2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1}[\hat{f}(n) + \hat{f}(-n)]\cos n\theta + i[\hat{f}(n) - \hat{f}(-n)]\sin n\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

3. We return to the problem of the plucked string discussed in Chapter 1. Show that the initial condition f is *equal* to its Fourier sine series

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{with} \quad A_m = \frac{2h}{m^2}\frac{\sin mp}{p(\pi - p)}.$$

[Hint: Note that $|A_m| \leq C/m^2$.]

4. Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$.

(a) Draw the graph of f.

(b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi}\sum_{k \text{ odd } \geq 1} \frac{\sin k\theta}{k^3}.$$

5. On the interval $[-\pi, \pi]$ consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta & \text{if } |\theta| \leq \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty}\frac{1 - \cos n\delta}{n^2\pi\delta}\cos n\theta.$$

6. Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) Draw the graph of f.

(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0, \\[2ex] \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

See also Example 2 in Section 1.1.

7. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.

(a) Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

(b) Deduce from this formula Dirichlet's test for convergence of a series: if the partial sums of the series $\sum b_n$ are bounded, and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.

8. Verify that $\dfrac{1}{2i} \sum_{n \neq 0} \dfrac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function illustrated in Figure 6, defined by $f(0) = 0$, and

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } -\pi < x < 0, \\[2ex] \dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } 0 < x < \pi. \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.

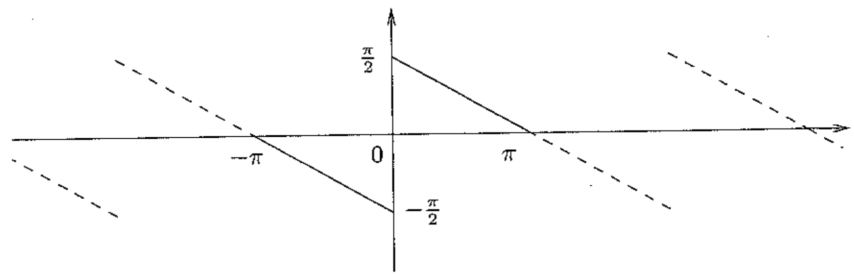

Figure 6. The sawtooth function

[Hint: Use Dirichlet's test for convergence of a series $\sum a_n b_n$.]

9. Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.

(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x. [Hint: It suffices to prove that for many values of n one has $|\sin n\theta_0| \geq c > 0$ where $\theta_0 = (b - a)/2$.]

(c) However, prove that the Fourier series converges at every point x. What happens if $a = -\pi$ and $b = \pi$?

10. Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n) = O(1/|n|^k) \quad \text{as } |n| \to \infty.$$

This notation means that there exists a constant C such $|\hat{f}(n)| \leq C/|n|^k$. We could also write this as $|n|^k \hat{f}(n) = O(1)$, where $O(1)$ means bounded.
[Hint: Integrate by parts.]

11. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0, 1]$ such that

$$\int_0^1 |f_k(x) - f(x)|\, dx \to 0 \quad \text{as } k \to \infty.$$