MATH-266
Exam #1 Extra Problems
Due: 07/02/14 **Name:** Key

Directions: You have 60 minutes to complete this exam. Complete each of the following problems. Points will be awarded for both completeness and clarity of solutions. Partial credit will be awarded for correct partial solutions. Be sure to show all of your work, and explain yourself thoroughly. Avoid saying things that are untrue, ambiguous, or nonsensical. There is a total of 80 points possible in this exam. Your exam will be scored out of 75 points.

Solve each of the following differential equations or IVP's. You will have exactly one of each type of equation (Separable, Linear, Exact, Bernoulli, or Homogeneous). You must identify each problem as one of these types of equations and only need to solve the type of problems you did not get full credit for on the exam. You can earn back 1/3 of the points that you lost on the exam problems 1-5, but your solutions must be perfect. You must hand these solved problems with your original exam.

(15 points) 1. $\dfrac{dy}{d\theta} = \dfrac{\theta\sec(y/\theta) + y}{\theta}$ $\Rightarrow \dfrac{dy}{d\theta} = \dfrac{y}{\theta} + \sec(y/\theta)$ Homogeneous ✓

$\Rightarrow \dfrac{dy}{d\theta} - \dfrac{1}{\theta}y = \sec(y/\theta)$ Not linear
Not Bernoulli

$\theta\, dy = (\theta\sec(y/\theta) + y)\,d\theta$

$\underbrace{(\theta\sec(y/\theta) + y)}_{M}d\theta \underbrace{- \theta}_{N} dy = 0$

$M(x\theta, xy) = x\theta\sec(xy/x\theta) + xy$
$= x(\theta\sec(y/\theta) + y)$
$= xM(\theta, y)$

$N(x\theta, xy) = x\theta$
$= x\,N(\theta, y)$

Let $y = u\theta$. Then $dy = u\,d\theta + \theta\,du$

$(\theta\sec(u\theta/\theta) + \cancel{u\theta})\,d\theta - \theta(\cancel{u\,d\theta} + \theta\,du) = 0$

$\Rightarrow \theta\sec(u)\,d\theta = \theta^2 du$

$\Rightarrow \frac{1}{\theta}d\theta = \cos(u)\,du$

$\Rightarrow \int \frac{1}{\theta}d\theta = \int \cos(u)\,du$

$\Rightarrow \ln|\theta| + C = \sin(u)$

$\Rightarrow \sin^{-1}(\ln|\theta| + C) = u$

$\Rightarrow \sin^{-1}(\ln|\theta| + C) = y/\theta$

$\boxed{y = \theta\sin^{-1}(\ln|\theta| + C)}$

(15 points) 2. $\underbrace{(e^t y + te^t y)}_{M}dt + \underbrace{(te^t + 2)}_{N}dy = 0$

$-(e^t y + te^t y)dt = (te^t + 2)dy$

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y} = e^t + te^t \\ \dfrac{\partial N}{\partial t} = e^t + te^t\end{array}\right\}\ \text{Exact.}$$

$$\frac{\partial F}{\partial t} = e^t y + te^t y$$

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t$$

$u = t \quad dv = e^t$

$du = dt \quad v = e^t$

$$F(x,y) = \int (e^t y + te^t y)dt$$

$$= e^t y + y\int te^t dt$$

$$= e^t y + y(te^t - e^t) + g(y)$$

$$= yte^t + g(y)$$

$$\frac{\partial F}{\partial y} = te^t + g'(y) = te^t + 2$$

$$\Rightarrow g'(y) = 2$$

$$\Rightarrow g(y) = 2y + C$$

$$F(x,y) = yte^t + 2y + C$$

$$\Rightarrow \boxed{yte^t + 2y = C}$$

$$\Rightarrow y(te^t + 2) = C$$

$$\Rightarrow \boxed{y = \frac{C}{te^t + 2}}$$

(15 points) 3. $(ye^{-2x} + y^3)dx - e^{-2x}dy = 0$

Bernoulli

$$\frac{ye^{-2x}+y^3}{e^{-2x}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - y = e^{2x}y^3$$

$$y^{-3}\left(\frac{dy}{dx} - y = e^{2x}y^3\right)$$

$$\Rightarrow y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}$$

$$w = y^{-2}$$

$$\frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{dw}{dx} - w = e^{2x}$$

$$\Rightarrow \frac{dw}{dx} + 2w = -2e^{2x}$$

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

$$\Rightarrow e^{2x}\frac{dw}{dx} + 2e^{2x}w = -2e^{2x}\cdot e^{2x}$$

$$\Rightarrow \frac{d}{dx}(e^{2x}w) = -2e^{4x}$$

$$\Rightarrow e^{2x}w = \int -2e^{4x}dx$$

$$\Rightarrow e^{2x}w = -\frac{2}{4}e^{4x} + C$$

$$\Rightarrow w = -\frac{2}{4}e^{2x} + Ce^{-2x}$$

$$\Rightarrow \frac{1}{y^2} = -\frac{2}{4}e^{2x} + Ce^{-2x}$$

$$\Rightarrow y^2 = \frac{1}{-\frac{1}{2}e^{2x} + Ce^{-2x}} \Rightarrow \boxed{y^2 = \frac{2}{-e^{2x} + Ce^{-2x}}}$$

(15 points) 4. $\frac{dy}{dx} = x^2 e^{-4x} - 4y$

$\frac{dy}{dx} + 4y = x^2 e^{-4x}$ Linear

$\mu(x) = e^{\int 4dx} = e^{4x}$

$\Rightarrow e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2 e^{-4x} e^{4x}$

$\Rightarrow \frac{d}{dx}(e^{4x}y) = x^2$

$\Rightarrow e^{4x}y = \int x^2$

$\Rightarrow e^{4x}y = \frac{x^3}{3} + C$

$\Rightarrow \boxed{y = \frac{x^3}{3}e^{-4x} + Ce^{-4x}}$

MATH-266
Exam #1 Extra Problems
Due: 07/02/14

(10 points) 5. $\frac{dy}{dx} = \frac{y}{x^2+2}$

$$\frac{1}{y}\,dy = \frac{1}{x^2+2}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x^2+2}\,dx \qquad \begin{array}{l} x = \sqrt{2}\tan\theta \\ dx = \sqrt{2}\sec^2\theta\, d\theta \end{array}$$

$$\Rightarrow \ln|y| = \int \frac{1}{2\tan^2\theta + 2} \cdot \sqrt{2}\sec^2\theta$$

$$\Rightarrow \ln|y| = \int \frac{1}{2\sec^2\theta} \cdot \sqrt{2}\sec^2\theta\, d\theta$$

$$\Rightarrow \ln|y| = \int \frac{\sqrt{2}}{2}\, d\theta$$

$$\Rightarrow \ln|y| = \frac{\sqrt{2}}{2}\theta + C$$

$$\Rightarrow \ln|y| = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$\Rightarrow \boxed{y = Ce^{\frac{\sqrt{2}}{2}\tan^{-1}(x/\sqrt{2})}}$$