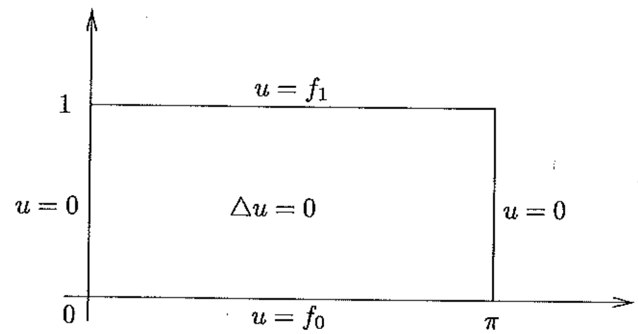


Figure 11. Dirichlet problem in a rectangle

4 Problem

1. Consider the Dirichlet problem illustrated in Figure 11.

More precisely, we look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x,y) : \ 0 \le x \le \pi, \ \ 0 \le y \le 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x,0) = f_0(x) \quad \text{and} \quad u(x,1) = f_1(x)\,,$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \quad \text{and} \quad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x,y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Compare this result with the solution of the Dirichlet problem in the strip obtained in Problem 3, Chapter 5.

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.

18. If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\Delta u = 0$ in the disc.

(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ.

However, u is not identically zero.

19. Solve Laplace's equation $\Delta u = 0$ in the semi infinite strip

$$S = \{(x,y) : 0 < x < 1,\ 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0,y) = 0 & \text{when } 0 \le y, \\ u(1,y) = 0 & \text{when } 0 \le y, \\ u(x,0) = f(x) & \text{when } 0 \le x \le 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y}\sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula (6).

20. Consider the Dirichlet problem in the annulus defined by $\{(r,\theta) : \rho < r < 1\}$, where $0 < \rho < 1$ is the inner radius. The problem is to solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions

$$\begin{cases} u(1,\theta) = f(\theta), \\ u(\rho,\theta) = g(\theta), \end{cases}$$

where f and g are given continuous functions.

Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

$$u(r,\theta) = \sum c_n(r) e^{in\theta}$$

with $c_n(r) = A_n r^n + B_n r^{-n}$, $n \neq 0$. Set

$$f(\theta) \sim \sum a_n e^{in\theta} \quad \text{and} \quad g(\theta) \sim \sum b_n e^{in\theta}.$$

We want $c_n(1) = a_n$ and $c_n(\rho) = b_n$. This leads to the solution

$$u(r,\theta) = \sum_{n \neq 0} \left(\frac{1}{\rho^n - \rho^{-n}} \right) \left[\left((\rho/r)^n - (r/\rho)^n \right) a_n + (r^n - r^{-n}) b_n \right] e^{in\theta}$$
$$+ a_0 + (b_0 - a_0) \frac{\log r}{\log \rho}.$$

Show that as a result we have

$$u(r,\theta) - (P_r * f)(\theta) \to 0 \quad \text{as } r \to 1 \text{ uniformly in } \theta,$$

and

$$u(r,\theta) - (P_{\rho/r} * g)(\theta) \to 0 \quad \text{as } r \to \rho \text{ uniformly in } \theta.$$

7 Problems

1. One can construct Riemann integrable functions on $[0,1]$ that have a dense set of discontinuities as follows.

(a) Let $f(x) = 0$ when $x < 0$, and $f(x) = 1$ if $x \geq 0$. Choose a countable dense sequence $\{r_n\}$ in $[0,1]$. Then, show that the function

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$. [Hint: F is monotonic and bounded.]

(b) Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),$$

where $g(x) = \sin 1/x$ when $x \neq 0$, and $g(0) = 0$. Then F is integrable, discontinuous at each $x = r_n$, and fails to be monotonic in any subinterval of $[0,1]$. [Hint: Use the fact that $3^{-k} > \sum_{n>k} 3^{-n}$.]

(c) The original example of Riemann is the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x) = x$ for $x \in (-1/2, 1/2]$ and (x) is continued to $\mathbb{R}$ by periodicity, that is, $(x+1) = (x)$. It can be shown that F is discontinuous whenever $x = m/2n$, where $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$.

2. Let D_N denote the Dirichlet kernel

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta.$$

(a) Prove that

$$L_N \geq c \log N$$

for some constant $c > 0$. [Hint: Show that $|D_N(\theta)| \geq c\frac{\sin((N+1/2)\theta)}{|\theta|}$, change variables, and prove that

$$L_N \geq c \int_{\pi}^{N\pi} \frac{|\sin \theta|}{|\theta|}\, d\theta + O(1).$$

Write the integral as a sum $\sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi}$. To conclude, use the fact that $\sum_{k=1}^{n} 1/k \geq c \log n$.] A more careful estimate gives

$$L_N = \frac{4}{\pi^2} \log N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c' \log n$. [Hint: The function g_n which is equal to 1 when D_n is positive and -1 when D_n is negative has the desired property but is not continuous. Approximate g_n in the integral norm (in the sense of Lemma 3.2) by continuous functions h_k satisfying $|h_k| \leq 1$.]

3.* Littlewood provided a refinement of Tauber's theorem:

This function is complex-valued as opposed to the examples R and W above, and so the nowhere differentiability of f_α does not imply the same property for its real and imaginary parts. However, a small modification of our proof shows that, in fact, the real part of f_α,

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x,$$

as well as its imaginary part, are both nowhere differentiable. To see this, observe first that by the same proof, Lemma 3.2 has the following generalization: if g is a continuous function which is differentiable at x_0, then

$$\triangle_N(g)'(x_0 + h) = O(\log N) \quad \text{whenever } |h| \leq c/N.$$

We then proceed with $F(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x$, noting as above that $\triangle_{2N}(F) - \triangle_N(F) = 2^{-n\alpha} \cos 2^n x$; as a result, assuming that F is differentiable at x_0, we get that

$$|2^{n(1-\alpha)} \sin(2^n(x_0 + h))| = O(\log N)$$

when $2N = 2^n$, and $|h| \leq c/N$. To get a contradiction, we need only choose h so that $|\sin(2^n(x_0 + h))| = 1$; this is accomplished by setting δ equal to the distance from $2^n x_0$ to the nearest number of the form $(k + 1/2)\pi$, $k \in \mathbb{Z}$ (so $\delta \leq \pi/2$), and taking $h = \pm\delta/2^n$.

Clearly, when $\alpha > 1$ the function f_α is continuously differentiable since the series can be differentiated term by term. Finally, the nowhere differentiability we have proved for $\alpha < 1$ actually extends to $\alpha = 1$ by a suitable refinement of the argument (see Problem 8 in Chapter 5). In fact, using these more elaborate methods one can also show that the Weierstrass function W is nowhere differentiable if $ab \geq 1$.

4 The heat equation on the circle

As a final illustration, we return to the original problem of heat diffusion considered by Fourier.

Suppose we are given an initial temperature distribution at $t = 0$ on a ring and that we are asked to describe the temperature at points on the ring at times $t > 0$.

The ring is modeled by the unit circle. A point on this circle is described by its angle $\theta = 2\pi x$, where the variable x lies between 0 and 1. If $u(x,t)$ denotes the temperature at time t of a point described by the angle θ, then considerations similar to the ones given in Chapter 1 show that u satisfies the differential equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}. \tag{7}$$

The constant c is a positive physical constant which depends on the material of which the ring is made (see Section 2.1 in Chapter 1). After rescaling the time variable, we may assume that $c = 1$. If f is our initial data, we impose the condition

$$u(x, 0) = f(x).$$

To solve the problem, we separate variables and look for special solutions of the form

$$u(x, t) = A(x)B(t).$$

Then inserting this expression for u into the heat equation we get

$$\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)}.$$

Both sides are therefore constant, say equal to λ. Since A must be periodic of period 1, we see that the only possibility is $\lambda = -4\pi^2n^2$, where $n \in \mathbb{Z}$. Then A is a linear combination of the exponentials $e^{2\pi inx}$ and $e^{-2\pi inx}$, and $B(t)$ is a multiple of $e^{-4\pi^2n^2t}$. By superposing these solutions, we are led to

$$u(x, t) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2n^2t}e^{2\pi inx}, \tag{8}$$

where, setting $t = 0$, we see that $\{a_n\}$ are the Fourier coefficients of f.

Note that when f is Riemann integrable, the coefficients a_n are bounded, and since the factor $e^{-4\pi^2n^2t}$ tends to zero extremely fast, the series defining u converges. In fact, in this case, u is twice differentiable and solves equation (7).

The natural question with regard to the boundary condition is the following: do we have $u(x, t) \to f(x)$ as t tends to 0, and in what sense? A simple application of the Parseval identity shows that this limit holds in the mean square sense (Exercise 11). For a better understanding of the properties of our solution (8), we write it as

$$u(x, t) = (f * H_t)(x),$$

where H_t is the **heat kernel for the circle**, given by

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}, \tag{9}$$

and where the convolution for functions with period 1 is defined by

$$(f * g)(x) = \int_0^1 f(x-y)g(y)\,dy.$$

An analogy between the heat kernel and the Poisson kernel (of Chapter 2) is given in Exercise 12. However, unlike in the case of the Poisson kernel, there is no elementary formula for the heat kernel. Nevertheless, it turns out that it is a good kernel (in the sense of Chapter 2). The proof is not obvious and requires the use of the celebrated Poisson summation formula, which will be taken up in Chapter 5. As a corollary, we will also find that H_t is everywhere positive, a fact that is also not obvious from its defining expression (9). We can, however, give the following heuristic argument for the positivity of H_t. Suppose that we begin with an initial temperature distribution f which is everywhere ≤ 0. Then it is physically reasonable to expect $u(x,t) \leq 0$ for all t since heat travels from hot to cold. Now

$$u(x,t) = \int_0^1 f(x-y)H_t(y)\,dy.$$

If H_t is negative for some x_0, then we may choose $f \leq 0$ supported near x_0, and this would imply $u(x_0, t) > 0$, which is a contradiction.

5 Exercises

1. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a parametrization for the closed curve Γ.

(a) Prove that γ is a parametrization by arc-length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ is precisely $s - a$, that is,

$$\int_a^s |\gamma'(t)|\,dt = s - a.$$

(b) Prove that any curve Γ admits a parametrization by arc-length. [Hint: If η is any parametrization, let $h(s) = \int_a^s |\eta'(t)|\,dt$ and consider $\gamma = \eta \circ h^{-1}$.]

2. Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a parametrization for a closed curve Γ, with $\gamma(t) = (x(t), y(t))$.

(a) If f is continuous and satisfies $\int_0^1 f(x)\,dx = 0$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n) = 0 \quad \text{uniformly in } x.$$

[Hint: Establish this result first for trigonometric polynomials.]

(b) If f is merely integrable on $[0,1]$ and satisfies $\int_0^1 f(x)\,dx = 0$, then

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right|^2 dx = 0.$$

11. Show that if $u(x,t) = (f * H_t)(x)$ where H_t is the heat kernel, and f is Riemann integrable, then

$$\int_0^1 |u(x,t) - f(x)|^2\,dx \to 0 \qquad \text{as } t \to 0.$$

12. A change of variables in (8) leads to the solution

$$u(\theta,\tau) = \sum a_n e^{-n^2\tau} e^{in\theta} = (f * h_\tau)(\theta)$$

of the equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \theta^2} \qquad \text{with } 0 \le \theta \le 2\pi \text{ and } \tau > 0,$$

with boundary condition $u(\theta, 0) = f(\theta) \sim \sum a_n e^{in\theta}$. Here $h_\tau(\theta) = \sum_{n=-\infty}^{\infty} e^{-n^2\tau} e^{in\theta}$. This version of the heat kernel on $[0,2\pi]$ is the analogue of the Poisson kernel, which can be written as $P_r(\theta) = \sum_{n=-\infty}^{\infty} e^{-|n|\tau} e^{in\theta}$ with $r = e^{-\tau}$ (and so $0 < r < 1$ corresponds to $\tau > 0$).

13. The fact that the kernel $H_t(x)$ is a good kernel, hence $u(x,t) \to f(x)$ at each point of continuity of f, is not easy to prove. This will be shown in the next chapter. However, one can prove directly that $H_t(x)$ is "peaked" at $x = 0$ as $t \to 0$ in the following sense:

(a) Show that $\int_{-1/2}^{1/2} |H_t(x)|^2\,dx$ is of the order of magnitude of $t^{-1/2}$ as $t \to 0$. More precisely, prove that $t^{1/2}\int_{-1/2}^{1/2} |H_t(x)|^2\,dx$ converges to a non-zero limit as $t \to 0$.

(b) Prove that $\int_{-1/2}^{1/2} x^2 |H_t(x)|^2\,dx = O(t^{1/2})$ as $t \to 0$.

[Hint: For (a) compare the sum $\sum_{-\infty}^{\infty} e^{-cn^2t}$ with the integral $\int_{-\infty}^{\infty} e^{-cx^2t}\,dx$ where $c > 0$. For (b) use $x^2 \leq C(\sin \pi x)^2$ for $-1/2 \leq x \leq 1/2$, and apply the mean value theorem to e^{-cx^2t}.]

6 Problems

1.* This problem explores another relationship between the geometry of a curve and Fourier series. The diameter of a closed curve Γ parametrized by $\gamma(t) = (x(t), y(t))$ on $[-\pi, \pi]$ is defined by

$$d = \sup_{P,\, Q \in \Gamma} |P - Q| = \sup_{t_1,\, t_2 \in [-\pi,\pi]} |\gamma(t_1) - \gamma(t_2)|.$$

If a_n is the n^{th} Fourier coefficient of $\gamma(t) = x(t) + iy(t)$ and ℓ denotes the length of Γ, then

(a) $2|a_n| \leq d$ for all $n \neq 0$.

(b) $\ell \leq \pi d$, whenever Γ is convex.

Property (a) follows from the fact that $2a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\gamma(t) - \gamma(t + \pi/n)]e^{-int}\,dt$.

The equality $\ell = \pi d$ is satisfied when Γ is a circle, but surprisingly, this is not the only case. In fact, one finds that $\ell = \pi d$ is equivalent to $2|a_1| = d$. We re-parametrize γ so that for each t in $[-\pi, \pi]$ the tangent to the curve makes an angle t with the y-axis. Then, if $a_1 = 1$ we have

$$\gamma'(t) = ie^{it}(1 + r(t)),$$

where r is a real-valued function which satisfies $r(t) + r(t + \pi) = 0$, and $|r(t)| \leq 1$. Figure 7 (a) shows the curve obtained by setting $r(t) = \cos 5t$. Also, Figure 7 (b) consists of the curve where $r(t) = h(3t)$, with $h(s) = -1$ if $-\pi \leq s \leq 0$ and $h(s) = 1$ if $0 < s < \pi$. This curve (which is only piecewise of class C^1) is known as the Reuleaux triangle and is the classical example of a convex curve of constant width which is not a circle.

2.* Here we present an estimate of Weyl which leads to some interesting results.

(a) Let $S_N = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Show that for $H \leq N$, one has

$$|S_N|^2 \leq c\,\frac{N}{H}\sum_{h=0}^{H}\left|\sum_{n=1}^{N-h} e^{2\pi i(f(n+h)-f(n))}\right|,$$

for some constant $c > 0$ independent of N, H, and f.

(b) Use this estimate to show that the sequence $\langle n^2\gamma\rangle$ is equidistributed in $[0, 1)$ whenever γ is irrational.

will happen if and only if the slope γ of the initial direction of the light (determined with respect to one of the sides of the square) is rational. In the second situation, when γ is irrational, the density follows from Kronecker's theorem. What stronger conclusion does one get from the equidistribution theorem?

3 A continuous but nowhere differentiable function

There are many obvious examples of continuous functions that are not differentiable at one point, say $f(x) = |x|$. It is almost as easy to construct a continuous function that is not differentiable at any given finite set of points, or even at appropriate sets containing countably many points. A more subtle problem is whether there exists a continuous function that is *nowhere* differentiable. In 1861, Riemann guessed that the function defined by

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} \tag{5}$$

was nowhere differentiable. He was led to consider this function because of its close connection to the theta function which will be introduced in Chapter 5. Riemann never gave a proof, but mentioned this example in one of his lectures. This triggered the interest of Weierstrass who, in an attempt to find a proof, came across the first example of a continuous but nowhere differentiable function. Say $0 < b < 1$ and a is an integer > 1. In 1872 he proved that if $ab > 1 + 3\pi/2$, then the function

$$W(x) = \sum_{n=1}^{\infty} b^n \cos(a^n x)$$

is nowhere differentiable.

But the story is not complete without a final word about Riemann's original function. In 1916 Hardy showed that R is not differentiable at all irrational multiples of π, and also at certain rational multiples of π. However, it was not until much later, in 1969, that Gerver completely settled the problem, first by proving that the function R *is* actually differentiable at all the rational multiples of π of the form $\pi p/q$ with p and q odd integers, and then by showing that R is not differentiable in all of the remaining cases.

In this section, we prove the following theorem.

Theorem 3.1 *If $0 < \alpha < 1$, then the function*

$$f_\alpha(x) = f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$$

is continuous but nowhere differentiable.

The continuity is clear because of the absolute convergence of the series. The crucial property of f which we need is that it has many vanishing Fourier coefficients. A Fourier series that skips many terms, like the one given above, or like $W(x)$, is called a **lacunary Fourier series.**

The proof of the theorem is really the story of three methods of summing a Fourier series. First, there is the ordinary convergence in terms of the partial sums $S_N(g) = g * D_N$. Next, there is Cesàro summability $\sigma_N(g) = g * F_N$, with F_N the Fejér kernel. A third method, clearly connected with the second, involves the **delayed means** defined by

$$\triangle_N(g) = 2\sigma_{2N}(g) - \sigma_N(g).$$

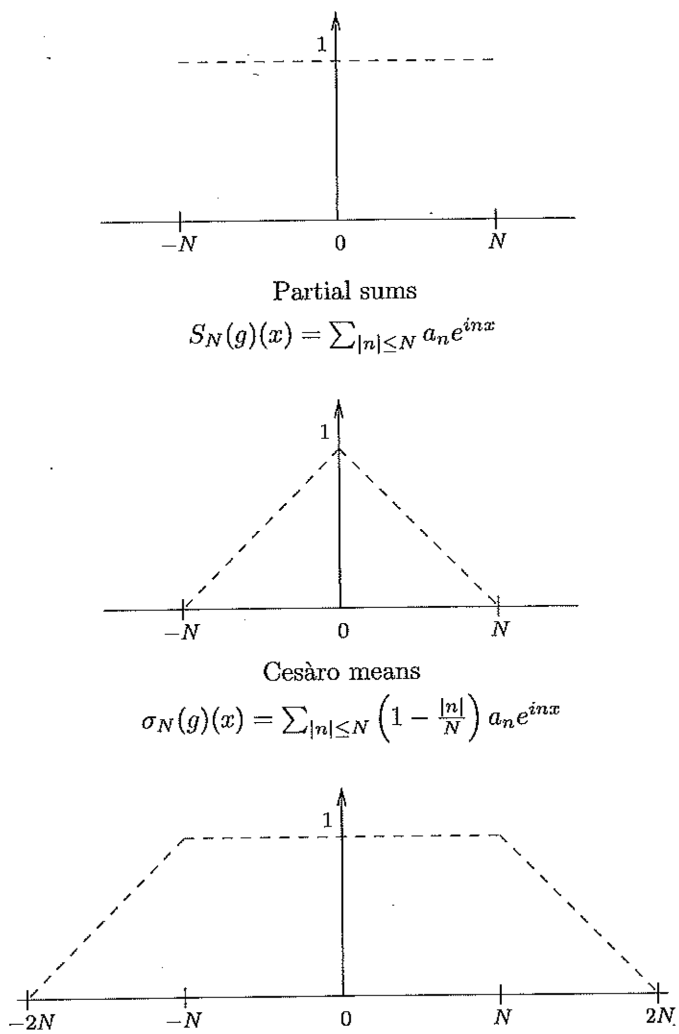
Hence $\triangle_N(g) = g * [2F_{2N} - F_N]$. These methods can best be visualized as in Figure 5.

Suppose $g(x) \sim \sum a_n e^{inx}$. Then:

- S_N arises by multiplying the term $a_n e^{inx}$ by 1 if $|n| \leq N$, and 0 if $|n| > N$.
- σ_N arises by multiplying $a_n e^{inx}$ by $1 - |n|/N$ for $|n| \leq N$ and 0 for $|n| > N$.
- $\triangle_N$ arises by multiplying $a_n e^{inx}$ by 1 if $|n| \leq N$, by $2(1 - |n|/(2N))$ for $N \leq |n| \leq 2N$, and 0 for $|n| > 2N$.

For example, note that

$$\begin{aligned}
\sigma_N(g)(x) &= \frac{S_0(g)(x) + S_1(g)(x) + \cdots + S_{N-1}(g)(x)}{N} \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} \sum_{|k| \leq \ell} a_k e^{ikx} \\
&= \frac{1}{N} \sum_{|n| \leq N} (N - |n|) a_n e^{inx} \\
&= \sum_{|n| \leq N} \left(1 - \frac{|n|}{N}\right) a_n e^{inx}.
\end{aligned}$$



Partial sums

$S_N(g)(x) = \sum_{|n| \le N} a_n e^{inx}$

Cesàro means

$\sigma_N(g)(x) = \sum_{|n| \le N} \left(1 - \frac{|n|}{N}\right) a_n e^{inx}$

Delayed means

$\Delta_N(g)(x) = 2\sigma_{2N}(g)(x) - \sigma_N(g)(x)$

Figure 5. Three summation methods

The proof of the other assertion is similar.

The delayed means have two important features. On the one hand, their properties are closely related to the (good) features of the Cesàro means. On the other hand, for series that have lacunary properties like those of f, the delayed means are essentially equal to the partial sums. In particular, note that for our function $f = f_\alpha$

$$S_N(f) = \triangle_{N'}(f), \tag{6}$$

where N' is the largest integer of the form 2^k with $N' \leq N$. This is clear by examining Figure 5 and the definition of f.

We turn to the proof of the theorem proper and argue by contradiction; that is, we assume that $f'(x_0)$ exists for some x_0.

Lemma 3.2 *Let g be any continuous function that is differentiable at x_0. Then, the Cesàro means satisfy $\sigma_N(g)'(x_0) = O(\log N)$, therefore*

$$\triangle_N(g)'(x_0) = O(\log N).$$

Proof. First we have

$$\sigma_N(g)'(x_0) = \int_{-\pi}^{\pi} F_N'(x_0 - t)g(t)\, dt = \int_{-\pi}^{\pi} F_N'(t)g(x_0 - t)\, dt,$$

where F_N is the Fejér kernel. Since F_N is periodic, we have $\int_{-\pi}^{\pi} F_N'(t)dt = 0$ and this implies that

$$\sigma_N(g)'(x_0) = \int_{-\pi}^{\pi} F_N'(t)[g(x_0 - t) - g(x_0)]\, dt.$$

From the assumption that g is differentiable at x_0 we get

$$|\sigma_N(g)'(x_0)| \leq C \int_{-\pi}^{\pi} |F_N'(t)|\, |t|\, dt.$$

Now observe that F_N' satisfies the two estimates

$$|F_N'(t)| \leq AN^2 \quad \text{and} \quad |F_N'(t)| \leq \frac{A}{|t|^2}.$$

For the first inequality, recall that F_N is a trigonometric polynomial of degree N whose coefficients are bounded by 1. Therefore, F_N' is a trigonometric polynomial of degree N whose coefficients are no bigger than N. Hence $|F'(t)| \leq (2N + 1)N \leq AN^2$.

For the second inequality, we recall that

$$F_N(t) = \frac{1}{N}\frac{\sin^2(Nt/2)}{\sin^2(t/2)}.$$

Differentiating this expression, we get two terms:

$$\frac{\sin(Nt/2)\cos(Nt/2)}{\sin^2(t/2)} - \frac{1}{N}\frac{\cos(t/2)\sin^2(Nt/2)}{\sin^3(t/2)}.$$

If we then use the facts that $|\sin(Nt/2)| \leq CN|t|$ and $|\sin(t/2)| \geq c|t|$ (for $|t| \leq \pi$), we get the desired estimates for $F_N'(t)$.

Using all of these estimates we find that

$$\begin{aligned}
|\sigma_N(g)'(x_0)| &\leq C\int_{|t|\geq 1/N} |F_N'(t)|\,|t|\,dt + C\int_{|t|\leq 1/N} |F_N'(t)|\,|t|\,dt \\
&\leq CA\int_{|t|\geq 1/N} \frac{dt}{|t|} + CAN\int_{|t|\leq 1/N} dt \\
&= O(\log N) + O(1) \\
&= O(\log N).
\end{aligned}$$

The proof of the lemma is complete once we invoke the definition of $\triangle_N(g)$.

Lemma 3.3 *If* $2N = 2^n$, *then*

$$\triangle_{2N}(f) - \triangle_N(f) = 2^{-n\alpha}e^{i2^n x}.$$

This follows from our previous observation (6) because $\triangle_{2N}(f) = S_{2N}(f)$ and $\triangle_N(f) = S_N(f)$.

Now, by the first lemma we have

$$\triangle_{2N}(f)'(x_0) - \triangle_N(f)'(x_0) = O(\log N),$$

and the second lemma also implies

$$|\triangle_{2N}(f)'(x_0) - \triangle_N(f)'(x_0)| = 2^{n(1-\alpha)} \geq cN^{1-\alpha}.$$

This is the desired contradiction since $N^{1-\alpha}$ grows faster than $\log N$.

A few additional remarks about our function $f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha}e^{i2^n x}$ are in order.

This function is complex-valued as opposed to the examples R and W above, and so the nowhere differentiability of f_α does not imply the same property for its real and imaginary parts. However, a small modification of our proof shows that, in fact, the real part of f_α,

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x,$$

as well as its imaginary part, are both nowhere differentiable. To see this, observe first that by the same proof, Lemma 3.2 has the following generalization: if g is a continuous function which is differentiable at x_0, then

$$\triangle_N(g)'(x_0 + h) = O(\log N) \qquad \text{whenever } |h| \leq c/N.$$

We then proceed with $F(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x$, noting as above that $\triangle_{2N}(F) - \triangle_N(F) = 2^{-n\alpha} \cos 2^n x$; as a result, assuming that F is differentiable at x_0, we get that

$$|2^{n(1-\alpha)} \sin(2^n(x_0 + h))| = O(\log N)$$

when $2N = 2^n$, and $|h| \leq c/N$. To get a contradiction, we need only choose h so that $|\sin(2^n(x_0 + h))| = 1$; this is accomplished by setting δ equal to the distance from $2^n x_0$ to the nearest number of the form $(k + 1/2)\pi$, $k \in \mathbb{Z}$ (so $\delta \leq \pi/2$), and taking $h = \pm\delta/2^n$.

Clearly, when $\alpha > 1$ the function f_α is continuously differentiable since the series can be differentiated term by term. Finally, the nowhere differentiability we have proved for $\alpha < 1$ actually extends to $\alpha = 1$ by a suitable refinement of the argument (see Problem 8 in Chapter 5). In fact, using these more elaborate methods one can also show that the Weierstrass function W is nowhere differentiable if $ab \geq 1$.

4 The heat equation on the circle

As a final illustration, we return to the original problem of heat diffusion considered by Fourier.

Suppose we are given an initial temperature distribution at $t = 0$ on a ring and that we are asked to describe the temperature at points on the ring at times $t > 0$.

The ring is modeled by the unit circle. A point on this circle is described by its angle $\theta = 2\pi x$, where the variable x lies between 0 and 1. If $u(x, t)$ denotes the temperature at time t of a point described by the