HW #11

a) $y'' - 6y' + 9y = 5t^2 e^{3t}$

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$

$\boxed{Y_h = c_1 e^{3t} + c_2 t e^{3t}}$

$f(t) = 5t^2 e^{3t}$

$Y_p = t^s (At^2 + Bt + C) e^{3t}$

since 3 is a double root, $s = 2$.

$Y_p = t^2 (At^2 + Bt + C) e^{3t}$

$= At^4 e^{3t} + Bt^3 e^{3t} + Ct^2 e^{3t}$

$Y_p' = 4At^3 e^{3t} + 3At^4 e^{3t} + 3Bt^2 e^{3t} + 3Bt^3 e^{3t} + 2Ct e^{3t} + 3Ct^2 e^{3t}$

$= 3At^4 e^{3t} + (4A + 3B)t^3 e^{3t} + (3B + 3C)t^2 e^{3t} + 2Ct e^{3t}$

$Y_p'' = 12At^2 e^{3t} + 12At^3 e^{3t} + 12At^3 e^{3t} + 9At^4 e^{3t} + 6Bt e^{3t} + 9Bt^2 e^{3t} + 9Bt^2 e^{3t} + 9Bt^3 e^{3t} + 2Ce^{3t} + 6Ct e^{3t} + 6Ct e^{3t} + 9Ct^2 e^{3t}$

$= 9At^4 e^{3t} + (24A + 9B)t^3 e^{3t} + (12A + 18B + 9C)t^2 e^{3t} + (6B + 12C)t e^{3t} + 2Ce^{3t}$

$Y_p'' - 6Y_p' + 9Y_p = (\overset{0}{\cancel{9A - 6(3A) + 9A}})t^4 e^{3t}$

$+ (\overset{0}{\cancel{24A} + \cancel{9B} - 6(\cancel{4A} + \cancel{3B}) + \cancel{9B}})t^3 e^{3t}$

$+ (12A + 18B + 9C - 6(3B + 3C) + 9C)t^2 e^{3t}$

$+ (6B + 12C - 6(2C))t e^{3t}$

$+ (2C) e^{3t}$

$12A = 5 \Rightarrow A = 5/12$

$6B = 0 \Rightarrow B = 0$

$2C = 0 \Rightarrow C = 0$

$\boxed{Y = \frac{5}{12} t^4 e^{3t} + c_1 e^{3t} + c_2 t e^{3t}}$

b) $y'' - y' - 12y = 2te^{-3t}$

$\lambda^2 - \lambda - 12 = 0 \rightarrow (\lambda - 4)(\lambda + 3) = 0$

$\rightarrow \lambda = 4, -3$

$\boxed{Y_h = c_1 e^{4t} + c_2 e^{-3t}}$

$f(t) = 2te^{-3t}$

$Y_p = t^s (At + B) e^{-3t}$

since -3 is a root, $s = 1$.

$Y_p = t(At + B)e^{-3t}$

$= At^2 e^{-3t} + Bte^{-3t}$

$Y_p' = 2Ate^{-3t} - 3At^2 e^{-3t} + Be^{-3t} - 3Bte^{-3t}$

$= -3At^2 e^{-3t} + (2A - 3B)te^{-3t} + Be^{-3t}$

$Y_p'' = -6Ate^{-3t} + 9At^2 e^{-3t} + (2A - 3B)e^{-3t} - 3(2A - 3B)te^{-3t} - 3Be^{-3t}$

$= 9At^2 e^{-3t} + (-12A + 9B)te^{-3t} + (2A - 6B)e^{-3t}$

$Y_p'' - Y_p' - 12Y_p = (\overset{0}{\cancel{9A + 3A - 12A}})t^2 e^{-3t}$

$+ (-12A + \cancel{9B} - 2A + \cancel{3B} - \cancel{12B})te^{-3t}$

$+ (2A - 6B - B)e^{-3t}$

$-14A = 2 \longrightarrow A = -1/7$

$2A - 7B = 0$

$-\frac{2}{7} - 7B = 0 \longrightarrow 7B = -\frac{2}{7} \longrightarrow B = -\frac{2}{49}$

$\boxed{Y = -\frac{1}{7} t^2 e^{-3t} - \frac{2}{49} te^{-3t} + c_1 e^{4t} + c_2 e^{-3t}}$

c) $y''+2y'+2y = 8e^{-t}\sin t$

$\lambda^2+2\lambda+2=0$

$\Rightarrow \lambda = \dfrac{-2\pm\sqrt{4-4(1)(2)}}{2(1)} = \dfrac{-2\pm 2i}{2} = -1\pm i$

$\boxed{y_h = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)}$

$f(t) = 8e^{-t}\sin t$

$y_p = t^s(A)e^{-t}\sin t$
$\quad + t^s(B)e^{-t}\cos t$

since $-1\pm i$ is a root, $s=1$

$y_p = Ate^{-t}\sin t + Bte^{-t}\cos t$

$y_p' = Ae^{-t}\sin t - Ate^{-t}\sin t + Ate^{-t}\cos t$
$\quad + Be^{-t}\cos t - Bte^{-t}\cos t - Bte^{-t}\sin t$
$\quad = Ae^{-t}\sin t + (-A-B)te^{-t}\sin t$
$\quad + (A-B)te^{-t}\cos t + Be^{-t}\cos t$

$y_p'' = -Ae^{-t}\sin t + Ae^{-t}\cos t + (-A-B)e^{-t}\sin t$
$\quad + (A+B)te^{-t}\sin t + (-A-B)te^{-t}\cos t$
$\quad + (A-B)e^{-t}\cos t - (A-B)te^{-t}\cos t$
$\quad - (A-B)te^{-t}\sin t - Be^{-t}\cos t - Be^{-t}\sin t$
$\quad = (-2A-2B)e^{-t}\sin t + (2A-2B)e^{-t}\cos t$
$\quad + 2Bte^{-t}\sin t - 2Ate^{-t}\cos t$

$y_p''+2y_p'+2y_p = (-2A-2B+2(A))e^{-t}\sin t$
$\quad + (2A-2B+2B)e^{-t}\cos t$
$\quad + (2B+2(-A-B)+2A)te^{-t}\sin t$
$\quad + (-2A+2(A-B)+2B)te^{-t}\cos t$

$-2A-2B+2A=8 \rightarrow -2B=8 \Rightarrow \boxed{B=-4}$

$2A-2B+2B=0 \rightarrow 2A=0 \Rightarrow \boxed{A=0}$

$2B-2A-2B+2A=0$

$-2A+2A-2B+2B=0$

$\boxed{y = -4te^{-t}\cos t + C_1e^{-t}\cos t + C_2e^{-t}\sin t}$