MATH 724
FALL 2010
HOMEWORK 2

Due Friday, February 19, 2010.

1. Let R be an integral domain. A nonzero nonunit element $z \in R$ is said to be a *universal side divisor* if given any $x \in R$ there is a $r \in R$ such that

$$x = rz + v$$

where v is either 0 or a unit in R. Let R be a Euclidean domain with norm function ϕ.

a) (5 pt) Show that any nonunit in R of minimal norm is a universal side divisor.
b) (5 pt) Show that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is not Euclidean.

2. Let d be a squarefree integer. We define

$$R = \mathbb{Z}[\omega] \text{ where } \omega = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \bmod(4); \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \bmod(4). \end{cases}$$

a) (5 pt) Show that R is integral over $\mathbb{Z}$.
b) (5 pt) We define a *norm* to be a map $N : R \longrightarrow \mathbb{N}_0$ satisfying $N(0) = 0$ and $N(ab) = N(a)N(b)$. Show that $N : \mathbb{Z}[\omega] \longrightarrow \mathbb{N}_0$ defined by $N(a + b\omega) = (a + b\omega)(a + b\overline{\omega})$ is a norm.
c) (5 pt) Use the norm to show that $\mathbb{Z}[\omega]$ is atomic.
d) (5 pt) Show that the ring $\mathbb{Z}[\sqrt{-14}]$ is not a UFD.

3. Let R be a domain and N a norm on R. We say that N is a *Dedekind-Hasse norm* if N is positive and for every nonzero $x, y \in R$ either y is divisible by x or we can find $a, b \in R$ such that

$$0 < N(ax + by) < N(x).$$

a) (5 pt) Show that R is a PID if and only if R has a Dedekind-Hasse norm.
b) (5 pt) Show that the norm defined in problem 2 for the ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a Dedekind-Hasse norm (hence $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a PID that is not Euclidean).

4. Suppose that R is a UFD.

a) (5 pt) Show that $R[[x]]$ is atomic.
b) (5 pt) Show that if $f(x) \in R[[x]]$ is such that $f(0) = \prod_{i=1}^{n} p_i^{a_i}$ (with the p_i's distinct nonzero prime elements of R and each $a_i > 0$) and $f(x) = \prod_{j=1}^{t} f_j(x)$ (with each $f_j(x)$ irreducible) then $1 \leq t \leq \sum_{i=1}^{n} a_i$. Give examples to show that both bounds can be achieved.
c) (5 pt) Suppose that R is a PID. Show that if $f(x) \neq x$ is irreducible in $R[[x]]$ then $f(x) = p^n + xg(x)$ with p a nonzero prime in R and $g(x) \in R[[x]]$ (is the converse true?).
d) (5 pt) With the notation as above, show that if R is a PID, then $n \leq t \leq \sum_{i=0}^{n} a_i$.

5. Let R be a domain with quotient field K. $\omega \in K$ is called almost integral over R if there is a nonzero $r \in R$ such that $rx^n \in R$ for all $n \geq 0$. If R contains all of the elements $\omega \in K$ that are almost integral over R, we say that R is completely integrally closed.

a) (5 pt) Show that any UFD is completely integrally closed.

b) (5 pt) Suppose that $A \subseteq B$ are integral domains. Completely characterize when the domain $A + xB[x]$ is a UFD.