MATH 725
FALL 2005
HOMEWORK 3

Due Friday, October 13, 2006.

1. Let $F \subseteq K$ be fields.

   a) (5 pt) Show that $F + xK[x]$ is always an HFD (note that the analogous result for $F + xK[[x]]$ was done in class).

   b) (5 pt) Show that if $(F + xK[x])[t]$ is an HFD, then F must be algebraically closed in K.

2. (5 pt) Let d be a square-free integer and $n \in \mathbb{N}$. Consider the domain

$$R := \begin{cases} \mathbb{Z}[n\sqrt{d}], & \text{if } d \equiv 2, 3 \bmod(4); \\ \mathbb{Z}[n(\frac{1+\sqrt{d}}{2})], & \text{if } d \equiv 1 \bmod(4). \end{cases}$$

We will let $\alpha = \sqrt{d}$ if $d \equiv 2, 3 \bmod(4)$ and $\alpha = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \bmod(4)$.
Define the norm map $N : R \longrightarrow \mathbb{Z}$ by

$$N(a + bn\alpha) = (a + bn\alpha)(a + bn\overline{\alpha})$$

where $\overline{\alpha} = -\sqrt{d}$ if $d \equiv 2, 3 \bmod(4)$ and $\overline{\alpha} = \frac{1-\sqrt{d}}{2}$ if $d \equiv 1 \bmod(4)$. Prove the following properties of the norm map.

   a) (5 pt) $N(x) = 0$ if and only if $x = 0$.

   b) (5 pt) $N(xy) = N(x)N(y)$.

   c) (5 pt) $N(x) = \pm 1$ if and only if x is a unit in R.

   d) (5 pt) If $N(x)$ is prime, then x is irreducible in R (does the converse hold?).

3. (5 pt) Find all nonprime irreducibles in $\mathbb{Z}[\sqrt{-3}]$ (you will find, in fact that this domain is extremely "close" to being a UFD).

4. (5 pt) Consider the ring of integers $R := \mathbb{Z}[\sqrt{-89}]$. Find an element of R that has one irreducible factorization of length 2 and another of length 12.