MATH 756
HW0: Due Friday Jan 19

1. Cesaro Summability

(a) Give an epsilon proof of the fact that if (x_n) is a convergent sequence of real numbers, then the sequence (y_n) defined by

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example of a non-convergent sequence (x_n) such that the corresponding sequence (y_n) is convergent.

2. Abel Summability

Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. In Real Analysis it is shown that that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is uniformly convergent on $[-y, y]$ for every $y \in (0, 1)$. To obtain the uniform convergence all the way to $y = 1$, we need to use a Lema by Abel. We introduce the following definition: A series $\sum_{n=0}^{\infty} a_n$ is called **Abel summable** if $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$ exists.

(a) In the same way as in the Cesaro summability, Abel summability is stronger than just regular summability of a series: Show that the divergent series $1 - 2 + 3 - 4 + 5 - 6 + \cdots$ is Abel summable.

(b) Find another example of a sequence that is Abel summable but not summable. (Hint: Consider Taylor series that are convergent on $(-1, 1)$ but not for $x = 1$).

(c) Prove Abel's Summation by Parts Formula: Let $(a_n), (b_n)$ be two sequences of real numbers, and define $B_n = \sum_{k=1}^{n} b_n$.

$$\sum_{n=1}^{N} a_n b_n = a_{N+1} B_N - \sum_{n=1}^{N} B_n (a_{n+1} - a_n),$$

(d) Show that if the series $\sum_{n=1}^{\infty} c_n$ converges to s, then it is Abel-summable to s. Hints: First show that it is enough to prove it for $s=0$, then use the summation by parts formula to prove that

$$\sum_{n=1}^{N} c_n r^n = r^N S_N + (1-r)\sum_{n=1}^{N-1} S_n r^n,$$

where $S_N = c_1 + \ldots + c_N$.

(e) Show that if the series $\sum_{n=1}^{\infty} c_n$ is Cesaro-summable to s, then it is Abel-summable to s. Hint: Apply summation by parts again to show that

$$\sum_{n=1}^{N} c_n r^n = r^N S_N + N\sigma_N(1-r)r^N + (1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n,$$

where $\sigma_N = \frac{s_1 + \cdots s_N}{N}$.

(f) Show that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} n$$

is not Cesaro-summable, but it is Abel-summable, and find the value of its Abel sum.

(g) Use Abel summability to compute the exact value of the following two series:

$$\frac{1}{2\cdot 1} - \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} - \frac{1}{5\cdot 4} + \cdots$$

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots$$

3. The Laplacian

Abel summability is very useful in the solution of the Laplace equation on the unit disc. (The Laplace operator has great importance in mathematics and physics, and appears in the three main second order linear partial differential equations: The heat equation $\Delta u - \frac{\partial u}{\partial t} = 0$, the wave equation $\Delta u - \frac{\partial^2 u}{\partial t^2} = 0$, and the Laplace equation $\Delta u = 0$, which represents the steady-state heat equation, among other things).

The Laplacian Δ is a differential operator defined for a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

The functions whose Laplacian is zero are called **harmonic functions**, and play an important role in PDEs, complex analysis and harmonic analysis.

(a) Find examples of harmonic polynomials in two variables $p(x, y)$ of degrees 0, 1, 2, 3 and 8.

(b) Let $x = r\cos(\theta), y = r\sin(\theta)$ be the polar coordinates in $\mathbb{R}^2$. Use the chain rule to derive the formula for the Laplacian in polar coordinates,

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial^2 \theta}.$$

Show all your work.

(c) Check that $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ are solutions to the Laplace equation $\Delta f = 0$ in polar coordinates, for any $n \in \mathbb{N}$.

We will continue studying how to apply Abel's summability to solve the Laplace equation in the next classes and assignments.