Complex 1 point per part.

1) $z = 1+i$ & $w = -2+2\sqrt{3}i$

a) $z+w = (1+i)+(-2+2\sqrt{3}i) = -1+(1+2\sqrt{3})i$

b) $zw = (1+i)(-2+2\sqrt{3}i) = -2+2\sqrt{3}i - 2i - 2\sqrt{3} = (-2-2\sqrt{3}) + (2\sqrt{3}-2)i$

c) $\arg(z)$ is the angle θ_1 for which $z = |z|e^{i\theta_1}$

$\Rightarrow 1+i = \sqrt{2}\, e^{i\theta_1} = \sqrt{2}(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \cos\theta_1 = \frac{\sqrt{2}}{2}$ & $\sin\theta_1 = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta_1 = \pi/4$

d) $\arg(w)$ is the angle θ_2 for which $w = |w|e^{i\theta_2}$

$\Rightarrow -2+2\sqrt{3}i = \sqrt{(-2)^2+(2\sqrt{3})^2}\, e^{i\theta_2} = 4(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow -2 = 4\cos\theta_2$ & $2\sqrt{3} = 4\sin\theta_2$

$\Rightarrow \cos\theta_2 = -\frac{1}{2}$ & $\sin\theta_2 = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta_2 = \pi/3$

e) $\arg(zw)$ is the angle θ for which $zw = |zw|e^{i\theta}$.

From notes: $|zw| = |z||w|$ & (argument of zw is $\theta_1+\theta_2$ where θ_1 is argument of z & θ_2 is argument of w)

$\Rightarrow \theta = \theta_1 + \theta_2 = \pi/4 + \pi/3 = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}$

f) $\bar{z} = 1-i$

g) $|z| = \sqrt{1^2+1^2} = \sqrt{2}$

h) $|w| = \sqrt{(-2)^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

i) $\frac{z}{w} = \frac{1+i}{-2+2\sqrt{3}i} \cdot \frac{-2-2\sqrt{3}i}{-2-2\sqrt{3}i} = \frac{-2-2\sqrt{3}i-2i+2\sqrt{3}}{4+12} = \boxed{\frac{(-2+2\sqrt{3})}{16} + \frac{(-2\sqrt{3}-2)}{16}i}$

j) $\frac{z\bar{w}}{|w|^2} = \frac{(1+i)(-2-2\sqrt{3}i)}{4^2} = \frac{-2-2\sqrt{3}i-2i+2\sqrt{3}}{16} = \frac{(2\sqrt{3}-2)}{16} + \frac{(-2\sqrt{3}-2)}{16}i$

$= \boxed{\frac{\sqrt{3}-1}{8} - \frac{\sqrt{3}+1}{8}i}$

2) a) $e^{1+\frac{\pi}{2}i} = e \cdot e^{\frac{\pi}{2}i} = e(\cos(\pi/2) + i\sin(\pi/2)) = \boxed{0 + ei}$

b) $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = \boxed{-1 + 0i}$

3) a) θ is a real # and $z = e^{i\theta}$

Show: $|z| = 1$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow |z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$ ✓

b) Let $z = a+bi$. Show: $|e^z| = e^a$.

$\Rightarrow e^z = e^{a+bi} = e^a \cdot e^{bi} = e^a(\cos(b) + i\sin(b)) = e^a\cos(b) + ie^a\sin(b)$

$$|e^z| = \sqrt{(e^a\cos(b))^2 + (e^a\sin(b))^2} = \sqrt{e^{2a}\cos^2(b) + e^{2b}\sin^2(b)}$$

$$= \sqrt{e^{2a}(\cos^2 b + \sin^2 b)}$$

$$= \sqrt{e^{2a}(1)}$$

$$= \sqrt{e^{2a}}$$

$$= e^a$$ ✓

4) Find complex numbers z s.t. $z^2 = \bar{z}$.

Let $z = a+bi$. Then $\bar{z} = a - bi$.

Therefore, $z^2 = \bar{z}$

$\Rightarrow (a+bi)^2 = a - bi$

$\Rightarrow a^2 + 2abi - b^2 = a - bi$

$\Rightarrow a^2 - b^2 = a$ & $2ab = -b$

$\Rightarrow b^2 = a^2 - a$ & $(2a+1)b = 0$

$\Rightarrow b = 0$ or $2a+1 = 0$

If $b = 0 \Rightarrow 0 = a^2 - a \Rightarrow a(a-1) = 0 \Rightarrow a = 0$ or $a = 1$

If $2a+1 = 0 \Rightarrow a = -1/2 \Rightarrow b^2 = \frac{1}{4} - (-\frac{1}{2}) = \frac{3}{4} \Rightarrow b = \pm\frac{\sqrt{3}}{2}$

Possible Choices: $\boxed{a=0 \ \& \ b=0, \ a=1 \ \& \ b=0, \ a=-1/2 \ \& \ b=\sqrt{3}/2 \text{ or } a=-1/2 \ \& \ b=-\sqrt{3}/2}$

5) Show:

a) $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$

b) $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$

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a) $$\frac{e^{ix}+e^{-ix}}{2} = \frac{(\cos x + i\sin x) + (\cos(-x) + i\sin(-x)}{2}$$

$$= \frac{\cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}}{2}$$

$$= \frac{2\cos x}{2}$$

$$= \cos x \quad \checkmark$$

b) $$\frac{e^{ix}-e^{-ix}}{2i} = \frac{(\cos x + i\sin x) - (\cos(-x) + i\sin(-x))}{2i}$$

$$= \frac{\cos x + i\sin x - (\cos x - i\sin x)}{2i}$$

$$= \frac{\cos x + i\sin x - \cos x + i\sin x}{2i}$$

$$= \frac{2i\sin x}{2i}$$

$$= \sin x \quad \checkmark$$