HW #16

a) $\begin{cases} y''+6y'+5y=12e^t \\ y(0)=-1 \\ y'(0)=7 \end{cases}$

$$\mathcal{L}\{y''+6y'+5y\}(s)=\mathcal{L}\{12e^t\}(s)$$

$$(s^2Y(s)-sy(0)-y'(0))+6(sY(s)-y(0))+5Y(s)=\frac{12}{s-1}$$

$$(s^2+6s+5)Y(s)+s-7-6=\frac{12}{s-1}$$

$$(s^2+6s+5)Y(s)=\frac{12}{s-1}-s+13$$

$$(s+1)(s+5)Y(s)=\frac{12-s(s-1)+13(s-1)}{s-1}$$

$$(s+1)(s+5)Y(s)=\frac{12-s^2+s+13s-13}{s-1}$$

$$Y(s)=\frac{-s^2+14s-1}{(s-1)(s+1)(s+5)}$$

$$\frac{-s^2+14s-1}{(s-1)(s+1)(s+5)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+5}$$

$$-s^2+14s-1=A(s+1)(s+5)+B(s-1)(s+5)+C(s-1)(s+1)$$

$s=-1: -16=-8B \longrightarrow B=2$

$s=1: 12=12A \longrightarrow A=1$

$s=-5: -25-70-1=24C$

$-96=24C$

$-4=C$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\{\tfrac{1}{s-1}\}(t)+2\mathcal{L}^{-1}\{\tfrac{1}{s+1}\}(t)-4\mathcal{L}^{-1}\{\tfrac{1}{s+5}\}(t)$$

$$y(t)=\boxed{e^t+2e^{-t}-4e^{-5t}}$$

b) $\begin{cases} y''-2y'+y=\cos t-\sin t \\ y(0)=1 \\ y'(0)=3 \end{cases}$

$$\mathcal{L}\{y''-2y'+y\}(s)=\mathcal{L}\{\cos t-\sin t\}(s)$$

$$(s^2Y(s)-sy(0)-y'(0))-2(sY(s)-y(0))+Y(s)=\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$(s^2-2s+1)Y(s)-s-3+2=\frac{s-1}{s^2+1}$$

$$(s-1)^2Y(s)=\frac{s-1}{s^2+1}+s+1$$

$$Y(s)=\frac{s-1+(s+1)(s^2+1)}{(s^2+1)(s-1)^2}$$

$$Y(s)=\frac{s-1+s^3+s+s^2+1}{(s^2+1)(s-1)^2}$$

$$Y(s)=\frac{s^3+s^2+2s}{(s^2+1)(s-1)^2}$$

$$\frac{s^3+s^2+2s}{(s^2+1)(s-1)^2}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$s^3+s^2+2s=(As+B)(s-1)^2+C(s^2+1)(s-1)+D(s^2+1)$$

$s=1: 4=2D \longrightarrow D=2$

$s=0: 0=B-C+D$

$0=B-C+2 \longrightarrow B-C=-2$

$s=-1: -2=-4A+4B-4C+2D$

$-2=-4A+4(-2)+4$

$2=-4A \longrightarrow A=-\frac{1}{2}$

$s=2: 8+4+4=2A+B+5C+5D$

$16=2(-\frac{1}{2})+B+5C+10$

$7=B+5C$

$-1(B-C=-2) \longrightarrow -B+C=2$

$B+5C=7 \longrightarrow B+5C=7$

$6C=9$

$C=\frac{3}{2}$

$-B+\frac{3}{2}=2 \longrightarrow B=-\frac{1}{2}$

$$\mathcal{L}^{-1}\left\{-\frac{1}{2}\left(\frac{s+1}{s^2+1}\right)+\frac{3}{2}\left(\frac{1}{s-1}\right)+2\left(\frac{1}{(s-1)^2}\right)\right\}(t)$$

$$\boxed{y(t)=-\frac{1}{2}\cos(t)-\frac{1}{2}\sin(t)+\frac{3}{2}e^t+2te^t}$$

(c) $\begin{cases} y''' + 4y'' + y' - 6y = -12 \\ y(0) = 1 \\ y'(0) = 4 \\ y''(0) = -2 \end{cases}$

$$\mathcal{L}\{y''' + 4y'' + y' - 6y\} = \mathcal{L}\{-12\}$$

$$\left(\begin{array}{l} (s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)) \\ +4(s^2 Y(s) - s y(0) - y'(0)) \\ + (s Y(s) - y(0)) \\ -6 Y(s) \end{array} \right) = \frac{-12}{s}$$

$$(s^3 + 4s^2 + s - 6) Y(s) - s^2 - 4s + 2 - 4s - 16 - 1 = \frac{-12}{s}$$

$$(s-1)(s^2 + 5s + 6) Y(s) = -\frac{12}{s} + s^2 + 8s + 15$$

$$(s-1)(s+3)(s+2) Y(s) = \frac{-12 + s^3 + 8s^2 + 15s}{s}$$

$$Y(s) = \frac{s^3 + 8s^2 + 15s - 12}{s(s-1)(s+3)(s+2)}$$

$$\frac{s^3 + 8s^2 + 15s - 12}{s(s-1)(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} + \frac{D}{s+2}$$

$$s^3 + 8s^2 + 15s - 12 = A(s-1)(s+3)(s+2) + Bs(s+3)(s+2) + Cs(s-1)(s+2) + Ds(s-1)(s+3)$$

$s = 0$: $-12 = -6A \longrightarrow A = 2$

$s = 1$: $12 = 12B \longrightarrow B = 1$

$s = -2$: $-8 + 32 - 30 - 12 = 6D$

$-18 = 6D \longrightarrow D = -3$

$s = -3$: $-27 + 72 - 45 - 12 = -12C$

$-12 = -12C \longrightarrow C = 1$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{2}{s} + \frac{1}{s-1} + \frac{1}{s+3} - \frac{3}{s+2} \right\}$$

$$= \boxed{2 + e^{t} + e^{-3t} - 3e^{-2t}}$$

(d) $\begin{cases} y'' + 3ty' - 6y = 1 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$\mathcal{L}\{y'' + 3ty' - 6y\} = \mathcal{L}\{1\}$$

$$\Rightarrow \left(\begin{array}{l} (s^2 Y(s) - s\overset{0}{y(0)} - \overset{0}{y'(0)}) \\ + 3(-1) \frac{d}{ds}(\mathcal{L}\{y'\}) - 6Y(s) \end{array} \right) = \frac{1}{s}$$

$$\Rightarrow s^2 Y(s) - 3\frac{d}{ds}(sY(s) - \cancel{y(0)}) - 6Y(s) = \frac{1}{s}$$

$$\Rightarrow s^2 Y(s) - 3Y(s) - 3sY'(s) - 6Y(s) = \frac{1}{s}$$

$$\Rightarrow -3sY'(s) + (s^2 - 9)Y(s) = \frac{1}{s}$$

$$\Rightarrow Y'(s) + \left(-\frac{s}{3} + \frac{3}{s}\right) Y(s) = -\frac{1}{3s^2}$$

$$\mu(s) = e^{\int(-\frac{s}{3} + \frac{3}{s})ds} = e^{-s^2/6 + 3\ln(s)}$$

$$= s^3 e^{-s^2/6}$$

$$\frac{d}{ds}\left(s^3 e^{-s^2/6} Y(s)\right) = -\frac{1}{3s^2} \cdot s^3 e^{-s^2/6}$$

$$s^3 e^{-s^2/6} Y(s) = \int -\frac{s}{3} e^{-s^2/6} ds$$

$u = -s^2/6 \quad du = -\frac{1}{3}s$

$$s^3 e^{-s^2/6} Y(s) = e^{-s^2/6} + C$$

$$Y(s) = \frac{1}{s^3} + \frac{Ce^{s^2/6}}{s^3}$$

Since $\lim_{s \to \infty} Y(s) = 0 \Rightarrow C = 0$.

$$Y(s) = \frac{1}{s^3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \boxed{\frac{1}{2}t^2}$$