FORMAL POWER SERIES RINGS OVER ZERO-DIMENSIONAL SFT-RINGS

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ABSTRACT: Let R be a zero-dimensional SFT-ring. It is proved that the minimal prime ideals of the formal power series ring $A=R[[x_1, \dots, x_n]]$ are the ideals of the form $\mathcal{M}[[x_1, \dots, x_n]]$, where $\mathcal{M}$ is a (minimal) prime of R . It follows that A has Krull dimension n and is catenarian. If $R \subseteq T$ where T is also a zero-dimensional SFT-ring, the lying-over, going-up, incomparable, and going-down properties are studied for the extension $A \subseteq T[[x_1, \dots, x_n]]$.

1. Introduction. Throughout, all rings are commutative with identity; $\{x_1, \dots, x_n\}$ is a finite, nonempty set of analytically independent indeterminates over any relevant coefficient ring(s); $\text{Min}(D)$ denotes the set of minimal prime ideals of a ring D ; and $\dim(\text{ension})$ means Krull dimension. The general purpose of this paper is to increase our knowledge about the prime spectrum of formal power series rings $A=R[[x_1, \dots, x_n]]$. In contrast to the well-studied

case of polynomial rings in n variables (cf. [20], [7]), there remain many open questions concerning dimension and catenarity for formal power series. To be sure, the Noetherian case is rather well understood. Indeed, if R is a Noetherian ring, then $\dim(A) = \dim(R) + n$ [12, Lemma 2.6]; and if R is also an integral domain, then A is catenarian if and only if R is universally catenarian (combine [16, Proposition 2.5] and [20, (2.6)]). If R is not Noetherian, the analogous questions are more problematic. In fact, if R is not an SFT-ring (in the sense of [2]), then $\dim(A) = \infty$ [2, Theorem 1]; and if R is an m -dimensional SFT-Prüfer domain, then $\dim(A) = mn + 1$ [4, Theorem 3.6]. Moreover, if R is a finite-dimensional SFT-Prüfer domain, then in contrast with the situation for polynomial rings [20], Arnold [5] has shown that the possible catenarity of $R[[x_1, \dots, x_n]]$ depends on $\dim(R)$ and n . Similarly, if R is neither Noetherian nor Prüfer, the catenarity of A seems to have been studied only for certain classical types of integral domain pullbacks (cf. [1, Theorem 2.6 and Corollary 3.5]). Our first contribution (see Corollaries 2.2 and 2.4) is to prove that if R is a zero-dimensional SFT-ring, then A is n -dimensional and catenarian. The key step (Theorem 2.1) is to show, for any SFT-ring R , that $\text{Min}(A) = \{\mathcal{M}[[x_1, \dots, x_n]] : \mathcal{M} \in \text{Min}(R)\}$.

The second half of this paper concerns extensions $A = R[[x_1, \dots, x_n]] \subseteq B = T[[x_1, \dots, x_n]]$ arising from an extension $R \subseteq T$ of zero-dimensional SFT-rings. For such extensions, we study the lying-over, going-up, incomparable, and going-down properties (denoted LO, GU, INC, and GD, respectively, as in [15, p. 28]). The analogous studies for polynomial extensions have already been carried out: for LO, see [13, Proposition 3.6.1 (ii), p. 244] and [18, Proposition 1]; for GU, see [9, Lemma, p. 160] and [14, Theorem]; for INC, see [21, section 2] and [11]; for GD, see [18] and [10]. We show in Theorems 2.6 and 2.7 that any extension $A \subseteq B$ of the above type satisfies LO and GD. Our corresponding result for GU is partial, only for the case $n=1$ (see Proposition 2.5). Although $A \subseteq B$ satisfies INC for $n=1$ (see Proposition 2.5), Example 2.9 shows, in contrast to the situation for extensions of polynomial rings, that $A \subseteq B$ need not satisfy INC if $n \geq 2$.

Background definitions and results will be recalled as needed in section 2. Any unexplained material is standard, as in [15].

2. Results. It is convenient to begin by recalling the following two definitions from [2] and [3]. An ideal I of a ring D is called an *SFT-ideal* if there is a finitely generated ideal $J \subseteq I$ and an integer $k \geq 1$ such that $d^k \in J$ for each $d \in I$. A ring D is called an *SFT-ring* if each of its ideals (resp. prime ideals) is an SFT-ideal. A useful example of a zero-dimensional SFT-ring to keep in mind is $R = \mathbf{F}_2[\{Y_i\}]/(\{Y_i^2\})$, where $\{Y_i\}$ is a set of algebraically independent indeterminates over the field $\mathbf{F}_2$ with two elements. The verification that this R is a zero-dimensional SFT-ring is easy, after one observes that $r^2 \in \mathbf{F}_2$ for each $r \in R$.

THEOREM 2.1. Let R be an SFT-ring and let $A = R[[x_1, \dots, x_n]]$ for some $n \geq 1$. Then $\text{Min}(A) = \{\mathcal{M}[[x_1, \dots, x_n]] : \mathcal{M} \in \text{Min}(R)\}$.

Proof. Observe that $D \subseteq D[[x_1, \dots, x_n]]$ satisfies LO for each ring D . Accordingly, if D is an SFT-ring and $\mathcal{M} \in \text{Min}(D)$, it follows easily from a result of Arnold [4, Proposition 2.1] that $\mathcal{M}[[x_1, \dots, x_n]] \in \text{Min}(D[[x_1, \dots, x_n]])$. It remains to prove the converse; namely, if $\mathcal{P} \in \text{Min}(A)$, then there exists $\mathcal{M} \in \text{Min}(R)$ such that $\mathcal{P} = \mathcal{M}[[x_1, \dots, x_n]]$.

We proceed by induction on n . Consider the induction basis, with $x = x_1$. Given $\mathcal{P} \in \text{Min}(R[[x]])$, let $\mathcal{M} = \mathcal{P} \cap R$. Of course, $\mathcal{M}A \subseteq \mathcal{P}$, and so $\text{rad}_A(\mathcal{M}A) \subseteq \text{rad}_A(\mathcal{P}) = \mathcal{P}$. Since R is an SFT-ring, another result of Arnold [2, Theorem 1] ensures that $\text{rad}_A(\mathcal{M}A) = \mathcal{M}[[x]]$. Hence, $\mathcal{M}[[x]] \subseteq \mathcal{P}$. By the minimality of $\mathcal{P}$, we have $\mathcal{M}[[x]] = \mathcal{P}$, establishing the induction basis.

For the induction step, we suppose the assertion for $B = R[[x_1, \dots, x_{n-1}]]$ for some $n \geq 2$ and consider $\mathcal{P} \in \text{Min}(A)$, with $A = B[[x_n]]$. As every prime ideal contains a minimal prime (cf. [15, Theorem 10]), the induction hypothesis yields $\mathcal{M} \in \text{Min}(R)$ such that

$\mathcal{M}[[x_1, \dots, x_{n-1}]] \subseteq \mathcal{P} \cap B$. Of course, $\mathcal{M}[[x_1, \dots, x_{n-1}]]A \subseteq \mathcal{P}$ and so

$$\text{rad}_A(\mathcal{M}[[x_1, \dots, x_{n-1}]]A) \subseteq \text{rad}_A(\mathcal{P}) = \mathcal{P}.$$

It suffices to show that $\text{rad}_A(\mathcal{M}[[x_1, \dots, x_{n-1}]]A) = \mathcal{M}[[x_1, \dots, x_{n-1}]][[x_n]]$, for then, as above, minimality of $\mathcal{P}$ would give $\mathcal{P} = \mathcal{M}[[x_1, \dots, x_{n-1}]][[x_n]] = \mathcal{M}[[x_1, \dots, x_n]]$. Now, by the proof that (3) $\implies$ (1) in [2, Theorem 1], it suffices to prove that $\mathcal{M}[[x_1, \dots, x_{n-1}]]$ is an SFT-ideal of B . Thus, by induction on the number of indeterminates, it suffices to observe that if I is an SFT-ideal of a ring D , then $I[[x]]$ is an SFT-ideal of $D[[x]]$. This, in turn, follows from two items: $J[[x]]$ is a finitely generated ideal of $D[[x]]$ if J is a finitely generated ideal of D ; and the necessary exponent k is available, after considering a ring of the form $(D/J)[[x]]$, via an application of [2, Lemma 4]. This completes the induction step.

COROLLARY 2.2. Let R be a zero-dimensional SFT-ring and let $A = R[[x_1, \dots, x_n]]$ for some $n \geq 1$. Then $\dim(A) = n$.

Proof. If $\mathcal{M} \in \text{Spec}(R)$, consider $\mathcal{N} = \mathcal{M}[[x_1, \dots, x_n]] \subseteq A$ and $K = R/\mathcal{M}$. Of course, $A/\mathcal{N} \cong K[[x_1, \dots, x_n]]$. As $\mathcal{M}$ is a maximal ideal of R , K is a field, and so $K[[x_1, \dots, x_n]]$ is an n -dimensional regular local (Noetherian) ring; in particular, $\dim(A/\mathcal{N}) = n$. Since $\dim(A) = \sup\{\dim(A/\mathcal{Q}) : \mathcal{Q} \in \text{Min}(A)\}$, an application of Theorem 2.1 gives $\dim(A) = \sup\{\dim(A/\mathcal{M}[[x_1, \dots, x_n]]) : \mathcal{M} \in \text{Spec}(R)\} = \sup\{n\} = n$.

COROLLARY 2.3. Let R be a zero-dimensional SFT-ring and let $A = R[[x]]$. Then the prime ideals of A are of two kinds, $\mathcal{M}[[x]] \subsetneq (\mathcal{M}, x)$ corresponding to the $\mathcal{M}$ in $\text{Spec}(R)$, with no other proper inclusions among the members of $\text{Spec}(A)$.

Proof. The assertion follows easily from Theorem 2.1 (cf. also [1, Proposition 2.2]).

Following [7], a ring R is said to be *catenarian* if, for each pair $\mathcal{P} \subseteq \mathcal{Q}$ of prime ideals of R , all saturated chains of primes from $\mathcal{P}$ to $\mathcal{Q}$ have a common finite length; and R is

said to be *universally catenarian* if the polynomial rings $R[x_1, \dots, x_n]$ are catenarian for each $n \geq 1$. It is known that a Noetherian ring is universally catenarian if (and only if) $R[x]$ is catenarian [20, (2.6)]; and a going-down domain is universally catenarian if and only if its integral closure is a locally finite-dimensional Prüfer domain [7, Theorem 6.2]. Surely, the most important class of universally catenarian rings is that of all Cohen-Macaulay rings (cf. [17, Theorem 31]). In particular, if K is a field, then $K[[x_1, \dots, x_n]]$ is catenarian for each $n \geq 1$. We next generalize this fact to some other zero-dimensional coefficient rings. The next result is to be contrasted with earlier work on catenarity of formal power series rings (cf. [1]), where the coefficient rings were certain integral domains having a going-down flavor.

COROLLARY 2.4. Let R be an SFT-ring and let $A=R[[x_1, \dots, x_n]]$ for some $n \geq 1$. Suppose that $(R/\mathcal{M})[[x_1, \dots, x_n]]$ is catenarian for each $\mathcal{M} \in \text{Min}(R)$ (for instance, suppose that $\dim(R)=0$). Then A is catenarian.

Proof. Since every prime ideal of a ring contains a minimal prime (cf. [15, Theorem 10]), a ring D is catenarian if (and only if) $D/\mathcal{Q}$ is catenarian for each $\mathcal{Q} \in \text{Min}(D)$. Thus, by Theorem 2.1, it suffices to know the catenarity of $A/\mathcal{M}[[x_1, \dots, x_n]] \cong (R/\mathcal{M})[[x_1, \dots, x_n]]$ for each $\mathcal{M} \in \text{Min}(R)$. This, however, follows from the above comments if $\dim(R)=0$ since each such $R/\mathcal{M}$ is then a field.

We turn next to the study of LO, GU, INC, and GD for extensions of formal power series rings, beginning with the case of one variable in Proposition 2.5. The reader may wish to contrast the following results with an example in [14] of a nonintegral ring extension $R \subseteq T$ which satisfies GU and is such that the polynomial ring extensions $R[x_1, \dots, x_n] \subseteq T[x_1, \dots, x_n]$ satisfy LO, GU, and GD for each $n \geq 1$.

PROPOSITION 2.5. Let R be a zero-dimensional SFT-ring, T a ring extension of R , and x an analytic indeterminate. Set $A=R[[x]]$ and $B=T[[x]]$. Then $A \subseteq B$ satisfies LO, GU, and GD. Moreover, if T is also a zero-dimensional SFT-ring, then $A \subseteq B$ also satisfies INC.

Proof. We verify the assertion about GD first. Suppose we are given prime ideals $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of A and a prime $\mathcal{P}_1$ of B such that $\mathcal{P}_1 \cap A = \mathcal{Q}_1$. Our task is to find $\mathcal{P}_2 \in \text{Spec}(B)$ such that $\mathcal{P}_2 \subseteq \mathcal{P}_1$ and $\mathcal{P}_2 \cap A = \mathcal{Q}_2$. Without loss of generality, $\mathcal{Q}_2 \neq \mathcal{Q}_1$, and so Corollary 2.3 yields $\mathcal{M} \in \text{Spec}(R)$ such that $\mathcal{Q}_2 = \mathcal{M}[[x]]$ and $\mathcal{Q}_1 = (\mathcal{M}, x)$. As $x \in \mathcal{P}_1$, it follows that $\mathcal{P}_1 = (\mathcal{N}, x)$ for some ideal $\mathcal{N}$ of T ; and since $\mathcal{P}_1$ is a prime ideal, so is $\mathcal{N}$. Notice that $\mathcal{N} \cap R = \mathcal{M}$. Accordingly, it suffices to choose $\mathcal{P}_2 = \mathcal{N}[[x]]$.

For the assertion about GU, we need only consider distinct primes $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ of A and a prime $\mathcal{P}_1$ of B such that $\mathcal{P}_1 \cap A = \mathcal{Q}_1$. We must find $\mathcal{P}_2 \in \text{Spec}(B)$ such that $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{P}_2 \cap A = \mathcal{Q}_2$. As above, there exists $\mathcal{M} \in \text{Spec}(R)$ such that $\mathcal{Q}_1 = \mathcal{M}[[x]]$ and $\mathcal{Q}_2 = (\mathcal{M}, x)$. Observe (directly or by [19, Theorem 15.1]) that $I = (\mathcal{P}_1, x)$ is a proper ideal of B which properly contains $\mathcal{P}_1$. Let $\mathcal{P} \in \text{Spec}(B)$ be minimal among primes containing I . As $x \in \mathcal{P} \cap A$, it follows that $\mathcal{P} \cap A$ is the unique prime ideal of A which properly contains $\mathcal{Q}_1$; that is, $\mathcal{P} \cap A = \mathcal{Q}_2$. Thus, it suffices to choose $\mathcal{P}_2 = \mathcal{P}$, and the assertion about GU has been proved. The assertion about LO is now immediate, since $\text{GU} \implies \text{LO}$ in general [15, Theorem 42].

Finally, suppose that T is also a zero-dimensional SFT-ring. If $A \subseteq B$ does not satisfy INC, there exist distinct primes $\mathcal{P}_1 \subseteq \mathcal{P}_2$ of B such that $\mathcal{P}_1 \cap A = \mathcal{P}_2 \cap A = \mathcal{Q}$. By the above comments, there exists $\mathcal{N} \in \text{Spec}(T)$ such that $\mathcal{P}_1 = \mathcal{N}[[x]]$ and $\mathcal{P}_2 = (\mathcal{N}, x)$. Putting $\mathcal{M} = \mathcal{N} \cap R$, we have that $\mathcal{Q} = \mathcal{M}[[x]]$. Consider the field extension $k = R/\mathcal{M} \subseteq K = T/\mathcal{N}$. The induced extension $k[[x]] \subseteq K[[x]]$ of formal power series rings may be identified with $A/\mathcal{Q} \subseteq B/\mathcal{P}_1$. However, both prime ideals of $B/\mathcal{P}_1$ meet $A/\mathcal{Q}$ in 0, although $xK[[x]] \cap k[[x]] = xk[[x]]$, which is nonzero, giving the desired contradiction.

If $n \geq 2$, we can say more about LO, GD, and INC for the ring extensions

$R[[x_1, \dots, x_n]] \subseteq T[[x_1, \dots, x_n]]$. The first two of these sustain positive results, given in Theorems 2.6 and 2.7; the situation for INC is addressed in Example 2.9.

THEOREM 2.6. Let R be a zero-dimensional SFT-ring and T a ring extension of R . Set $A=R[[x_1, \dots, x_n]]$ and $B=T[[x_1, \dots, x_n]]$ for some $n \geq 1$. Then $A \subseteq B$ satisfies LO.

Proof. Consider $\mathcal{Q} \in \text{Spec}(A)$; our task is to find $\mathcal{P} \in \text{Spec}(B)$ such that $\mathcal{P} \cap A = \mathcal{Q}$. Of course, $\mathcal{Q}$ contains a minimal prime, and so Theorem 2.1 yields $\mathcal{M} \in \text{Spec}(R)$ such that $\mathcal{M}[[x_1, \dots, x_n]] \subseteq \mathcal{Q}$. As $\mathcal{M} \in \text{Min}(R)$, a standard application of Zorn's Lemma ensures that there exists $\mathcal{N} \in \text{Spec}(T)$ such that $\mathcal{N} \cap R = \mathcal{M}$. Thus, the question concerning LO may be viewed inside the extension $A/(\mathcal{M}[[x_1, \dots, x_n]]) \subseteq B/(\mathcal{N}[[x_1, \dots, x_n]])$, which may be identified with the extension $D=k[[x_1, \dots, x_n]] \subseteq E=S[[x_1, \dots, x_n]]$, where $k=R/\mathcal{M}$ and $S=T/\mathcal{N}$. As k is a field and S is a (nontrivial) ring extension of k , it follows from [6, Exercise 17(a), p. 250] that E is a faithfully flat D -module. Hence, by [6, Corollary 4(i), p. 72], $D \subseteq E$ satisfies LO, completing the proof.

The techniques in the preceding proof may also be adapted to give the following result for GD.

THEOREM 2.7. Let R be a zero-dimensional SFT-ring and T an SFT-ring extension of R . Set $A=R[[x_1, \dots, x_n]]$ and $B=T[[x_1, \dots, x_n]]$ for some $n \geq 1$. Then $A \subseteq B$ satisfies GD.

Proof. Suppose we are given prime ideals $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of A and a prime $\mathcal{P}_1$ of B such that $\mathcal{P}_1 \cap A = \mathcal{Q}_1$. We must find $\mathcal{P}_2 \in \text{Spec}(B)$ such that $\mathcal{P}_2 \subseteq \mathcal{P}_1$ and $\mathcal{P}_2 \cap A = \mathcal{Q}_2$. First, consider $\mathcal{N}_1 = \mathcal{P}_1 \cap T \in \text{Spec}(T)$ and choose $\mathcal{N} \in \text{Min}(T)$ such that $\mathcal{N} \subseteq \mathcal{N}_1$. By the first paragraph of the proof of Theorem 2.1, the hypothesis on T ensures that $\mathcal{N}[[x_1, \dots, x_n]] \in \text{Min}(B)$. Of course, $\mathcal{N}[[x_1, \dots, x_n]] \subseteq \mathcal{N}_1[[x_1, \dots, x_n]]$. Moreover, by [4, Proposition 2.1], the hypothesis on T also ensures that $\mathcal{N}_1[[x_1, \dots, x_n]] \subseteq \mathcal{P}_1$. Hence, $\mathcal{N}[[x_1, \dots, x_n]] \subseteq \mathcal{P}_1$.

Next, let $\mathcal{M}=\mathcal{N}\cap R$. As $\mathcal{N}[[x_1, \dots, x_n]]\cap R=\mathcal{M}$ and $\dim(R)=0$, we have $\mathcal{P}_1\cap R=\mathcal{M}$, and so $\mathcal{Q}_1\cap R=\mathcal{M}$. Hence, since $\mathcal{Q}_2\cap R\subseteq \mathcal{M}$ and $\dim(R)=0$, it follows that $\mathcal{Q}_2\cap R=\mathcal{M}$. Now, since R is an SFT-ring, another application of [4, Proposition 2.1] yields that $\mathcal{M}[[x_1, \dots, x_n]] \subseteq \mathcal{Q}_2$. Thus, the question concerning GD may be viewed inside the extension $A/\mathcal{M}[[x_1, \dots, x_n]] \subseteq B/\mathcal{N}[[x_1, \dots, x_n]]$, which may be identified with the extension $D=k[[x_1, \dots, x_n]]\subseteq E=S[[x_1, \dots, x_n]]$, where $k=R/\mathcal{M}$ and $S=T/\mathcal{N}$. As k is a field, E is D -flat ([6, Exercise 17(a), p. 250]; see also [8, Remark (a)]), and so $D\subseteq E$ satisfies GD (cf. [15, Exercise 37, p. 44]), completing the proof.

The next remark indicates a number of alternate approaches to some special cases of the above results.

REMARK 2.8. (a) Suppose that R is an SFT-ring. Then, by [3, Proposition 2.5 and Corollary 2.6], $\text{Spec}(R)$ is a Noetherian space and $\text{Min}(R)$ is finite. If R is also zero-dimensional, then $\text{Spec}(R)$ is Hausdorff [6, Exercise 16(d), p. 143]. As Noetherian Hausdorff spaces are finite [6, p.98], it follows that if R is a zero-dimensional SFT-ring, then $\text{Spec}(R)=\text{Min}(R)$ is finite, say $\{\mathcal{P}_1, \dots, \mathcal{P}_m\}$; and the associated reduced ring, R_{red} (which is an absolutely flat SFT-ring), is isomorphic to a finite product of fields: $R_{red} \cong \prod_{j=1}^m K_j$, $K_j=R/\mathcal{P}_j$ by the Chinese Remainder Theorem.

(b) Let $R\subseteq T$ be a ring extension such that R is a zero-dimensional SFT-ring. Using the notation in (a), we have

$$\begin{aligned} \prod_{j=1}^m K_j[[x_1, \dots, x_n]] &\cong R_{red}[[x_1, \dots, x_n]] \cong R[[x_1, \dots, x_n]]/\sqrt{R}[[x_1, \dots, x_n]] \xrightarrow{f} \\ T_{red}[[x_1, \dots, x_n]] &\cong T[[x_1, \dots, x_n]]/\sqrt{T}[[x_1, \dots, x_n]], \end{aligned}$$

where f is induced by the canonical monomorphism $R_{red} \longrightarrow T_{red}$. Recall that Spec of a finite product is canonically the coproduct of Spec of the factors. Recall also (cf. [4, Proposition 2.1 (v)]) that if D is an SFT-ring and $\mathcal{P} \in \text{Spec}(R[[x_1, \dots, x_n]])$, then

$\sqrt{D}[[x_1, \dots, x_n]] \subseteq \mathcal{P}$. We thus find a way to reduce the proofs of Corollaries 2.2 and 2.3 to the case in which R is a field.

The above approach also recovers the zero-dimensional case of Corollary 2.4. Moreover, it can also serve to recover Theorems 2.6 and 2.7. Indeed, suppose $\mathcal{Q} \in \text{Spec}(T[[x_1, \dots, x_n]])$, $\mathcal{P} \in \text{Spec}(R[[x_1, \dots, x_n]])$, $\mathcal{Q}_1 = \mathcal{Q}/\sqrt{T}[[x_1, \dots, x_n]]$, and $\mathcal{P}_1 = \mathcal{P}/\sqrt{R}[[x_1, \dots, x_n]]$. It is easy to show that $f^{-1}(\mathcal{Q}_1) = \mathcal{P}_1$ if and only if $\mathcal{Q} \cap R[[x_1, \dots, x_n]] = \mathcal{P}$. Thus, for the “LO” results, it would suffice to know that f is faithfully flat; and if T is an SFT-ring, assertions about GU, GD, or INC would follow from corresponding assertions for f .

The fact is that $f: R_{red}[[x_1, \dots, x_n]] \longrightarrow T_{red}[[x_1, \dots, x_n]]$ is faithfully flat (thus leading to new proofs of Theorems 2.6 and 2.7, by the above remarks). For convenience, view f as an inclusion map. We show first that each maximal ideal $\mathcal{P}$ of $R_{red}[[x_1, \dots, x_n]]$ survives in $T_{red}[[x_1, \dots, x_n]]$. By [19, Theorem 15.1], $\mathcal{P} = (\mathcal{M}, x_1, \dots, x_n)$ for some $\mathcal{M} \in \text{Max}(R_{red}) = \text{Min}(R_{red})$. As minimal primes must be lain over [15, Exercise 1, p. 41], there exists $\mathcal{N} \in \text{Spec}(T_{red})$ such that $\mathcal{N} \cap R = \mathcal{M}$. Hence, $(\mathcal{N}, x_1, \dots, x_n)$ lies over $\mathcal{P}$, and so $\mathcal{P}$ survives in $T_{red}[[x_1, \dots, x_n]]$. By [6, Proposition 9 and Remark 2, p. 33], to show that f is faithfully flat, it now suffices to prove that f is flat.

We show that f is flat as follows. Identify $R_{red} = \prod_{j=1}^m K_j$, a finite product of fields, as above. Then T_{red} may be viewed as $\prod_{j=1}^m T_j$, with $T_j = T_{red} \otimes_{R_{red}} K_j$ for each j . By [6, Exercise 17(a), p. 250], $T_j[[x_1, \dots, x_n]]$ is flat over $K_j[[x_1, \dots, x_n]]$ for each j , and so $T_{red}[[x_1, \dots, x_n]] \cong \prod T_j[[x_1, \dots, x_n]]$ is flat over $\prod K_j[[x_1, \dots, x_n]] \cong R_{red}[[x_1, \dots, x_n]]$; that is, f is flat, completeing the proof.

If $k \subseteq K$ is an algebraic field extension, then the extension $k[x_1, \dots, x_n] \subseteq K[x_1, \dots, x_n]$ of polynomial rings is integral, and hence satisfies INC. (For a more general result, see [11, Theorem 2.2].) Our final result shows that the analogous statement for formal power series rings is false.

EXAMPLE 2.9. Let $k \subseteq K$ be a field extension (hence, an extension of zero-dimensional SFT-rings) such that some element $e \in K$ is not a root of any quadratic polynomial over k . Let $A = k[[x_1, x_2]]$ and $B = K[[x_1, x_2]]$. Then $A \subseteq B$ does not satisfy INC.

For a proof, consider $f = ex_1 + x_2 \in B$. By straightforward degree (“order”) arguments, one shows that f is an irreducible element of B . As B is a unique factorization domain (cf. [6, Proposition 8, p. 511]), it follows that $I = fB$ is a prime ideal of B . Since $I \neq 0$, it suffices to prove that $I \cap A = 0$.

Suppose not, and choose a nonzero element

$$h = fg = f(\sum a_{i,j} x_1^i x_2^j) \in I \cap A.$$

Since $h = (ex_1 + x_2)(\sum a_{i,j} x_1^i x_2^j) \in k[[x_1, x_2]]$, we see by equating the corresponding coefficients of $x_1^i x_2^j$, x_1^i , and x_2^j that

$$ea_{i-1,j} + a_{i,j-1}, ea_{i-1,0}, a_{0,j-1} \in k \text{ for all } i \geq 1, j \geq 1.$$

Observe that if $i \geq 1$, then

$$e^2 a_{i-1,1} + ea_{i,0} = e(ea_{i-1,1} + a_{i,0}) \in ek;$$

and since $ea_{i,0} \in k$, the hypothesis on e ensures that $a_{i-1,1} = 0$. It follows that $a_{i,0} \in k$ for each $i \geq 1$. However, since $ea_{i,0} \in k$ and $e \notin k$, we now have $a_{i,0} = 0$. Similarly, since $ea_{0,0}$ and $a_{0,0}$ are in k while $e \notin k$, we see that $a_{0,0} = 0$. Thus x_2 divides g in B and $h \in x_2 B \cap A = x_2 A$.

Consider $g_1 = gx_2^{-1} \in B$ and apply the preceding argument to the nonzero element

$$h_1 = hx_2^{-1} = fg_1 \in I \cap A.$$

The upshot is that x_2 divides g_1 in B and $h_1 \in x_2 A$. In particular, x_2^2 divides g in B .

Similarly, $g_2 = g_1 x_2^{-1} \in B$ and, by applying the above argument to $h_2 = h_1 x_2^{-1} = fg_2 \in I \cap A$, ■

we find that x_2 divides g_2 in B and $h_2 \in x_2 A$. Hence, x_2^2 divides g_1 and so x_2^3 divides g in B .

Iterating the above argument (or by induction), we have that x_2^n divides g in B for each $n \geq 1$. This (desired) contradiction to the fact that B is a unique factorization domain

completes the proof.

It remains an open question to find necessary and sufficient conditions on an extension $R \subseteq T$ of zero-dimensional SFT-rings for the induced extension $R[[x_1, \dots, x_n]] \subseteq T[[x_1, \dots, x_n]]$ to satisfy GU or INC.

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