MATH-266
Exam #1
06/27/14

Name: Key

Directions: You have 60 minutes to complete this exam. Complete each of the following problems. Points will be awarded for both completeness and clarity of solutions. Partial credit will be awarded for correct partial solutions. Be sure to show all of your work, and explain yourself thoroughly. Avoid saying things that are untrue, ambiguous, or nonsensical. There is a total of 80 points possible in this exam. Your exam will be scored out of 75 points.

Solve each of the following differential equations or IVP's. You will have exactly one of each type of equation (Separable, Linear, Exact, Bernoulli, or Homogeneous).

(15 points) 1.

$$\begin{cases} \left(3y^3e^{3xy} - 1\right) + \left(2ye^{3xy} + 3xy^2e^{3xy}\right) y' = 0 \\ y(0) = 1 \end{cases}$$

Exact

$$\underbrace{(3y^3e^{3xy} - 1)}_{M}\,dx + \underbrace{(2ye^{3xy} + 3xy^2e^{3xy})}_{N}\,dy = 0 \qquad (1)$$

$$\frac{\partial M}{\partial y} = 9y^2e^{3xy} + 9xy^3e^{3xy} \;\underset{\checkmark}{=}\; \frac{\partial N}{\partial x} = 6y^2e^{3xy} + 3y^2e^{3xy} + 9xy^3e^{3xy}$$

Exact

$$\frac{\partial F}{\partial x} = 3y^3e^{3xy} - 1 \implies F(x,y) = \int \left(3y^3e^{3xy} - 1\right)dx$$

$$= \frac{3y^3e^{3xy}}{3y} - x + g(y)$$

$$= y^2e^{3xy} - x + g(y)$$

$$\frac{\partial F}{\partial y} = 2ye^{3xy} + 3xy^2e^{3xy} + g'(y)$$

$$= 2ye^{3xy} + 3xy^2e^{3xy} \quad \text{by } (1)$$

$$\implies g'(y) = 0$$

$$\implies g(y) = C$$

$$F(x,y) = y^2e^{3xy} - x + C$$

$$\boxed{y^2e^{3xy} - x = C}$$

$$1^2e^0 - 0 = C$$

$$1 = C \implies \boxed{y^2e^{3xy} - x = 1}$$

(15 points) 2. $\dfrac{dy}{d\theta} = \dfrac{\theta \sec(y/\theta) + y}{\theta}$

Homogeneous

$$\frac{dy}{d\theta} = \frac{y}{\theta} + \sec(y/\theta)$$

$$\frac{dy}{d\theta} - \frac{1}{\theta} y = \sec(y/\theta)$$

Not linear
Not Bernoulli

Homogoneous?

$$\theta\, dy = (\theta \sec(y/\theta) + y)\, d\theta$$

$$\underbrace{(\theta \sec(y/\theta) + y)}_{M} d\theta - \underbrace{\theta}_{N} dy = 0$$

$$M(x\theta, xy) = x\theta \sec(xy/x\theta) + xy$$
$$= x(\theta \sec(y/\theta) + y)$$
$$= x\, M(\theta, y)$$
$$N(x\theta, xy) = x\theta$$
$$= x N(\theta, y)$$

Homogeneous of degree 1

Let $y = u\theta$. Then $dy = u\,d\theta + \theta\,du$.

$$\left(\theta \sec\left(\frac{u\theta}{\theta}\right) + u\theta\right) d\theta - \theta(u\,d\theta + \theta\,du) = 0$$

$$\Rightarrow \theta \sec(u)\, d\theta + \cancel{u\theta\, d\theta} - \cancel{u\theta\, d\theta} - \theta^2 du = 0$$

$$\Rightarrow \theta \sec(u)\, d\theta = \theta^2 du$$

$$\Rightarrow \frac{1}{\theta} d\theta = \cos(u)\, du$$

$$\Rightarrow \int \frac{1}{\theta} d\theta = \int \cos(u)\, du$$

$$\Rightarrow \ln|\theta| + C = \sin(u)$$

$$\Rightarrow \sin^{-1}(\ln|\theta| + C) = u$$

$$\Rightarrow \sin^{-1}(\ln|\theta| + C) = y/\theta$$

$$\Rightarrow \boxed{y = \theta \sin^{-1}(\ln|\theta| + C)}$$

(15 points) 3. $\frac{dr}{d\theta} = \frac{r^2 + 2r\theta}{\theta^2}$ $\Rightarrow$ $\frac{dr}{d\theta} = r^2\theta^{-2} + 2r\theta^{-1}$

$$\Rightarrow \frac{dr}{d\theta} - 2\theta^{-1}r = \theta^{-2}r^2$$

Bernoulli : $n=2$

$$r^{-2}\left(\frac{dr}{d\theta} - 2\theta^{-1}r = \theta^{-2}r^2\right) \Rightarrow r^{-2}\frac{dr}{d\theta} - 2\theta^{-1}r^{-1} = \theta^{-2}$$

$$\omega = r^{1-n} = r^{1-2} = r^{-1} \Rightarrow \frac{d\omega}{d\theta} = -r^{-2}\frac{dr}{d\theta}$$

$$\Rightarrow -\frac{d\omega}{d\theta} - 2\theta^{-1}\omega = \theta^{-2}$$

$$\Rightarrow \frac{d\omega}{d\theta} + \frac{2}{\theta}\omega = -\theta^{-2}$$ Linear

$$\mu(\theta) = e^{\int \frac{2}{\theta} d\theta} = e^{2\ln|\theta|} = \theta^2$$

$$\Rightarrow \theta^2\frac{d\omega}{d\theta} + \theta^2 \cdot \frac{2}{\theta}\omega = -\theta^2 \cdot \theta^{-2}$$

$$\Rightarrow \theta^2\frac{d\omega}{d\theta} + 2\theta\omega = -1$$

$$\Rightarrow \frac{d}{d\theta}(\theta^2\omega) = -1$$

$$\Rightarrow \theta^2\omega = \int -1\, d\theta$$

$$\Rightarrow \theta^2\omega = -\theta + C$$

$$\Rightarrow \omega = -\frac{1}{\theta} + \frac{C}{\theta^2}$$

$$\Rightarrow \frac{1}{r} = \frac{-1}{\theta} + \frac{C}{\theta^2}$$

$$\Rightarrow r = \frac{1}{-\frac{1}{\theta} + \frac{C}{\theta^2}}$$

$$\Rightarrow \boxed{r = \frac{\theta^2}{C-\theta}}$$

(15 points) 4. $x\dfrac{dy}{dx} + y = \dfrac{x^2+x+4}{x(x^2+4)}$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{x^2+x+4}{x^2(x^2+4)}$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$

$$x\frac{dy}{dx} + y = \frac{x^2+x+4}{x(x^2+4)}$$

$$\frac{d}{dx}(xy) = \frac{x^2+x+4}{x(x^2+4)}$$

$$xy = \int\left(\frac{1}{x} + \frac{1}{x^2+4}\right)dx$$

$$xy = \ln|x| + \int \frac{1}{x^2+4}dx$$

$x = 2\tan\theta$
$dx = 2\sec^2\theta$

$$xy = \ln|x| + \int \frac{1}{4\tan^2\theta+4} \cdot 2\sec^2\theta\, d\theta$$

$$xy = \ln|x| + \frac{1}{2}\int 1\, d\theta$$

$$xy = \ln|x| + \frac{1}{2}\theta + C$$

$$xy = \ln|x| + \frac{1}{2}\tan^{-1}(x/2) + C$$

$$\boxed{y = \frac{\ln|x| + \frac{1}{2}\tan^{-1}(x/2) + C}{x}}$$

$$\frac{x^2+x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow x^2+x+4 = A(x^2+4) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + 4A$$

$A+B=1 \longrightarrow B=0$
$C=1$
$4A=4 \Rightarrow A=1$

(10 points) 5. $y^{-1}dy + ye^{\cos x}\sin x dx = 0$

$\Rightarrow \frac{1}{y}dy = -ye^{\cos x}\sin x\, dx$

Separable.

$\Rightarrow \frac{dy}{dx} = -y^2 e^{\cos x}\sin x$

$\Rightarrow -\frac{1}{y^2}dy = e^{\cos x}\sin x\, dx$

$\int -\frac{1}{y^2}dy = \int e^{\cos x}\sin x\, dx$

$u = \cos x$, $du = -\sin x dx$

$\frac{1}{y} = -\int e^{u} du$

$\frac{1}{y} = -e^{\cos x} + C$

$\boxed{y = \frac{-1}{e^{\cos x} + C}}$

(10 points) 6. Show that if $y = y_1(t)$ is a solution to $\frac{dy}{dt} + p(t)y = r(t)$ and $y = y_2(t)$ is a solution of $\frac{dy}{dt} + p(t)y = q(t)$, then $y(t) = y_1(t) + y_2(t)$ is a solution of $\frac{dy}{dt} + p(t)y = r(t) + q(t)$.

$$\frac{dy_1}{dt} + p(t)y_1 = r(t)$$

$$\frac{dy_2}{dt} + p(t)y_2 = q(t).$$

$$y(t) = y_1(t) + y_2(t) \quad \& \quad \frac{dy}{dt} = y_1'(t) + y_2'(t).$$

$$\begin{aligned}\frac{dy}{dt} + p(t)y &= \left[y_1'(t) + y_2'(t)\right] + p(t)\left[y_1(t) + y_2(t)\right] \\ &= \left(y_1'(t) + p(t)y_1(t)\right) + \left(y_2'(t) + p(t)y_2(t)\right) \\ &= r(t) + q(t)\end{aligned}$$