HW #14

1ⓐ $\mathcal{L}\{e^{2t}\cos(3t)\}(s) = \int_0^\infty e^{-st} e^{2t}\cos(3t)\,dt$

$= \int_0^\infty e^{-(s-2)t}\cos(3t)\,dt$

$u = e^{-(s-2)t} \quad dv = \cos(3t)$

$du = -(s-2)e^{-(s-2)t} \quad v = \frac{1}{3}\sin(3t)$

$= \lim_{b\to\infty}\left[\frac{1}{3}e^{-(s-2)t}\sin(3t) - \int_0^\infty -\frac{(s-2)}{3}e^{-(s-2)t}\sin(3t)\,dt\right]_0^b$

$u = e^{-(s-2)t} \quad dv = \sin(3t)\,dt$

$du = -(s-2)e^{-(s-2)t}\,dt \quad v = -\frac{1}{3}\cos(3t)$

$= \lim_{b\to\infty}\left[\frac{1}{3}e^{-(s-2)t}\sin(3t) + \left(\frac{s-2}{3}\right)\left(-\frac{1}{3}e^{-(s-2)t}\cos(3t) - \int_0^\infty\left(\frac{s-2}{3}\right)e^{-(s-2)t}\cos(3t)\,dt\right)\right]_0^b$

$\Rightarrow \left(1+\left(\frac{s-2}{3}\right)^2\right)\int_0^\infty e^{-(s-2)t}\cos(3t)\,dt = \lim_{b\to\infty}\left[\frac{1}{3}e^{-(s-2)t}\sin(3t) - \left(\frac{s-2}{9}\right)e^{-(s-2)t}\cos(3t)\right]_0^b$

$\Rightarrow \int_0^\infty e^{-(s-2)t}\cos(3t) = \frac{(0-0)-\left(0-\frac{s-2}{9}\right)}{\left(1+\left(\frac{s-2}{3}\right)^2\right)}$

$= \frac{\frac{s-2}{9}}{1+\frac{(s-2)^2}{9}}\cdot\frac{9}{9}$

$= \boxed{\frac{s-2}{(s-2)^2+9}}$

ⓑ $\mathcal{L}\{e^{-t}\sin(2t)\}(s) = \int_0^\infty e^{-st}e^{-t}\sin(2t)\,dt$

$= \int_0^\infty e^{-(s+1)t}\sin(2t)\,dt$

$u = e^{-(s+1)t} \quad dv = \sin(2t)\,dt$

$du = -(s+1)e^{-(s+1)t}\,dt \quad v = -\frac{1}{2}\cos(2t)$

$= \lim_{b\to\infty}\left[-\frac{e^{-(s+1)t}}{2}\cos(2t) - \int_0^\infty \frac{(s+1)}{2}e^{-(s+1)t}\cos(2t)\,dt\right]_0^b$

$u = e^{-(s+1)t} \quad dv = \cos(2t)\,dt$

$du = -(s+1)e^{-(s+1)t}\,dt \quad v = \frac{1}{2}\sin(2t)$

$= \lim_{b\to\infty}\left[-\frac{e^{-(s+1)t}}{2}\cos(2t) - \left(\frac{s+1}{2}\right)\left(\frac{e^{-(s+1)t}}{2}\sin(2t) - \int_0^\infty -\frac{(s+1)}{2}e^{-(s+1)t}\sin(2t)\,dt\right)\right]_0^b$

$\Rightarrow \left(1+\left(\frac{s+1}{2}\right)^2\right)\int_0^\infty e^{-(s+1)t}\sin(2t)\,dt = \lim_{b\to\infty}\left[-\frac{e^{-(s+1)t}}{2}\cos(2t) - \frac{s+1}{4}e^{-(s+1)t}\sin(2t)\right]_0^b$

$\Rightarrow \int_0^\infty e^{-(s+1)t}\sin(2t)\,dt = \frac{(0-0)-\left(-\frac{1}{2}-0\right)}{\left(1+\left(\frac{s+1}{2}\right)^2\right)}$

$= \frac{\frac{1}{2}}{\left(1+\left(\frac{s+1}{2}\right)^2\right)}\cdot\frac{4}{4}$

$= \boxed{\frac{2}{(s+1)^2+4}}$

ⓒ $f(t) = \begin{cases} e^{2t} & 0<t<3 \\ 1 & t>3 \end{cases}$

$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}f(t)\,dt$

$= \int_0^3 e^{-st}e^{2t}\,dt + \int_3^\infty e^{-st}\,dt$

$= \int_0^3 e^{-(s-2)t}\,dt + \int_3^\infty e^{-st}\,dt$

$= -\frac{1}{s-2}e^{-(s-2)t}\Big|_0^3 + \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_0^b$

$= \frac{-1}{s-2}e^{-3(s-2)} + \frac{1}{s-2} + \left(0-\left(-\frac{1}{s}\right)\right)$

$= \boxed{\frac{-1}{s-2}e^{-3(s-2)} + \frac{1}{s-2} + \frac{1}{s}}$

2ⓐ $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$

$= \boxed{\frac{6}{s+3} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{8}{s}}$

ⓑ $\mathcal{L}\{t^3 - te^t + e^{4t}\cos(t)\}$

$= \boxed{\frac{6}{s^4} - \frac{1}{(s-1)^2} + \frac{(s-4)}{(s-4)^2+1}}$

ⓒ $\mathcal{L}\{t^2 - 3t - 2e^{-t}\sin(3t)\}$

$= \frac{2}{s^3} - \frac{3}{s^2} - 2\left(\frac{3}{(s+1)^2+9}\right)$

$= \boxed{\frac{2}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2+9}}$

ⓓ $\mathcal{L}\{te^{2t}\cos(5t)\}$

$= (-1)^1\cdot\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right)$

$= -\frac{((s-2)^2+25)(1) - (s-2)\cdot 2(s-2)(1)}{((s-2)^2+25)^2}$

$= -\frac{(s^2-4s+29) - 2(s^2-4s+4)}{((s-2)^2+25)^2} = \boxed{\frac{s^2-4s-21}{((s-2)^2+25)^2}}$

(e) $\mathcal{L}\{e^{-t}t\sin(2t) + e^{6t} - 1\}$

$= \mathcal{L}\{te^{-t}\sin(2t)\} + \mathcal{L}\{e^{6t}\} - \mathcal{L}\{1\}$

$= (-1)^1 \frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right) + \frac{1}{s-6} - \frac{1}{s}$

$= -\frac{0 - 2(2(s+1)(1))}{((s+1)^2+4)^2} + \frac{1}{s-6} - \frac{1}{s}$

$= \boxed{\frac{4(s+1)}{((s+1)^2+4)^2}}$

(f) $\mathcal{L}\{t^2\cos(bt)\}$

$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+b^2}\right)$

$= \frac{d}{ds}\left(\frac{(s^2+b^2)(1) - s(2s)}{(s^2+b^2)^2}\right)$

$= \frac{d}{ds}\left(\frac{-s^2+b^2}{(s^2+b^2)^2}\right)$

$= \frac{(s^2+b^2)^2(-2s) - (-s^2+b^2)\cdot 2(s^2+b^2)\cdot 2s}{(s^2+b^2)^4}$

$= \frac{(s^2+b^2)\left[-2s(s^2+b^2) - (-s^2+b^2)\cdot 4s\right]}{(s^2+b^2)^4}$

$= \frac{-2s^3 - 2sb^2 - (-4s^3 + 4sb^2)}{(s^2+b^2)^3}$

$= \boxed{\frac{2s^3 - 6sb^2}{(s^2+b^2)^3}}$