MATH-266
Exam #3
07/25/14

Name: Key

Directions: You have 60 minutes to complete this exam. Complete each of the following problems. Points will be awarded for both completeness and clarity of solutions. Partial credit will be awarded for correct partial solutions. Be sure to show all of your work, and explain yourself thoroughly. Avoid saying things that are untrue, ambiguous, or nonsensical. There is a total of 80 points possible in this exam. Your exam will be scored out of 75 points.

(20 points) 1. Find the Laplace transform of each of the following functions.

(a) $g(t) = e^{-4t}(3t^4 + t\cos(t))$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{3t^4e^{-4t} + te^{-4t}\cos t\}(s)$$
$$= \mathcal{L}\{3t^4e^{-4t}\}(s) + \mathcal{L}\{te^{-4t}\cos t\}(s)$$
$$= 3\cdot\frac{4!}{(s+4)^5} + (-1)^1\frac{d}{ds}\left(\mathcal{L}\{e^{-4t}\cos t\}(s)\right)$$
$$= \frac{72}{(s+4)^5} - \frac{d}{ds}\left(\frac{s+4}{(s+4)^2+1}\right)$$
$$= \frac{72}{(s+4)^5} - \frac{((s+4)^2+1)(1) - (s+4)(2(s+4)(1))}{((s+4)^2+1)^2}$$
$$= \frac{72}{(s+4)^5} - \frac{(s^2+8s+17 - 2(s^2+8s+16)}{((s+4)^2+1)^2}$$
$$= \frac{72}{(s+4)^5} - \frac{-s^2-8s-15}{((s+4)^2+1)^2}$$

(b) $g(t) = t^2\mathcal{U}(t-1)$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{t^2\mathcal{U}(t-1)\}(s)$$
$$= \mathcal{L}\{((t-1)+1)^2\mathcal{U}(t-1)\}(s)$$
$$= e^{-s}\mathcal{L}\{(t+1)^2\}(s)$$
$$= e^{-s}\mathcal{L}\{t^2+2t+1\}(s)$$
$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

(20 points) 2. Find the inverse Laplace transform of each of the following functions.

(a) $F(s) = \dfrac{8s - 2s^2 - 14}{(s+1)(s^2 - 2s + 5)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2-2s+5}$

$\Rightarrow -2s^2+8s-14 = A(s^2-2s+5)+(Bs+C)(s+1)$

$s=-1: \ -2-8-14 = 8A \Rightarrow -24 = 8A \Rightarrow A=-3$

$s=0: \ -14 = 5A + C \Rightarrow -14 = -15 + C \Rightarrow C=1$

$s=1: \ -8 = 4A + 2B + 2C \Rightarrow -8 = 4(-3)+2B+2 \Rightarrow B=1$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{-3}{s+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-2s+5}\right\}(t)$$

$$= -3e^{-t} + \mathcal{L}^{-1}\left\{\frac{s-1+2}{(s-1)^2+4}\right\}(t)$$

$$= -3e^{-t} + e^{t}\cos(2t) + e^{t}\sin(2t)$$

(b) $F(s) = \dfrac{s+3}{s^2+2s+10} = \dfrac{s+1+2}{(s+1)^2+9}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}(t) + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}(t)$$

$$= e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)$$

(20 points) 3. Use Laplace transforms to solve the IVP:

$$y'' - 2y' = \underbrace{\begin{cases} 6, & \text{if } 0 \le t < 4 \\ 0, & \text{if } t \ge 4 \end{cases}}_{f(t)}$$

given that $y(0) = -1$ and $y'(0) = 7$.

$f(t) = 6(\mathcal{U}(t) - \mathcal{U}(t-4)) = 6 - 6\mathcal{U}(t-4)$

$\mathcal{L}\{y''-2y'\}(t) = \mathcal{L}\{f(t)\}$

$\Rightarrow (s^2 Y(s) - s\overset{-1}{y(0)} - \overset{7}{y'(0)}) - 2(sY(s) - \overset{-1}{y(0)}) = 6\mathcal{L}\{1\}(s) - 6\mathcal{L}\{\mathcal{U}(t-4)\}(s)$

$\Rightarrow (s^2-2s)Y(s) + s - 9 = \frac{6}{s} - \frac{6e^{4s}}{s}$

$\Rightarrow s(s-2)Y(s) = \frac{6}{s}(1 - e^{4s}) - (s-9)$

$\Rightarrow Y(s) = \frac{6}{s^2(s-2)}(1-e^{4s}) - \frac{s^2-9s}{s^2(s-2)}$

$\Rightarrow Y(s) = \frac{6-s^2+9s}{s^2(s-2)} - \frac{6e^{4s}}{s^2(s-2)}$

$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$

$\Rightarrow 1 = As(s-2) + B(s-2) + Cs^2$

$s=0: \quad 1 = -2B \longrightarrow B = -\tfrac{1}{2}$

$s=2: \quad 1 = 4C \longrightarrow C = \tfrac{1}{4}$

$s=1: \quad 1 = -A - B + C \rightarrow -A = 1 - \tfrac{1}{2} - \tfrac{1}{4}$

$A = -\tfrac{1}{4}$

$\mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left(\frac{-s^2+9s+6}{s^2(s-2)} - \frac{6e^{4s}}{s^2(s-2)}\right)$

$\frac{-s^2+9s+6}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} \Rightarrow -s^2+9s+6 = As(s-2) + B(s-2) + Cs^2$

$s=0: \quad 6 = -2B \longrightarrow B = -3$

$s=2: \quad 20 = 4C \longrightarrow C = 5$

$s=1: \quad 14 = -A - B + C \longrightarrow A = -6$

$y(t) = \left(-6\mathcal{L}^{-1}\{\tfrac{1}{s}\}(t) - 3\mathcal{L}^{-1}\{\tfrac{1}{s^2}\}(t) + 5\mathcal{L}^{-1}\{\tfrac{1}{s-2}\}(t)\right)$
$\quad -6\left(\mathcal{L}^{-1}\{\tfrac{1}{s^2(s-2)}\}(t+4)\,\mathcal{U}(t+4)\right)$

$y(t) = -6 - 3t + 5e^{2t} - 6\left(-\tfrac{3}{2} - \tfrac{1}{2}t + \tfrac{1}{4}e^{2t}\right)\mathcal{U}(t+4)$

(20 points) 4. Use the Method of Undetermined Coefficients to solve the following ODE. (Do not use variation of parameters.)

$$y''' + 3y'' + 3y' + y = e^{-t}$$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$(\lambda+1)(\lambda^2+2\lambda+1) = 0$

$(\lambda+1)^3 = 0$

$\lambda = -1$

$y_h = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$

$y_p = t^3(Ae^{-t}) = At^3 e^{-t}$

$y_p' = 3At^2 e^{-t} - At^3 e^{-t} = (3At^2 - At^3)e^{-t}$

$y_p'' = (6At - 3At^2)e^{-t} - (3At^2 - At^3)e^{-t} = (6At - 6At^2 + At^3)e^{-t}$

$y_p''' = (6A - 12At + 3At^2)e^{-t} - (6At - 6At^2 + At^3)e^{-t} = (6A - 18At + 9At^2 - At^3)e^{-t}$

$y_p''' + 3y_p'' + 3y_p' + y_p = (6A - 18At + 9At^2 - At^3)e^{-t} + 3(6At - 6At^2 + At^3)e^{-t}$
$\quad + 3(3At^2 - At^3)e^{-t} + At^3 e^{-t}$

$= 6Ae^{-t}$

$6A = 1 \implies A = 1/6$

$y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + \frac{1}{6} t^3 e^{-t}$

- $\mathcal{L}\{af + bg\}(s) = a\mathcal{L}\{f\}(s) + b\mathcal{L}\{g\}(s)$
- $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$
- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
- $\mathcal{L}^{-1}\{aF + bG\}(t) = a\mathcal{L}^{-1}\{F\}(t) + b\mathcal{L}^{-1}\{G\}(t)$

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$\dfrac{1}{s}$, $s > 0$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$, $s > a$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$, $s > k$
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$, $s > k$
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$