that $\ell = \pi d$ is equivalent to $2|a_1| = d$. We re-parametrize γ so that for each t in $[-\pi, \pi]$ the tangent to the curve makes an angle t with the y-axis. Then, if $a_1 = 1$ we have

$$\gamma'(t) = ie^{it}(1 + r(t)),$$

where r is a real-valued function which satisfies $r(t) + r(t + \pi) = 0$, and $|r(t)| \leq 1$. Figure 7 (a) shows the curve obtained by setting $r(t) = \cos 5t$. Also, Figure 7 (b) consists of the curve where $r(t) = h(3t)$, with $h(s) = -1$ if $-\pi \leq s \leq 0$ and $h(s) = 1$ if $0 < s < \pi$. This curve (which is only piecewise of class C^1) is known as the Reuleaux triangle and is the classical example of a convex curve of constant width which is not a circle.

2.* Here we present an estimate of Weyl which leads to some interesting results.

(a) Let $S_N = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Show that for $H \leq N$, one has

$$|S_N|^2 \leq c\,\frac{N}{H}\sum_{h=0}^{H}\left|\sum_{n=1}^{N-h} e^{2\pi i(f(n+h)-f(n))}\right|,$$

for some constant $c > 0$ independent of N, H, and f.

(b) Use this estimate to show that the sequence $\langle n^2\gamma\rangle$ is equidistributed in $[0, 1)$ whenever γ is irrational.

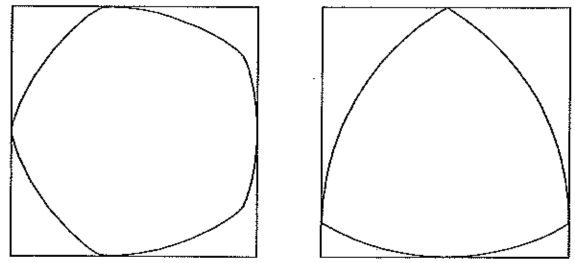

Figure 7. Some curves with maximal length for a given diameter

(c) More generally, show that if $\{\xi_n\}$ is a sequence of real numbers so that for all positive integers h the difference $\langle \xi_{n+h} - \xi_n \rangle$ is equidistributed in $[0, 1)$, then $\langle \xi_n \rangle$ is also equidistributed in $[0, 1)$.

(d) Suppose that $P(x) = c_n x^n + \cdots + c_0$ is a polynomial with real coefficients, where at least one of $c_1, \ldots, c_n$ is irrational. Then the sequence $\langle P(n) \rangle$ is equidistributed in $[0, 1)$.

[Hint: For (a), let $a_n = e^{2\pi i f(n)}$ when $1 \leq n \leq N$ and 0 otherwise. Then write $H \sum_n a_n = \sum_{k=1}^{H} \sum_n a_{n+k}$ and apply the Cauchy-Schwarz inequality. For (b), note that $(n + h)^2\gamma - n^2\gamma = 2nh\gamma + h^2\gamma$, and use the fact that for each integer h, the sequence $\langle 2nh\gamma \rangle$ is equidistributed. Finally, to prove (d), assume first that $P(x) = Q(x) + c_1 x + c_0$ where c_1 is irrational, and estimate the exponential sum $\sum_{n=1}^{N} e^{2\pi i k P(n)}$. Then, argue by induction on the highest degree term which has an irrational coefficient, and use part (c).]

3.* If $\sigma > 0$ is not an integer and $a \neq 0$, then $\langle a n^\sigma \rangle$ is equidistributed in $[0, 1)$. See also Exercise 8.

4. An elementary construction of a continuous but nowhere differentiable function is obtained by "piling up singularities," as follows.

On $[-1, 1]$ consider the function

$$\varphi(x) = |x|$$

and extend φ to $\mathbb{R}$ by requiring it to be periodic of period 2. Clearly, φ is continuous on $\mathbb{R}$ and $|\varphi(x)| \leq 1$ for all x so the function f defined by

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

is continuous on $\mathbb{R}$.