[Hint: Show that $\int_a^b (\log\theta)^2\, d\theta \to 0$ if $0 < a < b$ and $b \to 0$, by using the fact that the derivative of $\theta(\log\theta)^2 - 2\theta\log\theta + 2\theta$ is equal to $(\log\theta)^2$.]

6. Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by

$$a_k = \begin{cases} 1/k & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0. \end{cases}$$

Note that $\{a_k\} \in \ell^2(\mathbb{Z})$, but that no Riemann integrable function has k^{th} Fourier coefficient equal to a_k for all k.

7. Show that the trigonometric series

$$\sum_{n\geq 2} \frac{1}{\log n} \sin nx$$

converges for every x, yet it is not the Fourier series of a Riemann integrable function.

The same is true for $\sum \frac{\sin nx}{n^\alpha}$ for $0 < \alpha < 1$, but the case $1/2 < \alpha < 1$ is more difficult. See Problem 1.

8. Exercise 6 in Chapter 2 dealt with the sums

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Similar sums can be derived using the methods of this chapter.

(a) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sums of the following two series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

In fact, they are $\pi^4/96$ and $\pi^4/90$, respectively.

(b) Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Remark. The general expression when k is even for $\sum_{n=1}^{\infty} 1/n^k$ in terms of π^k is given in Problem 4. However, finding a formula for the sum $\sum_{n=1}^{\infty} 1/n^3$, or more generally $\sum_{n=1}^{\infty} 1/n^k$ with k odd, is a famous unresolved question.

9. Show that for α not an integer, the Fourier series of

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

Apply Parseval's formula to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

10. Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x,t)$ of the string at time t satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \tau/\rho.$$

The string is subject to the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

where we assume that $f \in C^1$ and g is continuous. We define the total **energy** of the string by

$$E(t) = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The first term corresponds to the "kinetic energy" of the string (in analogy with $(1/2)mv^2$, the kinetic energy of a particle of mass m and velocity v), and the second term corresponds to its "potential energy."

Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,

$$E(t) = E(0) = \frac{1}{2}\rho \int_0^L g(x)^2\, dx + \frac{1}{2}\tau \int_0^L f'(x)^2\, dx.$$

11. The inequalities of Wirtinger and Poincaré establish a relationship between the norm of a function and that of its derivative.

(a) If f is T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)\,dt = 0$, show that

$$\int_0^T |f(t)|^2\,dt \leq \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt,$$

with equality if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$. [Hint: Apply Parseval's identity.]

(b) If f is as above and g is just C^1 and T-periodic, prove that

$$\left|\int_0^T \overline{f(t)}g(t)\,dt\right|^2 \leq \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\,dt \int_0^T |g'(t)|^2\,dt.$$

(c) For any compact interval $[a, b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\,dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved. [Hint: Extend f to be odd with respect to a and periodic of period $T = 2(b-a)$ so that its integral over an interval of length T is 0. Apply part a) to get the inequality, and conclude that equality holds if and only if $f(t) = A\sin(\pi\frac{t-a}{b-a})$].

12. Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

[Hint: Start with the fact that the integral of $D_N(\theta)$ equals 2π, and note that the difference $(1/\sin(\theta/2)) - 2/\theta$ is continuous on $[-\pi, \pi]$. Apply the Riemann-Lebesgue lemma.]

13. Suppose that f is periodic and of class C^k. Show that

$$\hat{f}(n) = o(1/|n|^k),$$

that is, $|n|^k\hat{f}(n)$ goes to 0 as $|n| \to \infty$. This is an improvement over Exercise 10 in Chapter 2.

[Hint: Use the Riemann-Lebesgue lemma.]

14. Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.

[Hint: Use the Cauchy-Schwarz inequality and Parseval's identity for f'.]

15. Let f be 2π-periodic and Riemann integrable on $[-\pi, \pi]$.

(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}\,dx$$

hence

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}\,dx.$$

(b) Now assume that f satisfies a Hölder condition of order α, namely

$$|f(x+h) - f(x)| \le C|h|^{\alpha}$$

for some $0 < \alpha \le 1$, some $C > 0$, and all x, h. Use part a) to show that

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies

$$|f(x+h) - f(x)| \le C|h|^{\alpha},$$

and $\hat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

[Hint: For (c), break up the sum as follows $f(x+h) - f(x) = \sum_{2^k \le 1/|h|} + \sum_{2^k > 1/|h|}$. To estimate the first sum use the fact that $|1 - e^{i\theta}| \le |\theta|$ whenever θ is small. To estimate the second sum, use the obvious inequality $|e^{ix} - e^{iy}| \le 2$.]

16. Let f be a 2π-periodic function which satisfies a Lipschitz condition with constant K; that is,

$$|f(x) - f(y)| \le K|x-y| \quad \text{for all } x, y.$$

This is simply the Hölder condition with $\alpha = 1$, so by the previous exercise, we see that $\hat{f}(n) = O(1/|n|)$. Since the harmonic series $\sum 1/n$ diverges, we cannot say anything (yet) about the absolute convergence of the Fourier series of f. The outline below actually proves that the Fourier series of f converges absolutely and uniformly.

(a) For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2 |\hat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2.$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly. [Hint: Use the Cauchy-Schwarz inequality to estimate the sum.]

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series of f converges absolutely.

17. If f is a bounded monotonic function on $[-\pi, \pi]$, then

$$\hat{f}(n) = O(1/|n|).$$

[Hint: One may assume that f is increasing, and say $|f| \leq M$. First check that the Fourier coefficients of the characteristic function of $[a, b]$ satisfy $O(1/|n|)$. Now show that a sum of the form

$$\sum_{k=1}^{N} \alpha_k \chi_{[a_k, a_{k+1}]}(x)$$

with $-\pi = a_1 < a_2 < \cdots < a_N < a_{N+1} = \pi$ and $-M \leq \alpha_1 \leq \cdots \leq \alpha_N \leq M$ has Fourier coefficients that are $O(1/|n|)$ uniformly in N. Summing by parts one gets a telescopic sum $\sum(\alpha_{k+1} - \alpha_k)$ which can be bounded by $2M$. Now approximate f by functions of the above type.]

18. Here are a few things we have learned about the decay of Fourier coefficients:

(a) if f is of class C^k, then $\hat{f}(n) = o(1/|n|^k)$;

(b) if f is Lipschitz, then $\hat{f}(n) = O(1/|n|)$;

(c) if f is monotonic, then $\hat{f}(n) = O(1/|n|)$;

(d) if f is satisfies a Hölder condition with exponent α where $0 < \alpha < 1$, then $\hat{f}(n) = O(1/|n|^{\alpha})$;

(e) if f is merely Riemann integrable, then $\sum |\hat{f}(n)|^2 < \infty$ and therefore $\hat{f}(n) = o(1)$.

Nevertheless, show that the Fourier coefficients of a continuous function can tend to 0 arbitrarily slowly by proving that for every sequence of nonnegative real numbers $\{\epsilon_n\}$ converging to 0, there exists a continuous function f such that $|\hat{f}(n)| \geq \epsilon_n$ for infinitely many values of n.
[Hint: Choose a subsequence $\{\epsilon_{n_k}\}$ so that $\sum_k \epsilon_{n_k} < \infty$.]

19. Give another proof that the sum $\sum_{0<|n|\leq N} e^{inx}/n$ is uniformly bounded in N and $x \in [-\pi, \pi]$ by using the fact that

$$\frac{1}{2i} \sum_{0<|n|\leq N} \frac{e^{inx}}{n} = \sum_{n=1}^{N} \frac{\sin nx}{n} = \frac{1}{2} \int_0^x (D_N(t) - 1)\, dt,$$

where D_N is the Dirichlet kernel. Now use the fact that $\int_0^\infty \frac{\sin t}{t}\, dt < \infty$ which was proved in Exercise 12.

20. Let $f(x)$ denote the sawtooth function defined by $f(x) = (\pi - x)/2$ on the interval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$. The Fourier series of f is

$$f(x) \sim \frac{1}{2i} \sum_{|n|\neq 0} \frac{e^{inx}}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n},$$

and f has a jump discontinuity at the origin with

$$f(0^+) = \frac{\pi}{2}, \quad f(0^-) = -\frac{\pi}{2}, \quad \text{and hence} \quad f(0^+) - f(0^-) = \pi.$$

Show that

$$\max_{0<x\leq \pi/N} S_N(f)(x) - \frac{\pi}{2} = \int_0^\pi \frac{\sin t}{t}\, dt - \frac{\pi}{2},$$

which is roughly 9% of the jump π. This result is a manifestation of Gibbs's phenomenon which states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump.
[Hint: Use the expression for $S_N(f)$ given in Exercise 19.]