(a) Draw the graph of f.

(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0, \\[2ex] \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

See also Example 2 in Section 1.1.

7. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.

(a) Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

(b) Deduce from this formula Dirichlet's test for convergence of a series: if the partial sums of the series $\sum b_n$ are bounded, and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.

8. Verify that $\dfrac{1}{2i} \sum_{n \neq 0} \dfrac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function illustrated in Figure 6, defined by $f(0) = 0$, and

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } -\pi < x < 0, \\[2ex] \dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } 0 < x < \pi. \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.

Theorem 2.2 *Suppose f and g are two integrable functions defined on the circle, and for some θ_0 there exists an open interval I containing θ_0 such that*

$$f(\theta) = g(\theta) \quad \text{for all } \theta \in I.$$

Then $S_N(f)(\theta_0) - S_N(g)(\theta_0) \to 0$ as N tends to infinity.

Proof. The function $f - g$ is 0 in I, so it is differentiable at θ_0, and we may apply the previous theorem to conclude the proof.

2.2 A continuous function with diverging Fourier series

We now turn our attention to an example of a continuous periodic function whose Fourier series diverges at a point. Thus, Theorem 2.1 fails if the differentiability assumption is replaced by the weaker assumption of continuity. Our counter-example shows that this hypothesis which had appeared plausible, is in fact false; moreover, its construction also illuminates an important principle of the theory.

The principle that is involved here will be referred to as "symmetry-breaking."[1] The symmetry that we have in mind is the symmetry between the frequencies $e^{in\theta}$ and $e^{-in\theta}$ which appear in the Fourier expansion of a function. For example, the partial sum operator S_N is defined in a way that reflects this symmetry. Also, the Dirichlet, Fejèr, and Poisson kernels are symmetric in this sense. When we break the symmetry, that is, when we split the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ into the two pieces $\sum_{n\geq 0} a_n e^{in\theta}$ and $\sum_{n<0} a_n e^{in\theta}$, we introduce new and far-reaching phenomena.

We give a simple example. Start with the sawtooth function f which is odd in θ and which equals $i(\pi - \theta)$ when $0 < \theta < \pi$. Then, by Exercise 8 in Chapter 2, we know that

$$f(\theta) \sim \sum_{n\neq 0} \frac{e^{in\theta}}{n}. \tag{4}$$

Consider now the result of breaking the symmetry and the resulting series

$$\sum_{n=-\infty}^{n=-1} \frac{e^{in\theta}}{n}.$$

Then, unlike (4), the above is no longer the Fourier series of a Riemann integrable function. Indeed, suppose it were the Fourier series of an

[1] We have borrowed this terminology from physics, where it is used in a very different context.

integrable function, say $\tilde{f}$, where in particular $\tilde{f}$ is bounded. Using the Abel means, we then have

$$|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n},$$

which tends to infinity as r tends to 1, because $\sum 1/n$ diverges. This gives the desired contradiction since

$$|A_r(\tilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| P_r(\theta)\, d\theta \leq \sup_\theta |\tilde{f}(\theta)|,$$

where $P_r(\theta)$ denotes the Poisson kernel discussed in the previous chapter.

The sawtooth function is the object from which we will fashion our counter-example. We proceed as follows. For each $N \geq 1$ we define the following two functions on $[-\pi, \pi]$,

$$f_N(\theta) = \sum_{1 \leq |n| \leq N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N \leq n \leq -1} \frac{e^{in\theta}}{n}.$$

We contend that:

(i) $|\tilde{f}_N(0)| \geq c \log N$.

(ii) $f_N(\theta)$ is uniformly bounded in N and θ.

The first statement is a consequence of the fact that $\sum_{n=1}^{N} 1/n \geq \log N$, which is easily established (see also Figure 2):

$$\sum_{n=1}^{N} \frac{1}{n} \geq \sum_{n=1}^{N-1} \int_n^{n+1} \frac{dx}{x} = \int_1^N \frac{dx}{x} = \log N.$$

To prove (ii), we argue in the same spirit as in the proof of Tauber's theorem, which says that if the series $\sum c_n$ is Abel summable to s and $c_n = o(1/n)$, then $\sum c_n$ actually converges to s (see Exercise 14 in Chapter 2). In fact, the proof of Tauber's theorem is quite similar to that of the lemma below.

Lemma 2.3 *Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum_{n=1}^{\infty} c_n$ are bounded as r tends to 1 (with $r < 1$). If $c_n = O(1/n)$, then the partial sums $S_N = \sum_{n=1}^{N} c_n$ are bounded.*

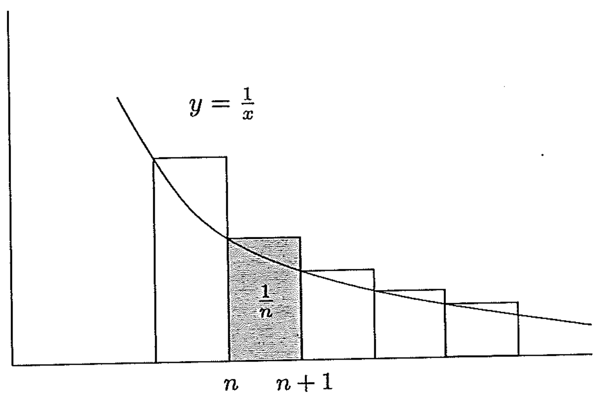

Figure 2. Comparing a sum with an integral

Proof. Let $r = 1 - 1/N$ and choose M so that $n|c_n| \le M$. We estimate the difference

$$S_N - A_r = \sum_{n=1}^{N}(c_n - r^n c_n) - \sum_{n=N+1}^{\infty} r^n c_n$$

as follows:

$$\begin{aligned}|S_N - A_r| &\le \sum_{n=1}^{N} |c_n|(1 - r^n) + \sum_{n=N+1}^{\infty} r^n |c_n| \\ &\le M \sum_{n=1}^{N}(1-r) + \frac{M}{N} \sum_{n=N+1}^{\infty} r^n \\ &\le MN(1-r) + \frac{M}{N}\frac{1}{1-r} \\ &= 2M,\end{aligned}$$

where we have used the simple observation that

$$1 - r^n = (1-r)(1 + r + \cdots + r^{n-1}) \le n(1-r).$$

So we see that if M satisfies both $|A_r| \le M$ and $n|c_n| \le M$, then $|S_N| \le 3M$.

We apply the lemma to the series

$$\sum_{n \ne 0} \frac{e^{in\theta}}{n},$$

which is the Fourier series of the sawtooth function f used above. Here $c_n = e^{in\theta}/n + e^{-in\theta}/(-n)$ for $n \neq 0$, so clearly $c_n = O(1/|n|)$. Finally, the Abel means of this series are $A_r(f)(\theta) = (f * P_r)(\theta)$. But f is bounded and P_r is a good kernel, so $S_N(f)(\theta)$ is uniformly bounded in N and θ, as was to be shown.

We now come to the heart of the matter. Notice that f_N and $\tilde{f}_N$ are trigonometric polynomials of degree N (that is, they have non-zero Fourier coefficients only when $|n| \leq N$). From these, we form trigonometric polynomials P_N and $\tilde{P}_N$, now of degrees $3N$ and $2N-1$, by displacing the frequencies of f_N and $\tilde{f}_N$ by $2N$ units. In other words, we define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$ and $\tilde{P}_N(\theta) = e^{i(2N)\theta}\tilde{f}_N(\theta)$. So while f_N has non-vanishing Fourier coefficients when $0 < |n| \leq N$, now the coefficients of P_N are non-vanishing for $N \leq n \leq 3N$, $n \neq 2N$. Moreover, while $n = 0$ is the center of symmetry of f_N, now $n = 2N$ is the center of symmetry of P_N. We next consider the partial sums S_M.

Lemma 2.4

$$S_M(P_N) = \begin{cases} P_N & \textit{if } M \geq 3N, \\ \tilde{P}_N & \textit{if } M = 2N, \\ 0 & \textit{if } M < N. \end{cases}$$

This is clear from what has been said above and from Figure 3.

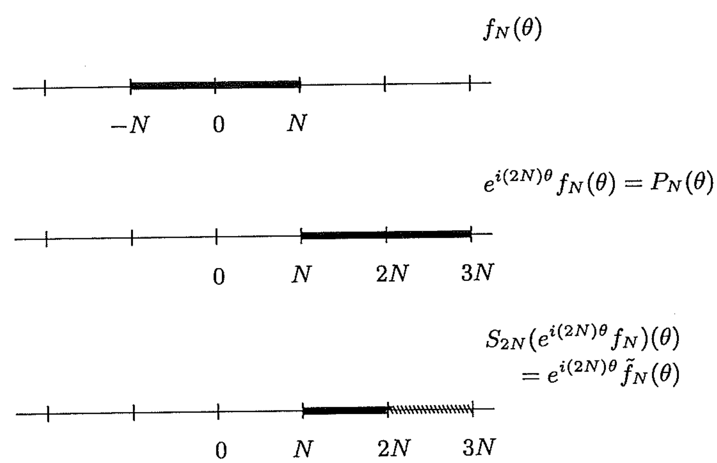

Figure 3. Breaking symmetry in Lemma 2.4

The effect is that when $M = 2N$, the operator S_M breaks the symmetry of P_N, but in the other cases covered in the lemma, the action of S_M

is relatively benign, since then the outcome is either P_N or 0.

Finally, we need to find a convergent series of positive terms $\sum \alpha_k$ and a sequence of integers $\{N_k\}$ which increases rapidly enough so that:

(i) $N_{k+1} > 3N_k$,

(ii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

We choose (for example) $\alpha_k = 1/k^2$ and $N_k = 3^{2^k}$ which are easily seen to satisfy the above criteria.

Finally, we can write down our desired function. It is

$$f(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta).$$

Due to the uniform boundedness of the P_N (recall that $|P_N(\theta)| = |f_N(\theta)|$), the series above converges uniformly to a continuous periodic function. However, by our lemma we get

$$|S_{2N_m}(f)(0)| \geq c\alpha_m \log N_m + O(1) \to \infty \qquad \text{as } m \to \infty.$$

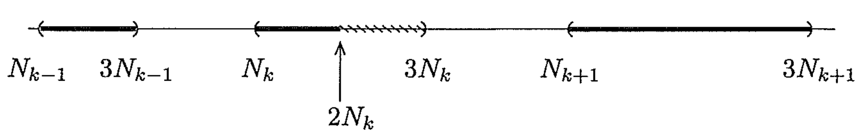

Figure 4. Symmetry broken in the middle interval $(N_k, 3N_k)$

Indeed, the terms that correspond to N_k with $k < m$ or $k > m$ contribute $O(1)$ or 0, respectively (because the P_N's are uniformly bounded), while the term that corresponds to N_m is in absolute value greater than $c\alpha_m \log N_m$ because $|\tilde{P}_N(\theta)| = |\tilde{f}_N(\theta)| \geq c \log N$. So the partial sums of the Fourier series of f at 0 are not bounded, and we are done since this proves the divergence of the Fourier series of f at $\theta = 0$. To produce a function whose series diverges at any other preassigned $\theta = \theta_0$, it suffices to consider the function $f(\theta - \theta_0)$.

3 Exercises

1. Show that the first two examples of inner product spaces, namely $\mathbb{R}^d$ and $\mathbb{C}^d$, are complete.