HW #13

1) ⓐ $y''' - 3y' + 2y = te^t$

$\lambda^3 - 3\lambda + 2 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2+\lambda-2)=0$

$(\lambda-1)(\lambda+2)(\lambda-1)=0$

$y_h = c_1e^t + c_2te^t + c_3e^{-2t}$

$$W(y_1,y_2,y_3) = \begin{vmatrix} e^t & te^t & e^{-2t} \\ e^t & e^t+te^t & -2e^{-2t} \\ e^t & 2e^t+te^t & 4e^{-2t} \end{vmatrix}$$

$= e^t(4e^{-t} + 4te^{-t} + 4e^{-t} + 2te^{-t})$

$-e^t(4te^{-t} - 2e^{-t} - te^{-t})$

$+e^t(-2te^{-t} - e^{-t} - te^{-t})$

$= (8 + \cancel{6t}) - (\cancel{3t} - 2) + (\cancel{-3t} - 1)$

$= 9$

$W_1(y_1,y_2,y_3) = \begin{vmatrix} te^t & e^{-2t} \\ e^t+te^t & -2e^{-2t} \end{vmatrix} = -2te^{-t} - e^{-t} - te^{-t}$

$= -3te^{-t} - e^{-t}$

$= (-3t-1)e^{-t}$

$W_2(y_1,y_2,y_3) = \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = -2e^{-t} - e^{-t}$

$= -3e^{-t}$

$W_3(y_1,y_2,y_3) = \begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t}$

$= e^{2t}$

$u_1(t) = \int (-1)^{3+1}\left(\frac{(-3t-1)\cancel{e^{-t}}}{9}\right) t\cancel{e^t}$

$= -\int \frac{1}{9}(3t^2+t)$

$= -\frac{1}{9}\left(t^3 + \frac{t^2}{2}\right)$

$= -\frac{1}{9}t^3 - \frac{t^2}{18}$

$u_2(t) = \int (-1)^{3+2}\left(\frac{-3e^{-t}}{9}\right) te^t$

$= \int \frac{1}{3}t$

$= \frac{1}{6}t^2$

$u_3(t) = \int (-1)^{3+3}\left(\frac{e^{2t}}{9}\right) te^t$

$= \int \frac{1}{9}te^{3t}$

$u = t \quad | \quad dv = e^{3t}dt$

$du = dt \quad | \quad v = \frac{1}{3}e^{3t}$

$= \frac{1}{9}\left[\frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t}dt\right]$

$= \frac{1}{27}te^{3t} - \frac{1}{81}e^{3t}$

$y = c_1e^t + c_2te^t + c_3e^{-2t} + \left(-\frac{1}{9}t^3 - \frac{1}{18}t^2\right)e^t$

$+ \left(\frac{1}{6}t^2\right)te^t + \left(\frac{1}{27}te^{3t} - \frac{1}{81}e^{3t}\right)e^{-2t}$

$= c_1e^t + c_2te^t + c_3e^{-2t} - \frac{2}{18}t^3e^t - \frac{1}{18}t^2e^t$

$+ \frac{3}{18}t^3e^t + \frac{1}{27}te^t - \frac{1}{81}e^t$

$= c_1e^t + c_2te^t + c_3e^{-2t} + \frac{1}{18}t^3e^t - \frac{1}{18}t^2e^t$

$\frac{1}{27}te^t - \frac{1}{81}e^t$

or

$$\boxed{y = c_1e^t + c_2te^t + c_3e^{-2t} + \frac{1}{18}t^3e^t - \frac{1}{18}t^2e^t}$$

(b) $y''' + 16y' = t\sin t$

$\lambda^3 + 16\lambda = 0$

$\lambda(\lambda^2 + 16) = 0$

$\lambda = 0, \pm 4i$

$$y_h = C_1 + C_2\cos(4t) + C_3\sin(4t)$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos(4t) & \sin(4t) \\ 0 & -4\sin(4t) & 4\cos(4t) \\ 0 & -16\cos(4t) & -16\sin(4t) \end{vmatrix}$$

$$= 64\sin^2(4t) + 64\cos^2(4t)$$
$$= 64$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{vmatrix}$$

$$= 4\cos^2(4t) + 4\sin^2(4t)$$
$$= 4$$

$$W_2(y_1, y_2, y_3) = \begin{vmatrix} 1 & \sin(4t) \\ 0 & 4\cos(4t) \end{vmatrix} = 4\cos(4t)$$

$$W_3(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos(4t) \\ 0 & -4\sin(4t) \end{vmatrix} = -4\sin(4t)$$

$$u_1(t) = \int (-1)^{3+1}\left(\frac{4}{64}\right) t\sin t\,dt = \frac{1}{16}\int t\sin t\,dt$$

$u = t$ | $dv = \sin t\,dt$
$du = dt$ | $v = -\cos t$

$$= \frac{1}{16}\left[-t\cos t - \int -\cos t\,dt\right]$$
$$= -\frac{1}{16}t\cos t + \frac{1}{16}\sin t$$

$$u_2(t) = \int (-1)^{3+2}\left(\frac{4\cos(4t)}{64}\right) t\sin t$$

$$= -\frac{1}{16}\int t\left(\frac{1}{2}\left(\sin(5t) - \sin(3t)\right)\right) dt$$

$$= -\frac{1}{32}\left[\int t\sin(5t)\,dt - \int t\sin(3t)\,dt\right]$$

$u = t$ | $dv = \sin(5t)$
$du = dt$ | $v = -\frac{1}{5}\cos(5t)$

$u = t$
$du = dt$
$dv = \sin(3t)$
$v = -\frac{1}{3}\cos(3t)$

$$= -\frac{1}{32}\left[-\frac{1}{5}t\cos(5t) - \int -\frac{1}{5}\cos(5t)\,dt\right]$$
$$+ \frac{1}{32}\left[-\frac{1}{3}t\cos(3t) - \int -\frac{1}{3}\cos(3t)\,dt\right]$$

$$\cos(a)\sin(b) = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right)$$

$$\Rightarrow \boxed{\frac{1}{160}t\cos(5t) - \frac{1}{160(5)}\sin(5t) - \frac{1}{96}t\cos(3t) + \frac{1}{96(3)}\sin(3t)}$$

$$u_3(t) = \int (-1)^{3+3}\left(\frac{-4\sin(4t)}{64}\right) t\sin t$$

$$= -\frac{1}{16}\int t\left(\frac{1}{2}\left(\cos(3t) - \cos(5t)\right)\right) dt$$

$$= -\frac{1}{36}\left[\int t\cos(3t)\,dt - \int t\cos(5t)\,dt\right]$$

$u = t$ | $dv = \cos(3t)$
$du = dt$ | $v = \frac{1}{3}\sin(3t)$

$u = t$ | $dv = \cos(5t)$
$du = dt$ | $v = \frac{1}{5}\sin(5t)$

$$= -\frac{1}{36}\left[\frac{1}{3}t\sin(3t) - \int \frac{1}{3}\sin(3t)\,dt\right]$$
$$+ \frac{1}{36}\left[\frac{1}{5}t\sin(5t) - \int \frac{1}{5}\sin(5t)\,dt\right]$$

$$= \boxed{-\frac{1}{108}t\sin(3t) + \frac{1}{324}\cos(3t) + \frac{1}{180}t\sin(5t) + \frac{1}{900}\cos(5t)}$$

$$\boxed{\begin{aligned} y &= C_1 + C_2\cos(4t) + C_3\sin(4t) \\ &+ \left(\frac{1}{16}\sin t - \frac{1}{16}t\cos t\right)\cdot 1 \\ &+ \left(\frac{1}{160}t\cos(5t) - \frac{1}{800}\sin(5t) - \frac{1}{96}t\cos(3t) + \frac{1}{288}\sin(3t)\right)\cos(4t) \\ &+ \left(-\frac{1}{108}t\sin(3t) - \frac{1}{324}\cos(3t) + \frac{1}{180}t\sin(5t) + \frac{1}{900}\cos(5t)\right)\sin(4t) \end{aligned}}$$

2(a) $y^{(4)} - 5y'' + 4y = 10\cos(t) - 20\sin(t)$

$\lambda^4 - 5\lambda^2 + 4 = 0$

$\Rightarrow (\lambda^2-4)(\lambda^2-1) = 0$

$\lambda = \pm 2, \pm 1$

$y_h = C_1 e^{-2t} + C_2 e^{-t} + C_3 e^{t} + C_4 e^{2t}$

$f(t) = 10\cos(t) - 20\sin(t)$

$\pm i$ not a solution to y_h.

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p''' = A\sin t - B\cos t$

$y_p^{(4)} = A\cos t + B\sin t$

$y^{(4)} - 5y'' + 4y = (A + 5A + 4A)\cos t$

$+ (B + 5B + 4B)\sin t$

$= 10A\cos t + 10B\sin t$

$10A = 10 \longrightarrow A = 1$

$10B = -20 \longrightarrow B = -2$

$$\boxed{y = C_1 e^{-2t} + C_2 e^{-t} + C_3 e^{t} + C_4 e^{2t} + \cos t - 2\sin t}$$

(b) $y^{(4)} - 3y''' + 3y'' - y' = 6t - 20.$

$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda = 0$

$\lambda(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0$

1	1	-3	3	-1
		1	-2	1
	1	-2	1	0

$\lambda(\lambda-1)(\lambda^2 - 2\lambda + 1) = 0$

$\lambda(\lambda-1)^3 = 0$

$\lambda = 0, 1$

$y_h = C_1 + C_2 e^{t} + C_3 te^{t} + C_4 t^2 e^{t}$

$f(t) = 6t - 20$

$y_p = t^s(At + B)e^{0t}$

0 is a solution, so let $s = 1$

$y_p = t(At + B) = At^2 + Bt$

$y_p' = 2At + B$

$y_p'' = 2A$

$y_p''' = 0$

$y_p^{(4)} = 0$

$y^{(4)} - 3y''' + 3y'' - y' = 0 - 3(0) + 3(2A) - (2At + B)$

$= 6A - B + 2At$

$6A - B = -20$

$-2A = 6 \longrightarrow A = -3$

$-18 - B = -20$

$-B = -2$

$B = 2$

$y_p = -3t^2 + 2t$

$y = C_1 + C_2 e^{t} + C_3 te^{t} + C_4 t^2 e^{t} - 3t^2 + 2t$