HW #3

1) a) $\frac{dy}{dx} - y = e^{3x}$

$$e^{\int -1\,dx} = e^{-x}$$

$\Rightarrow e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x}$

$\Rightarrow e^{-x}\frac{dy}{dx} - e^{-x}y = e^{2x}$

$\Rightarrow \frac{d}{dx}(e^{-x}y) = e^{2x}$

$\Rightarrow e^{-x}y = \int e^{2x}dx$

$\Rightarrow e^{-x}y = \frac{1}{2}e^{2x} + C$

$\Rightarrow \boxed{y = \frac{1}{2}e^{3x} + Ce^{x}}$

b) $\frac{dy}{dx} = \frac{y}{x} + 2x + 1 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = 2x+1$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = x^{-1}$$

$\Rightarrow x^{-1}\frac{dy}{dx} - x^{-1}\cdot\frac{1}{x}y = x^{-1}(2x+1)$

$\Rightarrow \frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}y = 2 + \frac{1}{x}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = 2 + \frac{1}{x}$

$\Rightarrow \frac{1}{x}y = \int\left(2 + \frac{1}{x}\right)dx$

$\Rightarrow \frac{1}{x}y = 2x + \ln|x| + C$

$\Rightarrow \boxed{y = 2x^2 + x\ln|x| + Cx}$

c) $\frac{dr}{d\theta} + r\tan\theta = \sec\theta$

$$e^{\int\tan\theta\,d\theta} = e^{\int\frac{\sin\theta}{\cos\theta}d\theta} = e^{-\int\frac{1}{u}du} = e^{-\ln|u|} = \frac{1}{\cos\theta}$$

$u = \cos\theta$, $du = -\sin\theta$

$\Rightarrow \frac{1}{\cos\theta}\frac{dr}{d\theta} + r\frac{\sin\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\Rightarrow \frac{d}{d\theta}\left(r\cdot\frac{1}{\cos\theta}\right) = \frac{1}{\cos^2\theta}$

$\Rightarrow r\cdot\frac{1}{\cos\theta} = \int\sec^2\theta\,d\theta$

$\Rightarrow r\cdot\frac{1}{\cos\theta} = \tan\theta + C$

$\Rightarrow r = \left(\frac{\sin\theta}{\cos\theta} + C\right)\cdot\cos\theta$

$\Rightarrow \boxed{r = \sin\theta + C\cos\theta}$

d) $(x^2+1)\frac{dy}{dx} + xy = x \Rightarrow \frac{dy}{dx} + \frac{x}{x^2+1}y = \frac{x}{x^2+1}$

$$e^{\int\frac{x}{x^2+1}dx} = e^{\frac{1}{2}\int\frac{1}{u}du} = e^{\frac{1}{2}\ln|u|} = u^{1/2} = (x^2+1)^{1/2}$$

$u = x^2+1$, $du = 2x\,dx$

$\Rightarrow (x^2+1)^{1/2}\frac{dy}{dx} + (x^2+1)^{1/2}\cdot\frac{x}{x^2+1}y = (x^2+1)^{1/2}\cdot\frac{x}{x^2+1}$

$\Rightarrow (x^2+1)^{1/2}\frac{dy}{dx} + \frac{x}{(x^2+1)^{1/2}}y = \frac{x}{(x^2+1)^{1/2}}$

$\Rightarrow \frac{d}{dx}\left((x^2+1)^{1/2}y\right) = \frac{x}{(x^2+1)^{1/2}}$

$\Rightarrow (x^2+1)^{1/2}y = \int\frac{x}{(x^2+1)^{1/2}}dx =$

$u = x^2+1$, $du = 2x$

$\Rightarrow (x^2+1)^{1/2}y = \int\frac{1}{u^{1/2}}du$

$\Rightarrow (x^2+1)^{1/2}y = 2u^{1/2} + C$

$\Rightarrow (x^2+1)^{1/2}y = 2(x^2+1)^{1/2} + C$

$\Rightarrow \boxed{y = 2 + C(x^2+1)^{-1/2}}$

2 a) $\begin{cases}\frac{dy}{dx} - \frac{y}{x} = xe^x \\ y(1) = e-1\end{cases}$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = x^{-1} = \frac{1}{x}$$

$\Rightarrow \frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{y}{x} = \frac{1}{x}\cdot xe^x$

$\Rightarrow \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = e^x$

$\Rightarrow \frac{1}{x}y = \int e^x dx$

$\Rightarrow \frac{1}{x}y = e^x + C$

$\Rightarrow y = xe^x + Cx$

$e - 1 = y(1) = e + C$

$\Rightarrow C = -1$

$\therefore \boxed{y = xe^x - x}$

b) $\begin{cases}t^3\frac{dx}{dt} + 3t^2x = t \Rightarrow \frac{dx}{dt} + \frac{3}{t}x = \frac{1}{t^2} \\ x(2) = 0\end{cases}$

$$e^{\int\frac{3}{t}dt} = e^{3\ln|t|} = t^3$$

$\Rightarrow t^3\frac{dx}{dt} + t^3\cdot\frac{3}{t}x = t^3\cdot\frac{1}{t^2}$

$\Rightarrow t^3\frac{dx}{dt} + 3t^2x = t$

$\Rightarrow \frac{d}{dt}(t^3x) = t$

$\Rightarrow t^3x = \int t\,dt$

$\Rightarrow t^3x = \frac{t^2}{2} + C$

$\Rightarrow \boxed{x = \frac{1}{2t} + \frac{C}{t^3}}$

$0 = x(2) = \frac{1}{4} + \frac{C}{8}$

$\Rightarrow -\frac{1}{4} = \frac{C}{8}$

$\Rightarrow 4C = -8$

$\Rightarrow C = -2$

$\therefore \boxed{x = \frac{1}{2t} - \frac{2}{t^3}}$

c) $\begin{cases}\cos x\frac{dy}{dx} + y\sin x = 2x\cos^2 x \Rightarrow \frac{dy}{dx} + \frac{\sin x}{\cos x}y = 2x\cos x \\ y(\pi/4) = \frac{-15\pi^2\sqrt{2}}{32}\end{cases}$

$$e^{\int\frac{\sin x}{\cos x}dx} = e^{-\int\frac{1}{u}du} = e^{-\ln|u|} = u^{-1} = \frac{1}{\cos x}$$

$u = \cos x$, $du = -\sin x\,dx$

$\Rightarrow \frac{1}{\cos x}\cdot\frac{dy}{dx} + \frac{\sin x}{\cos^2 x}y = 2x$

$\Rightarrow \frac{d}{dx}\left(y\cdot\frac{1}{\cos x}\right) = 2x$

$\Rightarrow \frac{y}{\cos x} = \int 2x\,dx$

$\Rightarrow \frac{y}{\cos x} = x^2 + C$

$\Rightarrow \boxed{y = x^2\cos x + C\cos x}$

$-\frac{15\pi^2\sqrt{2}}{32} = y(\pi/4) = \frac{\pi^2}{16}\cos(\pi/4) + C\cos(\pi/4)$

$\Rightarrow -\frac{15\pi^2\sqrt{2}}{32} = \frac{\pi^2\sqrt{2}}{32} + \frac{C\sqrt{2}}{2}$

$\Rightarrow -\frac{16\pi^2\sqrt{2}}{32} = \frac{C\sqrt{2}}{2}$

$\Rightarrow C = -\pi^2$

$\therefore \boxed{y = x^2\cos x - \pi^2\cos x}$

#3B in Section 2.3

$y = y_1(t)$ satisfies $\frac{dy}{dt} + p(t)y = 0 \implies \frac{dy_1}{dt} + p(t)y_1 = 0$

$y = y_2(t)$ satisfies $\frac{dy}{dt} + p(t)y = r(t) \Rightarrow \frac{dy_2}{dt} + p(t)y_2 = r(t)$

Show: $y(t) = y_1(t) + y_2(t)$ satisfies

$\frac{dy}{dt} + p(t)y = r(t)$.

Consider $y'(t) = y_1'(t) + y_2'(t)$.

Then
$$\begin{aligned}\frac{dy}{dt} + p(t)y &= y_1'(t) + y_2'(t) + p(t)(y_1(t) + y_2(t))\\ &= y_1'(t) + y_2'(t) + p(t)y_1(t) + p(t)y_2(t)\\ &= \underbrace{y_1'(t) + p(t)y_1(t)}_{} + \underbrace{y_2'(t) + p(t)y_2(t)}_{}\\ &= 0 + r(t)\\ &= r(t)\end{aligned}$$