MATH 756: Homework 1: Due Jan 31

1. Folland, Chapter 6, Ex. 2 (prove Theorem 6.8, about the properties of L^∞). Skip (d) and (e): The proof for (d) is identical than the one seen in class. We will see density results including (e) later on.

   (a) (Hölder's Ineq): If f $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$.

   (b) $\|\cdot\|_\infty$ is a norm on L^∞.

   (c) $\|f - f_n\|_\infty \to 0$ if and only if there exists a set E whose complement has measure 0 and such that f_n converges uniformly to f on E.

2. Use induction to prove the following generalization of Hölder's inequality: Let $(X, \mathcal{M}, \mu)$ be a measure space. Given $1 < p_1, p_2, \ldots, p_n < \infty$ such that $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \cdots + \dfrac{1}{p_n} = 1$, and measurable functions $f_j \in L^p_j(X)$, show that

$$\left| \int_X f_1 f_2 \cdots f_n \, d\mu \right| \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}.$$

3. Let $1 \le p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$.

4. If $\mu(X) = 1$, prove that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$, even in the case where $\|f\|_\infty = \infty$. (In fact, the result is also true for any space X with finite measure).

5. Let p be a real number, $0 < p < \infty$.

   (a) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{p<r<\infty} L^r$.

   (b) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{0<r<p} L^r$.

   (c) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{r \ne p} L^r$.

   Hint: Consider functions of the type $\frac{1}{x^a}$ *or* $\frac{1}{x^a(1+\log x)^b}$ *for appropriate powers a,b.*

6. Let p be a real number, $0 < p < \infty$.

   (a) Find a measure space (X, μ) and a function $f \in \left(\bigcap_{p<r<\infty} L^r \right) \setminus L^p$.

   (b) Find a measure space (X, μ) and a function $f \in \left(\bigcap_{0<r<p} L^r \right) \setminus L^p$.

   (c) Prove that it is impossible to find a measure space (X, μ) and a function $f \in \left(\bigcap_{r \ne p} L^r \right) \setminus L^p$.