MATH 754: Homework 1 (Due September 20)

Chapter I, Section 1:

1. Exercise I.1.3 in the book.

2. Let X be a vector space over $\mathbb{C}$, and let $u : X \times X \to \mathbb{C}$ be a sesquilineal function. Prove that u is hermitian (*i.e.*, for every $x, y \in X$, $u(x, y) = \overline{u(y, x)}$) if and only if for every $x \in X$, $u(x, x) \in \mathbb{R}$.

3. (a) Prove that a necessary and sufficient condition for a normed space X to be complete is that for every sequence of vectors $\{x_n\} \subset X$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the series $\sum_{n=1}^{\infty} x_n$ converges to an element $x \in X$.

   (b) Use the condition in (a) to show that the space

   $$A(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is bounded } \}$$

   is complete.

4. Let X be a normed space.

   (a) Let $M = [x_0]$ be the subspace generated by the vector $x_0 \in X$. Prove that M, with the metric inherited from X, is complete.

   (b) Assume that M_1 is a closed subspace of X. Let $x_1 \in X \setminus M_1$. Show that $M_2 = M_1 + [x_1]$ is a closed subspace of X.

   (c) Use (a) and (b) to prove by induction that any finite-dimensional subspace of a normed space is closed.

Chapter I, Section 2:

1. Exercises I.2.1 and III.1.4 in the book.

2. Consider $C[0, 1]$, the space of all continuous functions on $[0, 1]$ with the norm of uniform convergence. We define

   $$M = \{f \in C[0, 1] : \int_0^{1/2} f(t)dt - \int_{1/2}^{1} f(t)dt = 1\}$$

   (a) Prove that M is a closed, convex subset of $C[0, 1]$.

   (b) Show that for every $\varepsilon > 0$ there is an element of M with norm equal to $1+\varepsilon$, but that there is no element of M with norm 1. Does this contradict Theorem I.2.5?

3. Let A be a subspace of a Hilbert space X. Show that $(A^{\perp})^{\perp} = \overline{A}$.

4. Let $\{F_\alpha\}_{\alpha\in A}$ be a familiy of closed subspaces of the Hilbert space X. Prove that

$$\left(\bigcup_{\alpha\in A} F_\alpha\right)^\perp = \bigcap_{\alpha\in A} F_\alpha^\perp$$

and

$$\left(\bigcap_{\alpha\in A} F_\alpha\right)^\perp = \overline{\left[\bigcup_{\alpha\in A} F_\alpha^\perp\right]}$$

Remember that $[A]$ means the subspace spanned by A.