MATH 725
FALL 2005
HOMEWORK 1

Due Friday, September 8, 2006.

1. Let R be a domain, and $a, b \in R$. We define the *greatest common divisor* of a and b to be a common divisor $d := \gcd(a, b)$ with the property that if x is any other common divisor of a and b then x divides d. We also define the *least common multiple* to be a common multiple $L := \mathrm{lcm}(a, b)$ with the property that if y is any other common multiple of a and b then L divides y.

  a) (5 pt) Give an example of a domain, R, and two elements $a, b \in R$ such that $\gcd(a, b)$ exists but $\mathrm{lcm}(a, b)$ does not.
  b) (5 pt) Show that if $\mathrm{lcm}(a, b)$ exists, then so does $\gcd(a, b)$.

2. A *GCD domain* is an integral domain in which every two (nonzero) elements have a greatest common divisor. Show that in a GCD domain the following hold.

  a) (5 pt) $x(\gcd(a, b)) = \gcd(xa, xb)$.
  b) (5 pt) If $\gcd(a, b) = d$ then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.
  c) (5 pt) If $\gcd(x, a) = 1$ and $\gcd(x, b) = 1$ then $\gcd(x, ab) = 1$.
  d) (5 pt) If $\gcd(x, a) = 1$ and x divides ab then x divides b.
  e) (5 pt) Which of the above hold in a general integral domain?

3. A Bezout domain is a domain where every finitely generated ideal is principal.

  a) (5 pt) Show R is a valuation domain if and only if R is a quasi-local Bezout domain.
  b) (5 pt) Show that any Bezout domain is a GCD domain.
  c) (5 pt) Show that any UFD is a GCD domain.
  d) (5 pt) Give examples to show that the classes of Bezout domains and UFDs are distinct.