MATH 724
FALL 2010
HOMEWORK 3

Due Monday, March 8, 2010

1. Let $d < 0$ be a squarefree integer and R the ring of integers of the field $\mathbb{Q}(\sqrt{d})$.
   a) (5 pt) Show that if R is a UFD, then d must be prime (or -1).
   b) (5 pt) Show that if R is an HFD, then $d = -1, p$, or pq where p and q are distinct primes.
   c) (5 pt) What is the status to the converse of the statements in parts a) and b)?

2. Let R be an integral domain and K a field containing R. We consider the domain $D := R + xK[x]$
   a) (5 pt) Show that D is atomic if and only if R is a field.
   b) (5 pt) Show that if D is atomic, then D is an HFD.

3. We have shown in class that the domain $\mathbb{Z}[\sqrt{-3}]$ is an HFD.
   a) (5 pt) Find all nonprime irreducibles in $\mathbb{Z}[\sqrt{-3}]$.
   b) (5 pt) What is the status of $\mathbb{Z}[\sqrt{-3}][x]$? Is it an HFD?

4. The domain $\mathbb{Z}[\sqrt{-61}]$ has class group isomorphic to $\mathbb{Z}/6\mathbb{Z}$. You may use this fact (along with the fact that every ideal class contains infinitely many primes) to answer the following questions.
   a) (5 pt) Find all possible ideal factorizations of an irreducible in $\mathbb{Z}[\sqrt{-61}]$ (in terms of primes from classes in the class group).
   b) (5 pt) Use this information to construct (in terms of prime ideals) an element with irreducible factorizations of length 6 and length 2.
   c) (5 pt) Find a concrete example of such an element.