SUMMARY of PREREQUISITE TOPICS for MATH 266

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1. Fundamental Properties of $\mathbb{R}^n$

1.1 The (real) line $\mathbb{R}$

The real number system is a set $\mathbb{R}$ with addition and multiplication operations and the order relation "$<$" satisfying the following properties:

a) The addition and multiplication operations are commutative and associative, multiplication is distributive over addition. The additive identity is the number 0 and multiplicative identity is the number 1. Every real number a has an additive inverse $-a$; every non-zero real number b has a multiplicative inverse $b^{-1} = \frac{1}{b}$.
b) The properties of the order relation are
   (i) if $x < y$ and $y < z$, then $x < z$,
   (ii) if $x < y$, then $x + z < y + z$ for all $z \in \mathbb{R}$,
   (iii) if $x > 0$ and $y > 0$, then $xy > 0$, and
   (iv) For all $x, y \in \mathbb{R}$, either $x > y$, or $y > x$, or $x = y$. [Trichotomy]

Furthermore,

- The sum and product of any two real numbers is a real number (i.e., $\mathbb{R}$ is **closed** under addition and multiplication operations).
- Geometrically, $\mathbb{R}$ can be viewed as a **line**; elements $a \in \mathbb{R}$ are points on this line ordered following the properties in (b) above.
- $\mathbb{R}$ is **unbounded** (hence so are $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{Q}^c$).

1.2 The plane $\mathbb{R}^2$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs (a, b), where $a, b \in \mathbb{R}$. On $\mathbb{R}^2$ the **addition** and **scalar multiplication** operations are defined, respectively, as

$(a, b) + (a', b') = (a + a', b + b')$ for all $(a, b),\ (a', b') \in \mathbb{R}^2$, and
$\alpha(a, b) = (\alpha a, \alpha b)$ for all $(a, b) \in \mathbb{R}^2$ and for all $\alpha \in \mathbb{R}$.

Consequently,

$$(a, b) - (a', b') = (a, b) + (-1)(a', b') = (a - a', b - b')$$

Furthermore, for any $(a, b) \in \mathbb{R}^2$, we have $(a, b) = a(1, 0) + b(0, 1)$. Hence, any point in $\mathbb{R}^2$ can be expressed as a **linear combination** of $(1, 0)$ and $(0, 1)$.

Geometrically, $\mathbb{R}^2$ is viewed as a **plane**; elements $A = (a, b) \in \mathbb{R}^2$ are points on this plane. Caution: The points in this plane are **not ordered!**

For any pair of points $A = (a, b), B = (a', b') \in \mathbb{R}^2$, the distance between A and B is defined as $\sqrt{(a - a')^2 + (b - b')^2}$. The set of all points on the line passing through A and B and lying in between them is called as **the line segment** $\overline{AB}$. Hence the **length** of $\overline{AB}$ is given by $\sqrt{(a - a')^2 + (b - b')^2}$.

1.3 The space $\mathbb{R}^3$

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the set of all ordered triplets (a, b, c), where $a, b, c \in \mathbb{R}$. On $\mathbb{R}^3$ the **addition** and **scalar multiplication** operations are defined as in $\mathbb{R}^2$, i.e.,

$(a, b, c) + (a', b', c') = (a + a', b + b', c + c')$ for all $(a, b, c),\ (a', b', c') \in \mathbb{R}^3$, and
$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$ for all $(a, b, c) \in \mathbb{R}^3$ and for all $\alpha \in \mathbb{R}$.

Thus,

$$(a, b, c) - (a', b', c') = (a, b, c) + (-1)(a', b', c') = (a - a', b - b', c - c'),\ \text{and}$$
$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1).$$

Geometrically, $\mathbb{R}^3$ is viewed as the **space**; elements $(a, b, c) \in \mathbb{R}^3$ are points on the space. The points in the space are not ordered.

For any pair $A = (a, b, c), B = (a', b', c') \in \mathbb{R}^3$, the **the line segment** $\overline{AB}$ is the set of all points on the line passing through A and B and lying in between them. The length of $\overline{AB}$ is given by $\sqrt{(a - a')^2 + (b - b')^2 + (c - c')^2}$.

1.4 The n-dimensional space $\mathbb{R}^n,\ n \geq 4$

$\mathbb{R}^n$ (the cross product of $\mathbb{R}$ by itself n-times) is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_1 \in \mathbb{R},\ 1 \leq i \leq n$. On $\mathbb{R}^n$ the addition and scalar multiplication (with real numbers) operations are defined, pointwise; i.e.,

$(a_1, \ldots, a_n) + (a'_1, \ldots, a'_n) = (a_1 + a'_1, \ldots, a_n + a'_n)$ for all $(a_1, \ldots, a_n),\ (a'_1, \ldots, a'_n) \in \mathbb{R}^n$, and
$\alpha(a_1, \ldots, a_n) = (\alpha a_1, \ldots, \alpha a_n)$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}$.

Geometrically, $\mathbb{R}^n$ is referred to as **the n-dimensional space**; elements $(a_1, \ldots, a_n) \in \mathbb{R}^n$ are points on this space. The points in the n-dimensional space are not ordered.

2. $\mathbb{R}^n$ as a Set of Vectors

A line segment $\overline{AB}$ in $\mathbb{R}^n$ is called **directed line segment** if, say, A is the *initial point* and B is the *terminal point* of the segment. In that case we denote it by $\overrightarrow{AB}$. Two ordered line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called **equivalent** if $B - A = D - C$, denoted by $\overrightarrow{AB} \approx \overrightarrow{CD}$. Geometrically, equivalent line segments are those having the same length but lying on (possibly different) parallel lines. It follows that, if O denotes the origin, then $\overrightarrow{AB} \approx \overrightarrow{OC}$ if and only if $B - A = C$.

Since directed line segments have **initial point, magnitude (length)** and **direction**, they are used to represent **vectors**. Hence, mathematically, directed line segments in $\mathbb{R}^n$ are called **vectors in $\mathbb{R}^n$.** An arbitrary vector $\overrightarrow{AB}$ is also called as a **located vector**, whereas a vector of the type $\overrightarrow{OC}$ is called as a **position vector** and is denoted by $\overrightarrow{C}$. Observe that every located vector is equivalent to a position vector; indeed, many other located vectors may be equivalent to a single position vector. Since position vectors are easy to describe; in mathematics we study position vectors mostly (and transfer the properties obtained to located vectors by a simple translation). Hence, for the rest, when we say vector, it will always mean a position vector. First, note that the magnitude (also called the **norm**) of a vector $\overrightarrow{C}$ is the length of $\overline{OC}$, denoted by $\|C\|$. Any vector with norm equal to 1 is called a **unit vector**.

For any two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and a real number α, we define $\overrightarrow{A} + \overrightarrow{B}$ as the position vector $\overrightarrow{A + B}$, and $\alpha\overrightarrow{A}$ as the position vector αA. These define the addition and scalar multiplication on the set of vectors unambiguously. Consequently, the unit vector in the direction of a vector $\overrightarrow{A}$ is the vector $\frac{1}{\|A\|}\overrightarrow{A}$. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called **parallel**, denoted by $\overrightarrow{AB} \parallel \overrightarrow{CD}$, if $B-A=\alpha(D-C)$ for some $\alpha\in\mathbb{R}$. In the case position vectors, parallel vectors are coincident and one is a scalar multiple of the other.

Geometrically, from the properties of $\mathbb{R}^n$, if $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors, then $\overrightarrow{A}+\overrightarrow{B}$ is the (main) diagonal of the parallelogram defined by the sides $\overline{OA}$ and $\overline{OB}$, whereas $\alpha\overrightarrow{A}$ is the vector on the same line as $\overrightarrow{A}$ and rescaled by factor α. Note, if $\alpha>0$, then $\alpha\overrightarrow{A}$ has the same direction as $\overrightarrow{A}$; otherwise, the direction is reversed.

On $\mathbb{R}^n$, in addition to scalar multiplication, one can define other multiplication operations. A very important one is the **dot product** (or **inner product**). If $A=(a_1,a_2,\ldots,a_n)$ and $B=(b_1,b_2,\ldots,b_n)$ then we define the dot product by

$$\overrightarrow{A}\bullet\overrightarrow{B}=a_1b_1+a_2b_2+\cdots+a_nb_n.$$

Notice that, the outcome of dot product is a **scalar**, not a vector! The following are some of the important properties of the dot product.

a) $\overrightarrow{A}\bullet\overrightarrow{B}=\overrightarrow{B}\bullet\overrightarrow{A}$ and $\overrightarrow{A}\bullet(\overrightarrow{B}+\overrightarrow{C})=\overrightarrow{A}\bullet\overrightarrow{B}+\overrightarrow{A}\bullet\overrightarrow{C}$.
b) $\|A\|=\sqrt{\overrightarrow{A}\bullet\overrightarrow{A}}$.
c) If θ denotes the angle between the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, then
$$\cos\theta=\frac{\overrightarrow{A}\bullet\overrightarrow{B}}{\|A\|\,\|B\|}.$$
d) Two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ are **orthogonal** (denoted by $\overrightarrow{A}\perp\overrightarrow{B}$) if and only if $\overrightarrow{A}\bullet\overrightarrow{B}=0$.
e) $|\overrightarrow{A}\bullet\overrightarrow{B}|\le\|A\|\,\|B\|$. (Cauchy-Schwartz Inequality)
f) $\|\overrightarrow{A}+\overrightarrow{B}\|\le\|A\|+\|B\|$. (Triangle Inequality)
g) $\|\overrightarrow{A}+\overrightarrow{B}\|^2=\|A\|^2+\|B\|^2$ if and only if $\overrightarrow{A}\perp\overrightarrow{B}$. (Pythagorean Identity)
h) If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two (distinct) vectors, then the **projection (vector)** of $\overrightarrow{A}$ on $\overrightarrow{B}$ is a vector $P_B\overrightarrow{A}$ defined by $P_B\overrightarrow{A}=\frac{\overrightarrow{A}\bullet\overrightarrow{B}}{\|B\|^2}\,\overrightarrow{B}$.

3. Lines and Planes in $\mathbb{R}^n$

Lines and planes in $\mathbb{R}^n$ have a neat description in terms of vectors. A line passing through a point $P\in\mathbb{R}^n$ and in the direction of a vector $\overrightarrow{A}$ is defined as

$$L_{P,A}=\{Q\in\mathbb{R}^n:\ \overrightarrow{Q}=\overrightarrow{P}+t\overrightarrow{A},\ t\in\mathbb{R}\}.$$

The vector $\overrightarrow{A}$ is called the **direction** vector of the line. Accordingly, two lines $\overrightarrow{P}+t\overrightarrow{A}$ and $\overrightarrow{Q}+t\overrightarrow{B}$ are parallel iff $\overrightarrow{A}\parallel\overrightarrow{B}$, and are perpendicular iff $\overrightarrow{A}\perp\overrightarrow{B}$. In $\mathbb{R}^2$ any two perpendicular lines meet; however, this need not hold for perpendicular lines in $\mathbb{R}^n$ for $n\ge3$.

For any point $Q\in\mathbb{R}^n$ and a vector $\overrightarrow{N}$ in $\mathbb{R}^n$, the plane containing (or passing through) and perpendicular to $\overrightarrow{N}$ is defined as

$$P_{Q,N}=\{X\in\mathbb{R}^n:\ (\overrightarrow{X}-\overrightarrow{Q})\bullet\overrightarrow{N}\}=0.$$

The vector $\overrightarrow{N}$ is called the **normal** vector of the plane. Hence, two planes $P_{Q,N}$ and $P_{R,M}$ are parallel iff $\overrightarrow{N}\parallel\overrightarrow{M}$, and are perpendicular iff $\overrightarrow{N}\perp\overrightarrow{M}$. In $\mathbb{R}^3$ any pair of planes are either parallel (coincident) or do intersect.

It is well known in geometry that there is a unique plane passing through three distinct non-collinear points. This provided an alternative means of defining planes in $\mathbb{R}^n$ If A,B and C such distinct non-collinear points, then find a vector $\overrightarrow{N}$ orthogonal to any two of the vectors

$\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. Then, the plane desired is given by $P_{Q,N}$, where Q is any one of the points A, B or C. In $\mathbb{R}^3$, any plane passing through a point Q with normal vector $N = (a, b, c)$, has equation $ax + by + cz = d$, where $X = (x, y, z)$ and $d = \overrightarrow{N} \bullet \overrightarrow{Q}$.

4. Matrices

An $m \times n$ **(real) matrix** A is an array of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ & & \dots & & \\ & & \dots & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix},$$

where $a_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Each a_{ij} is called the ***ij*-th entry** of A. The terms $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$ are called the ***i*-th row** of A and the terms $a_{1j}, a_{2j}, a_{3j}, \dots, a_{nj}$ are called the ***j*-th column** of A. The i-th row and the j-th column of A are also denoted by A_i and A^j, respectively. When convenient, an $m \times n$ matrix A is also denoted by $A = (a_{ij})_{m \times n}$ or simply by $A = (a_{ij})$ when m and n are known. The set of all $m \times n$ real matrices is denoted by $\mathcal{M}_{m \times n}(\mathbb{R})$. A $1 \times n$ matrix is called a **row** matrix (vector) and an $n \times 1$ matrix is called a **column** matrix (vector). When $m = n$, an $n \times n$ matrix is called a **square** matrix; the set of all $n \times n$ real square matrices is denoted by $\mathcal{M}_n(\mathbb{R})$.

A $m \times n$ matrix whose all entries is 0 is called the **zero** matrix and is denoted by $O_{m,n}$, or simply O. The matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

is called the **identity** matrix of size n. When n is known, it is simply denoted by I. A matrix of the form

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ & & & \dots & & \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix},$$

where $*$ denotes not necessarily zero entries, is called an **upper triangular** matrix. Lower triangular matrices are defined similarly. A matrix of the form

$$\begin{bmatrix} * & 0 & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ & & \dots & & \\ & & \dots & & \\ 0 & 0 & 0 & \dots & * \end{bmatrix},$$

where $*$ denotes not necessarily zero entries, is called a **diagonal** matrix.

On $\mathcal{M}_{m\times n}(\mathbb{R})$ we define **addition** and **scalar multiplication** operations by

$$A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij}),\ \text{and}$$
$$\alpha A=\alpha(a_{ij})=(\alpha a_{ij}),$$

respectively, where $A=(a_{ij})$, $B=(b_{ij})\in\mathcal{M}_{m\times n}(\mathbb{R})$ and $\alpha\in\mathbb{R}$. In other words, these operations are performed entry-wise. Notice that both the addition and scalar multiplication operations are valid on matrices of the **same size**!

These operations satisfy the following properties: for any $A,B,C\in\mathcal{M}_{m\times n}(\mathbb{R})$ and $\alpha,\beta\in\mathbb{R}$,

a) $A+B=B+A$ (commutative) and $A+(B+C)=(A+B)+C$ (associative)
b) $A+O=A$ and $1A=A$
c) $\alpha(A+B)=\alpha A+\alpha B$
d) $(\alpha+\beta)A=\alpha A+\beta B$
e) $(\alpha\beta)A=\alpha(\beta A)=\beta(\alpha A)$.

Another operation on matrices is **matrix multiplication**, which is defined as, for any $A\in\mathcal{M}_{m\times n}(\mathbb{R})$ and $B\in\mathcal{M}_{n\times r}(\mathbb{R})$,

$$AB=(c_{ij})_{m\times r},\quad \text{where } c_{ij}=A_i\bullet B^j,$$

where A_i is the i-th row of A and B^j is the j-th column of B. Notice that $AB\in\mathcal{M}_{m\times r}(\mathbb{R})$ and $c_{ij}=A_i\bullet B^j=\sum_{k=1}^{n}a_{ik}b_{kj}$. Also, observe that, matrix multiplication AB is valid if the sizes of A and B match, in the sense that, *number of columns of A is equal to the number of rows of B.*

Matrix multiplication has the following properties: for any matrices A,B,C of appropriate size and $\alpha\in\mathbb{R}$,

a) $A(BC)=(AB)C$ (associative)
b) $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$ (distributive over addition)
c) $\alpha(AB)=(\alpha A)B=A(\alpha B)$
d) It is possible that $AB\neq BA$. (noncommutative)

A (square) matrix $A\in\mathcal{M}_n(\mathbb{R})$ is called **invertible** if there exists a matrix $B\in\mathcal{M}_n(\mathbb{R})$ such that

$$AB=I_{n\times n}=BA.$$

The matrix B is called the **inverse** of A, and is denoted by A^{-1}. Notice that not every matrix is invertible; non-invertible matrices are also called **singular**. Furthermore, if $A\in\mathcal{M}_n(\mathbb{R})$ is invertible,

a) the inverse A^{-1} is unique,
b) if B is also invertible, then $(AB)^{-1}=B^{-1}A^{-1}$,
c) $(A^{-1})^{-1}=A$,
d) if $AB=0$, then $B=0$; if $CA=0$, then $C=0$.

Determining invertibility of a matrix and (if so) finding the inverse matrix is a delicate process that requires some other tools, such as elementary row operations (EROs). These EROs are as follows

ERO-1: Interchanging any two rows of a matrix
ERO-2: Multiplying a row of a matrix by a scalar
ERO-3: Adding a scalar multiple of a row of a matrix to another row.

It turns out that applying EROs to a matrix does not alter its invertibility. Furthermore, if A is invertible, then the inverse A^{-1} is obtained via EROs using "auxiliary matrix". Namely,

$$[A|I] \ \overrightarrow{EROs} \ [I|A^{-1}] .$$

In the same manner, one can convert any matrix A into a upper (lower) triangular one by applying EROs:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ & & \dots & & \\ & & \dots & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \overrightarrow{EROs} \begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ & & & \dots & & \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} .$$

5. Systems of Linear Equations

Any set of linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ &\dots \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$

where $a_{ij} \in \mathbb{R}$ and $b_j \in \mathbb{R}$, for $1 \le i \le m$ and $1 \le j \le n$, is called a **system of m linear equations in n unknowns**, in short **SLEs**. Each a_{ij} is called the **ij-th coefficient** of the SLE. If $b_i = 0$ for all $1 \le i \le m$, the system is called **homogeneous**; otherwise, **non-homogeneous**. Any n-tuple $\mathbf{c} = (c_1, c_2, c_3, \dots, c_n)$, when substituted for $(x_1, x_2, x_3, \dots, x_n)$ in the SLE satisfies all the equations simultaneously, is called a **solution** of the SLE.

The SLE above can be expressed in a simplified manner by using matrix notation; namely, it can be rewritten as

$$AX = B,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ & & \dots & & \\ & & \dots & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

The matrix A is called the **coefficient** matrix of the SLE. The advantages of this notation are multi-fold; that it is easy to describe various properties of SLEs in a clear manner, it is also amenable to utilize matrix operations in their study, to name a few. One of these is the EROs when applied to the auxiliary matrix $[\ A \mid B\ ]$ leading to solution of the system, which is the matrix version of the Gauss Elimination process. In fact, the EROs method

$$[A|B] \ \overrightarrow{EROs} \ [U|B'] ,$$

where U is an upper triangular matrix, leads to a system $UX = B'$ that has the same solution(s) as the original SLE. Since upper triangular SLEs are very easy to solve, this is a very convenient method to solve a SLE! In some cases, one can achieve

$$[A|B] \ \overrightarrow{EROs} \ [D|B'] ,$$

where D is a diagonal matrix; of course, this diagonal SLE that has the same solution(s) as the original one and is trivial to solve!

There is another method of solving SLEs, suitable to $n \times n$-systems if the coefficient matrix A is invertible. In that case, if $AX = B$ is the SLE given, by multiplying both sides of the SLE from left with A^{-1}, we obtain

$$A^{-1}AX = A^{-1}B \;\Rightarrow\; IX = A^{-1}B \;\Rightarrow\; X = A^{-1}B,$$

which is the solution.

How about the existence and uniqueness of the solutions of SLEs? The answer to this question requires considering homogeneous and non-homogeneous cases separately.

a) *Homogeneous SLE* $AX = 0$. $X = 0$ is always a solution; hence a homogeneous SLE has at least one solution. This is the only (unique) solution if the SLE has the same number of unknowns as the number of equations **and** the coefficient matrix is invertible. If the number of unknowns is not the same as the number of equations, then the SLE has more than one solutions; of course the inverse matrix method does not apply, one needs to apply Gauss Elimination (EROs) to find these solutions.

   Geometrically, in $\mathbb{R}^3$, and equation of the form $ax_1 + bx_2 + cx_3 = d$ represents a plane; if d=0 it passes through origin, otherwise, does not contain origin. If the SLE is homogeneous,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0, \end{aligned}$$

   each equation is a plane passing through the origin; hence, 0 is always a solution. If the coefficient matrix is invertible, then 0 is the only solution, which means that these three planes are distinct. If there are other solutions, then there are infinitely many other solutions, which means that the planes intersect along a line passing through origin.

   If such a SLE has more unknowns (equations) than equations (unknowns), then it has infinitely many solutions.

b) *Non-homogeneous SLE* $AX = B,\ B \neq 0$. If A is invertible, then the system has unique solution $X = A^{-1}B$. If A is not invertible, then may have no, or multiple solutions. If the number of unknowns is not the same as the number of equations, then the SLE may have no or more than one solutions; of course the inverse matrix method does not apply, one needs to apply Gauss Elimination (EROs) to find these solutions, if they exist.

   Geometrically, if the SLE is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

   where at least one of b_i's is non-zero, the equations are planes (for at least one of them) not through origin. If the coefficient matrix is invertible, there is a unique solution, which is the point of intersections of the planes. If the coefficient matrix is not invertible, and all are distinct and at least two of them parallel to each other, they cannot have a common point of intersection; hence, no solution. If all are coincident or all meet along a line, they have infinitely many solutions.

   The cases when the number of unknowns is different than the number of equations are similar to the homogeneous case and the non-invertible case above. Detailed investigation of cases is left as an exercise. The picture in $\mathbb{R}^2$ is easier; hence, it is also left as an exercise.

6. Limits, Continuity and Derivative of Functions

Throughout we will consider functions $f : \mathbb{R} \to \mathbb{R}$.

A function f has **limit** L at a point $c \in \mathbb{R}$, denoted by

$$\lim_{x \to c} f(x) = L,$$

if, when x is sufficiently close to c, but not at c, the values $f(x)$ get arbitrarily close to L. If this is not the case, we say that f does not have a limit at c. Many functions have limits at every point $c \in \mathbf{R}$; namely, when f is a polynomial function, root function, trigonometric function $\sin x$ or $\cos x$, exponential function $a^x, a > 0$, or a logarithmic function $\log_a x$ for $c >$, then it has limit at every $c \in \mathbf{R}$. Furthermore, for these functions $\lim_{x \to c} f(x) = f(c)$. In general, the limit of f at c need not be equal to $f(c)$. The algebra of limits is as follows:

a) $\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
b) $\lim_{x \to c}[f(x)g(x)] = [\lim_{x \to c} f(x)]\ [\lim_{x \to c} g(x)]$
c) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$.

A function f is **continuous** at a point $c \in \mathbb{R}$ if $\lim_{x \to c} f(x) = f(c)$. If this is not the case, we say that f is discontinuous at c. Polynomial functions, root functions, trigonometric functions $\sin x$ or $\cos x$, exponential functions $a^x, a > 0$, and logarithmic functions $\log_a x$ are continuous at every c in their domains. If f and g are continuous at $c \in \mathbb{R}$, then

a) $f(x) \pm g(x)$ are continuous at c
b) $f(x)g(x)$ is continuous at c
c) $\frac{f(x)}{g(x)}$ is continuous at c, provided $g(c) \neq 0$.

The **derivative** of a function f at a point c, denoted by $f'(c)$, is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

if the limit exists; otherwise, f is called non-differentiable at c. If f has derivative at every point, it is called differentiable. In that case, the derivative function is denoted by $f'(x)$. Every differentiable function is continuous; however some continuous are not differentiable. Polynomial functions, trigonometric functions $\sin x$ or $\cos x$, exponential functions $a^x, a > 0$, and logarithmic functions $\log_a x$ are differentiable at every c in their domains; root functions are differentiable in their domains, except at 0. If f and g are differentiable, then

a) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
b) $[f(x)g(x)]' = f(x)'g(x) + f(x)g'(x)$
c) $[\frac{f(x)}{g(x)}]' = \frac{f(x)'g(x) - f(x)g'(x)}{[g(x)]^2}$
d) $[f(g(x))]' = f'(g(x))g'(x)$.

Derivatives of some special functions are:

a) $[x^a]' = ax^{a-1}$
b) $\sin' x = \cos x$ and $\cos' x = -\sin x$
c) $[e^x]' = e^x$
d) $[\ln |x|]' = \frac{1}{x}$.

7. Integration of Functions

Geometrically, for a function f, the **definite integral** on $[a, b]$, denoted by

$$\int_a^b f(x)dx,$$

is defined as the area bounded by the lines $x = a$, $x = b$, x-axis and the graph of f. The **indefinite integral** of a continuous function f is a (continuous) function $F(x)$ defined by

$$F(x) = \int f(x)dx + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant, by the Fundamental Theorem of Calculus. Furthermore,

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Every continuous and every piecewise-continuous functions, and discontinuous functions with only finitely many jump discontinuities are integrable.

If f and g are integrable, then

a) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
b) $\int cf(x)dx = c \int f(x)dx$, where c is a constant.

Indefinite integrals of some special functions are:

a) $\int x^a dx = \frac{x^{a+1}}{a+1} + C, \quad a \neq -1$
b) $\int \sin^x dx = -\cos x + C$ and $\int \cos x dx = \sin x + C$
c) $\int e^x dx = e^x + C$
d) $\int \frac{1}{x} dx = \ln |x| + C$
e) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$

Unfortunately, integral of a product (ratio) of functions is not equal to the product (ratio) of the integrals! Hence, for such functions we develop **techniques of integration**.

Substitution Method. If the integral is in the form $\int f(g(x))g'(x)dx$, then letting $u = g(x)$, it is converted into $\int f(u)du$ to integrate.

Example. $\int x(x^2 - 1)^{2020} dx$. (Exercise: Evaluate this integral.)

Integration by Parts. Utilize the identity $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$.

Example. $\int e^x \cos x dx$. (Exercise: Evaluate this integral.)

Integration by Partial Fractions. If the integral is of the form $\int \frac{f(x)}{g(x)} dx$, then express the rational function as a linear combination of rational functions whose denominators are functions of the form $(x - k)^\alpha$ or $(ax^2 + bx + c)^\alpha$ and integrate.

Example. $\int \frac{x^2+1}{x^2(x^2-2x-3)} dx$. (Exercise: Evaluate this integral.)

In addition to these, there are special means of integrating trigonometric functions and substitution methods involving trigonometric functions.

Examples. a) $\int \cos^2 x dx$. (Exercise: Evaluate this integral.)
b) $\int \sin^3 x \cos x dx$. (Exercise: Evaluate this integral.)
c) $\int \frac{dx}{\sqrt{x^2-4}}$. (Exercise: Evaluate this integral.)

8. Improper Integrals

Integrals of the form $\int_a^\infty f(x)dx$ or $\int_{-\infty}^a f(x)dx$, where $a \in \mathbb{R}$, are called **improper integrals**. Such integrals are evaluated as

$$\int_a^\infty f(x)dx = \lim_{r\to\infty} \int_a^r f(x)dx, \text{ or}$$
$$\int_{-\infty}^a f(x)dx = \lim_{r\to-\infty} \int_r^a f(x)dx,$$

provided that the limits exist. Improper integrals of the form $\int_{-\infty}^\infty f(x)dx$ are evaluated by using the property

$$\int_{-\infty}^\infty f(x)dx = \int_a^\infty f(x)dx + \int_{-\infty}^a f(x)dx.$$