HW #5

a) $\underbrace{(3x^2-y^2)}_{M}dx+\underbrace{(xy-x^3y^{-1})}_{N}dy=0$

$M(xt,yt)=3(xt)^2-(yt)^2=t^2(3x^2-y^2)=t^2M(x,y)$

$N(xt,yt)=(xt)(yt)-(xt)^3(yt)^{-1}$
$=xyt^2-x^3t^3y^{-1}t^{-1}$
$=xyt^2-x^3y^{-1}t^2$
$=t^2(xy-x^3y^{-1})$
$=t^2N(x,y)$

∴ Homogeneous of degree 2

Let $y=ux$. Then $dy=udx+xdu$

$(3x^2-(ux)^2)dx+(x(ux)-x^3(ux)^{-1})(udx+xdu)=0$

$\Rightarrow 3x^2dx-\cancel{u^2x^2dx}+\cancel{x^2u^2dx}+x^3udu-x^2dx-x^3u^{-1}du=0$

$\Rightarrow 2x^2dx=(x^3u^{-1}-x^3u)du$

$\Rightarrow 2x^2dx=x^3(u^{-1}-u)du$

$\Rightarrow \frac{2}{x}dx=(u^{-1}-u)du$

$\Rightarrow \int\frac{2}{x}dx=\int(u^{-1}-u)du$

$\Rightarrow 2\ln|x|=\ln|u|-\frac{u^2}{2}+C$

$\Rightarrow 2\ln|x|=\ln|y/x|-\frac{(y/x)^2}{2}+C$

$\Rightarrow 4\ln|x|=2(\ln y-\ln x)-(y/x)^2+C$

$\Rightarrow \boxed{6\ln|x|=2\ln y-(y/x)^2+C}$

b) $\frac{dx}{dt}=\frac{x^2+t\sqrt{t^2+x^2}}{tx}\Rightarrow(x^2+t\sqrt{t^2+x^2})dt-txdx=0$

$M(ut,ux)=(ux)^2+(ut)\sqrt{(ut)^2+(ux)^2}$
$=u^2x^2+ut\sqrt{u^2(t^2+x^2)}$
$=u^2x^2+u^2t\sqrt{t^2+x^2}$
$=u^2(x^2+t\sqrt{t^2+x^2})$
$=u^2M(t,x)$

$N(ut,ux)=(ut)(ux)=u^2tx=u^2N(t,x)$.

∴ Homogeneous of degree 1.

Let $x=ut$. Then $dx=udt+tdu$

$((ut)^2+t\sqrt{t^2+(ut)^2})dt-t(ut)(udt+tdu)=0$

$\Rightarrow \cancel{u^2t^2dt}+t^2\sqrt{1+u^2}\,dt-\cancel{u^2t^2dt}-ut^3du=0$

$\Rightarrow t^2\sqrt{1+u^2}\,dt=ut^3du$

$\Rightarrow \frac{t^2}{t^3}dt=\frac{u}{\sqrt{1+u^2}}du$

$\Rightarrow \int\frac{1}{t}dt=\int\frac{u}{\sqrt{1+u^2}}du$

$w=1+u^2,\ dw=2udu$

$\Rightarrow \ln|t|+C=\frac{1}{2}\int\frac{1}{\sqrt{w}}dw$

$\Rightarrow \ln|t|+C=\sqrt{w}$

$\Rightarrow \ln|t|+C=\sqrt{1+u^2}$

$\Rightarrow \boxed{\ln|t|+C=\sqrt{1+(x/t)^2}.}$

c) $\frac{dy}{dx}=\frac{y(\ln y-\ln x+1)}{x}\Rightarrow y(\ln y-\ln x+1)dx-xdy=0$

$\Rightarrow \underbrace{y(\ln(y/x)+1)}_{M}dx-\underbrace{x}_{N}dy=0$

$M(xt,yt)=yt(\ln(yt)-\ln(xt)+1)$
$=yt(\ln y+\cancel{\ln t}-\ln x-\cancel{\ln t}+1)$
$=yt(\ln(y/x)+1)$
$=tM(x,y)$

$N(xt,yt)=xt=tN(x,y)$

∴ Homogeneous of degree 1.

Let $y=ux$. Then, $dy=udx+xdu$.

$ux(\ln(\frac{ux}{x})+1)dx-x(udx+xdu)=0$

$\Rightarrow ux\ln(u)dx+\cancel{uxdx}-\cancel{uxdx}-x^2du=0$

$\Rightarrow ux\ln(u)dx=x^2du$

$\Rightarrow \frac{1}{x}dx=\frac{1}{u\ln u}du$

$\Rightarrow \int\frac{1}{x}dx=\int\frac{1}{u\ln u}du$ $\quad w=\ln u,\ dw=\frac{1}{u}$

$\Rightarrow \ln|x|+C=\int\frac{1}{w}dw$

$\Rightarrow \ln|x|+C=\ln|w|$

$\Rightarrow \ln|x|+C=\ln(\ln u)$

$\Rightarrow e^{\ln x+C}=e^{\ln(\ln u)}$

$\Rightarrow Cx=\ln(u)$

$\Rightarrow Cx=\ln(y/x)$

$\Rightarrow e^{Cx}=e^{\ln(y/x)}$

$\Rightarrow \boxed{y=xe^{Cx}}$

d) $\frac{dy}{dx} + \frac{y}{x} = x^2 y^2$ $\qquad w = y^{1-n} = y^{1-2} = y^{-1}$

$\frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$

Bernoulli : $n=2$

$y^{-2}\frac{dy}{dx} + \frac{y}{x}y^{-2} = x^2 y^2 \cdot y^{-2}$

$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x^2$

$\Rightarrow -\frac{dw}{dx} + \frac{1}{x}w = x^2$

$\Rightarrow \frac{dw}{dx} - \frac{1}{x}w = -x^2$

$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

$\Rightarrow \frac{1}{x}\frac{dw}{dx} - \frac{1}{x^2}w = -x$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}w\right) = -x$

$\Rightarrow \frac{1}{x}w = \int -x\,dx$

$\Rightarrow \frac{1}{x}w = -\frac{x^2}{2} + C$

$\Rightarrow w = -\frac{x^3}{2} + Cx$

$\Rightarrow \frac{1}{y} = -\frac{x^3}{2} + Cx$

$\Rightarrow y = \frac{1}{-\frac{x^3}{2} + Cx}$

$\Rightarrow \boxed{y = \frac{2}{-x^3 + 2Cx}} \Rightarrow \boxed{y = \frac{2}{-x^3 + Cx}}$

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e) $\frac{dy}{dx} - y = e^{2x}y^3$ $\qquad w = y^{1-n} = y^{1-3} = y^{-2}$

$\frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$

Bernoulli : $n=3$

$y^{-3}\frac{dy}{dx} - y^{-3}\cdot y = e^{2x}\cdot y^{-3}\cdot y^3$

$\Rightarrow y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}$

$\Rightarrow -\frac{1}{2}\frac{dw}{dx} - w = e^{2x}$

$\Rightarrow \frac{dw}{dx} + 2w = -2e^{2x}$

$\mu(x) = e^{\int 2dx} = e^{2x}$

$\Rightarrow e^{2x}\frac{dw}{dx} + 2e^{2x}w = -2e^{4x}$

$\Rightarrow \frac{d}{dx}(e^{2x}w) = -2e^{4x}$

$\Rightarrow e^{2x}w = \int -2e^{4x}dx$

$\Rightarrow e^{2x}w = -\frac{2}{4}e^{4x} + C$

$\Rightarrow w = -\frac{1}{2}e^{2x} + Ce^{-2x}$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2}e^{2x} + Ce^{-2x}$

$\Rightarrow y^2 = \frac{1}{-\frac{1}{2}e^{2x} + Ce^{-2x}}$

$\boxed{y^2 = \frac{-2}{e^{2x} - Ce^{-2x}}}$

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f) $\frac{dx}{dt} + tx^3 + \frac{x}{t} = 0$ $\qquad w = x^{1-n} = x^{1-3} = x^{-2}$

$\frac{dw}{dt} = -2x^{-3}\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} + \frac{x}{t} = -tx^3$

Bernoulli : $n=3$

$x^{-3}\frac{dx}{dt} + \frac{1}{t}x^{-2} = -t$

$\Rightarrow -\frac{1}{2}\frac{dw}{dt} + \frac{1}{t}w = -t$

$\Rightarrow \frac{dw}{dt} - \frac{2}{t}w = 2t$

$\mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln|t|} = t^{-2}$

$\Rightarrow t^{-2}\frac{dw}{dt} - \frac{2}{t}w\,t^{-2} = 2t\cdot t^{-2}$

$\Rightarrow t^{-2}\frac{dw}{dt} - 2wt^{-3} = 2t^{-1}$

$\Rightarrow \frac{d}{dt}(t^{-2}w) = \frac{2}{t}$

$\Rightarrow t^{-2}w = \int \frac{2}{t}dt$

$\Rightarrow t^{-2}w = 2\ln|t| + C$

$\Rightarrow w = 2t^2\ln|t| + Ct^2$

$\Rightarrow \boxed{x^{-2} = 2t^2\ln|t| + Ct^2}$