HW #17

1 (a) $\mathcal{L}\{-20\mathcal{U}(t-4)\} = -20e^{-4s}\mathcal{L}\{1\}(s)$

$= \boxed{\dfrac{-20e^{-4s}}{s}}$

(b) $\mathcal{L}\{3\mathcal{U}(t-5)+6\mathcal{U}(t-8)\}(s)$

$= 3e^{-5s}\mathcal{L}\{1\}(s) + 6e^{-8s}\mathcal{L}\{1\}(s)$

$= \boxed{\dfrac{3e^{-5s}}{s} + \dfrac{6e^{-8s}}{s}}$

(c) $\mathcal{L}\{-12(t+2)^3\mathcal{U}(t-3)\}(s)$

$= \mathcal{L}\{-12((t-3)+5)^3\mathcal{U}(t-3)\}(s)$

$= -12e^{-3s}\mathcal{L}\{(t+5)^3\}(s)$

$= -12e^{-3s}\mathcal{L}\{(t^2+10t+25)(t+5)\}(s)$

$= -12e^{-3s}\mathcal{L}\{t^3+15t^2+75t+125\}$

$= \boxed{-12e^{-3s}\left(\dfrac{3!}{s^4} + \dfrac{15\cdot 2!}{s^3} + \dfrac{75\cdot 1!}{s^2} + \dfrac{125}{s}\right)}$

(d) $\mathcal{L}\{-2\cos(t-\frac{\pi}{4})\mathcal{U}(t-\frac{\pi}{4})\}(s)$

$= e^{-\frac{\pi}{4}s}\mathcal{L}\{-2\cos(t)\}$

$= \boxed{-2e^{-\pi/4 s}\cdot\dfrac{s}{s^2+1}}$

(e) Don't Do.

2 (a) $\mathcal{L}^{-1}\left\{\dfrac{2e^{2s}-5}{se^{5s}}\right\}(t)$

$= \mathcal{L}^{-1}\{\frac{2}{s}e^{-3s} - \frac{5}{s}e^{-5s}\}(t)$

$= \boxed{2\mathcal{U}(t-3) - 5\mathcal{U}(t-5)}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+9}\right\}(t)$

$= \mathcal{U}(t-3)\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\}$

$= \boxed{\mathcal{U}(t-3)\cos(3t)}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4s+5}\right\}(t)$

$= \mathcal{U}(t-3)\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}(t)$

$= \mathcal{U}(t-3)\mathcal{L}^{-1}\left\{\dfrac{s+2-2}{(s+2)^2+1}\right\}(t)$

$= \mathcal{U}(t-3)\left[\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+1}\right\}(t) - 2\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2+1}\right\}(t)\right]$

$= \boxed{\mathcal{U}(t-3)\left[e^{-2t}\cos(t) - 2e^{-2t}\sin(t)\right]}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\}(t)$

$= \mathcal{U}(t-3)\mathcal{L}^{-1}\left\{\dfrac{s-5}{(s+1)(s+2)}\right\}(t)$

$\dfrac{s-5}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$\Rightarrow s-5 = A(s+2)+B(s+1)$

$s=-2: \ -7 = -B \Rightarrow B = 7$

$s=-1: \ -6 = A \Rightarrow A = -6$

$= \mathcal{U}(t-3)\mathcal{L}^{-1}\left\{\dfrac{-6}{s+1} + \dfrac{7}{s+2}\right\}$

$= \boxed{\mathcal{U}(t-3)(-6e^{-t} + 7e^{-2t})}$

3) (a) $y'' + 4y = \begin{cases} \sin t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$

$y(0) = 1$

$y'(0) = 3$

$f(t) = \sin t\,(\mathcal{U}(t) - \mathcal{U}(t-2\pi))$

$= \sin t - \mathcal{U}(t-2\pi)$

$\mathcal{L}\{y''+4y\}(s) = \mathcal{L}\{\sin t - \mathcal{U}(t-2\pi)\}$

$\rightarrow (s^2Y(s) - sy(0) - y'(0)) + (4Y(s)) = \dfrac{1}{s^2+1} - \dfrac{e^{-2\pi s}}{s}$

$(s^2+4)Y(s) - s - 3 = \dfrac{1}{s(s^2+1)}e^{-2\pi}$

$(s^2+4)Y(s) = (s+3) + \dfrac{1}{s^2+1} - \dfrac{e^{-2\pi s}}{s(s^2+1)}$

Continued →

$$Y(s) = \frac{s+3}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{s(s^2+1)(s^2+4)}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \cos(2t) + \frac{3}{2}\sin(2t) \quad (*)$$

$$* \quad \frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

① $A+C=0$
② $B+D=0$
③ $4A+C=0$
④ $4B+D=1$

① $(A+C=0)\cdot -1 \longrightarrow -A-C=0$
③ $4A+C=0 \qquad\qquad 4A+C=0$

$3A=0$

$A=0$

$C=0$

② $(B+D=0)\cdot -1 \longrightarrow -B-D=0$
④ $4B+D=1 \qquad\qquad 4B+D=1$

$3B=1$

$B=1/3$

$D=-1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{1}{s^2+4}\right)\right\}(t)$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin(2t) \quad (**)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi}}{s(s^2+1)(s^2+4)}\right\}$$

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{A}{s} + \frac{(Bs+C)}{s^2+1} + \frac{(Ds+E)}{s^2+4}$$

$$1 = A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)s(s^2+1)$$

$$1 = A(s^4+5s^2+4) + (Bs^4+4Bs^2+Cs^3+4Cs) + (Ds^4+Ds^2+Es^3+Es)$$

$$= (A+B+D)s^4 + (C+E)s^3 + (5A+4B+D)s^2 + (4C+E)s + 4A$$

① $A+B+D=0$
② $C+E=0$
③ $5A+4B+D=0$
④ $4C+E=0$

⑤ $4A=1 \longrightarrow A=1/4$

① $(B+D=-1/4)\cdot -1 \longrightarrow -B-D=1/4$
③ $4B+D=-5/4 \longrightarrow 4B+D=-5/4$

$3B=-1$

$B=-1/3$

$-1/3+D=-1/4 \longrightarrow D=1/12$

② $(C+E=0)\cdot -1 \longrightarrow -C-E=0$
④ $4C+E=0 \longrightarrow 4C+E=0$

$3C=0$

$C=0$

$E=0$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s(s^2+1)(s^2+4)}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{3}\left(\frac{s}{s^2+1}\right) + \frac{1}{12}\left(\frac{s}{s^2+4}\right)\right\}(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$= \left(\frac{1}{4} - \frac{1}{3}\cos(t-2\pi) + \frac{1}{12}\cos(2(t-2\pi))\right)\mathcal{U}(t-2\pi)$$

$$= \left(\frac{1}{4} - \frac{1}{3}\cos(t) + \frac{1}{12}\cos(2t)\right)\mathcal{U}(t-2\pi) \quad (***)$$

$$y(t) = (*) + (**) - (***)$$

$$= \boxed{\cos(2t) + \frac{3}{2}\sin(2t) + \frac{1}{3}\sin t - \frac{1}{6}\sin(2t) + \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos(2t)\right)\mathcal{U}(t-2\pi)}$$

ⓑ $y''+4y'+4y = \mathcal{U}(t-\pi)-\mathcal{U}(t-2\pi)$

$y(0)=y'(0)=0.$

$$\mathcal{L}\{y''+4y'+4y\} = \mathcal{L}\{\mathcal{U}(t-\pi)-\mathcal{U}(t-2\pi)\}$$

$$\to \left(\begin{array}{l} (s^2Y(s)-sy(0)-y'(0)) \\ +4(sY(s)-y(0)) \\ +4Y(s) \end{array}\right) = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$(s^2+4s+4)Y(s) = \frac{1}{s}\left(e^{-\pi s}-e^{-2\pi s}\right)$$

$$Y(s) = \frac{1}{(s+2)^2 s}\left(e^{-\pi s}-e^{-2\pi s}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 s}\left(e^{-\pi s}-e^{-2\pi s}\right)\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)^2}\right\}(t-\pi)\,\mathcal{U}(t-\pi) - \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)^2}\right\}(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$\frac{1}{s(s+2)^2} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + Bs(s+2) + Cs$$

$s=0$: $1=4A \to A=1/4$

$s=-2$: $1=-2C \to C=-1/2$

$s=1$: $1=9A+3B+C$

$1=\frac{9}{4}+3B-\frac{1}{2}$

$-\frac{3}{4}=3B \to B=-1/4$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{4}\left(\frac{1}{s+2}\right)-\frac{1}{2}\left(\frac{1}{(s+2)^2}\right)\right\}(t-\pi)\,\mathcal{U}(t-\pi)$$

$$-\mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{4}\left(\frac{1}{s+2}\right)-\frac{1}{2}\left(\frac{1}{(s+2)^2}\right)\right\}(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$= \boxed{\begin{array}{l}\left(\frac{1}{4}-\frac{1}{4}e^{-2(t-\pi)}-\frac{1}{2}e^{-2(t-\pi)}(t-\pi)\right)\mathcal{U}(t-\pi) \\ -\left(\frac{1}{4}-\frac{1}{4}e^{-2(t-2\pi)}-\frac{1}{2}e^{-2(t-2\pi)}(t-2\pi)\right)\mathcal{U}(t-2\pi)\end{array}}$$