

Directions: You have 60 minutes to complete this exam. Complete each of the following problems. Points will be awarded for both completeness and clarity of solutions. Partial credit will be awarded for correct partial solutions. Be sure to show all of your work, and explain yourself thoroughly. Avoid saying things that are untrue, ambiguous, or nonsensical. There is a total of 80 points possible in this exam. Your exam will be scored out of 75 points.

(20 points) 1. Solve the following ODE using the Method of Undetermined Coefficients:

$$y'' + 3y' + 2y = \sin(t)$$

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda + 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -1, -2$$

$$y_h = C_1 e^{-t} + C_2 e^{-2t}$$

$f(t) = \sin t \longrightarrow 0 \pm i$ is not a solution to the homogeneous eqn.

$$y_p(t) = A \sin t + B \cos t$$

$$y_p'(t) = A \cos t - B \sin t$$

$$y_p''(t) = -A \sin t - B \cos t$$

$$y_p'' + 3y_p' + 2y_p = -A \sin t - B \cos t + 3(A \cos t - B \sin t) + 2(A \sin t + B \cos t)$$

$$= (-A - 3B + 2A) \sin t + (-B + 3A + 2B) \cos t$$

$$= (A - 3B) \sin t + (B + 3A) \cos t$$

$$\Rightarrow \begin{cases} A - 3B = 1 \\ 3A + B = 0 \end{cases} \longrightarrow \begin{cases} A - 3B = 1 \\ 9A + 3B = 0 \end{cases}$$

$$\hline 10A = 1 \Rightarrow A = \frac{1}{10}$$

$$\frac{1}{10} - 3B = 1$$

$$-3B = \frac{9}{10}$$

$$B = -\frac{3}{10}$$

$$y_p(t) = \frac{1}{10} \sin t - \frac{3}{10} \cos t$$

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{10} \sin t - \frac{3}{10} \cos t$$

(20 points) 2. Solve the following IVP:

$$\begin{cases} y'' + 2y' + 17y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$$

$$\lambda^2 + 2\lambda + 17 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(17)}}{2(1)} = \frac{-2 \pm \sqrt{4(1-17)}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i$$

$$y_h = c_1 e^{-t} \cos(4t) + c_2 e^{-t} \sin(4t)$$

$$y' = -c_1 e^{-t} \cos(4t) - 4c_1 e^{-t} \sin(4t) - c_2 e^{-t} \sin(4t) + 4c_2 e^{-t} \cos(4t)$$

$$= (-4c_1 - c_2) e^{-t} \sin(4t) + (4c_2 - c_1) e^{-t} \cos(4t)$$

$$1 = y(0) = c_1$$

$$-1 = y'(0) = 4c_2 - c_1 \Rightarrow -1 = 4c_2 - 1 \Rightarrow c_2 = 0$$

$$y = e^{-t} \cos(4t)$$

(20 points) 3. Solve the following ODE using the Method of Variation of Parameters:

$$y'' - 10y' + 25y = e^{5t} \ln(2t)$$

$$\lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda - 5)^2 = 0 \Rightarrow \lambda = 5$$

$$y_h = c_1 e^{5t} + c_2 t e^{5t}$$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{5t} & t e^{5t} \\ 5e^{5t} & e^{5t} + 5t e^{5t} \end{vmatrix}$$

$$= e^{10t} + 5t e^{10t} - 5t e^{10t}$$

$$= e^{10t}$$

$$u_1(t) = \int \frac{-y_2 f(t)}{W(y_1, y_2)(t)} dt = \int \frac{-t e^{5t} (e^{5t} \ln(2t))}{e^{10t}} dt$$

$$= \int -t \ln(2t) dt$$

$$u = \ln(2t) \quad \left| \quad dv = -t dt \right.$$

$$du = \frac{1}{2t} \cdot 2 dt \quad \left| \quad v = -\frac{t^2}{2} \right.$$

$$= -\frac{t^2}{2} \ln(2t) - \int -\frac{t^2}{2} \cdot \frac{1}{t} dt$$

$$= -\frac{t^2}{2} \ln(2t) + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{y_1 f(t)}{W(y_1, y_2)(t)} dt = \int \frac{e^{5t} (e^{5t} \ln(2t))}{e^{10t}} dt$$

$$= \int \ln(2t) dt$$

$$u = \ln(2t) \quad \left| \quad dv = dt \right.$$

$$du = \frac{1}{2t} \cdot 2 dt \quad \left| \quad v = t \right.$$

$$= t \ln(2t) - \int t \cdot \frac{1}{t} dt$$

$$= t \ln(2t) - t$$

$$y = c_1 e^{5t} + c_2 t e^{5t} + \left(\frac{t^2}{4} - \frac{t^2}{2} \ln(2t) \right) e^{5t} + \left(t \ln(2t) - t \right) t e^{5t}$$

$$y = c_1 e^{5t} + c_2 t e^{5t} + \left(\frac{t^2}{2} \ln(2t) - \frac{3}{4} t^2 \right) e^{5t}$$

(20 points) 4. Write the given in the form $a + bi$ for real numbers a and b .

$$\begin{aligned}
 \text{(a) } e^{-2 + \frac{5\pi}{4}i} &= e^{-2} \left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right) \\
 &= e^{-2} \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) \\
 &= \boxed{-\frac{\sqrt{2}}{2} e^{-2} - e^{-2} \frac{\sqrt{2}}{2} i}
 \end{aligned}$$

(b) $(1 + i)^6$

(Hint: Rewrite $z = 1 + i$ in the form $z = |z|e^{i\theta}$)

$$\begin{aligned}
 \text{Let } z = 1 + i &\Rightarrow z = |1 + i| e^{i\theta} = \sqrt{2} (\cos\theta + i \sin\theta) = 1 + i \\
 &\Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \quad \& \quad \sin\theta = \frac{\sqrt{2}}{2} \\
 &\Rightarrow \theta = \pi/4
 \end{aligned}$$

$$z = \sqrt{2} e^{\frac{\pi}{4}i}$$

$$\begin{aligned}
 z^6 &= (\sqrt{2} e^{\frac{\pi}{4}i})^6 = 8 e^{\frac{3\pi}{2}i} = 8 \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) \right) \\
 &= 8(0 - i) \\
 &= \boxed{0 - 8i}
 \end{aligned}$$