MATH 725
FALL 2005
HOMEWORK 2

Due Friday, September 23, 2006.

1. Suppose that R is a UFD.
   a) (5 pt) Show that $R[[x]]$ is atomic.
   b) (5 pt) Show that if $f(x) \in R[[x]]$ is such that $f(0) = \prod_{i=1}^{n} p_i^{a_i}$ (with the p_i's distinct nonzero prime elements of R and each $a_i > 0$) and $f(x) = \prod_{j=1}^{t} f_j(x)$ (with each $f_j(x)$ irreducible) then $1 \leq t \leq \sum_{i=1}^{n} a_i$. Give examples to show that both bounds can be achieved.
   c) (5 pt) Suppose that R is a PID. Show that if $f(x) \neq x$ is irreducible in $R[[x]]$ then $f(x) = p^n + xg(x)$ with p a nonzero prime in R and $g(x) \in R[[x]]$ (is the converse true?).
   d) (5 pt) With the notation as above, show that if R is a PID, then $n \leq t \leq \sum_{i=0}^{n} a_i$.

2. Let R be a domain and $f(x) \in R[x]$. We define the content of the polynomial f to be the ideal (in R) generated by the coefficients of f (that is, if $f(x) = \sum_{i=0}^{n} a_i x^i$ then $c(f) = (a_0, a_1, \cdots, a_n)$). Show that if $f, g \in R[x]$ then $c(fg) \subseteq c(f)c(g)$ and give an example to show that this containment may be strict.

3. Let R be a domain with quotient field K. $\omega \in K$ is called almost integral over R if there is a nonzero $r \in R$ such that $rx^n \in R$ for all $n \geq 0$. If R contains all of the elements $\omega \in K$ that are almost integral over R, we say that R is completely integrally closed.
   a) (5 pt) Show that any UFD is completely integrally closed.
   b) (5 pt) Suppose that $A \subseteq B$ are integral domains. Completely characterize when the domain $A + xB[x]$ is a UFD.