MATH 720, Algebra I
Exercises 3
Due Fri 14 Sep

Exercise 1. Show that every finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.

Exercise 2. Let $f\colon G \to G'$ be a group homomorphism.
(a) If $H' \leqslant G'$, then $f^{-1}(H') \leqslant G$.
(b) If $H' \trianglelefteq G'$, then $f^{-1}(H') \trianglelefteq G$.
(c) If $H' \trianglelefteq G'$, then the function $\overline{f}\colon G/f^{-1}(H') \to G'/H'$ is a well-defined group monomorphism

Exercise 3. Find all groups of orders $\leqslant 7$ up to isomorphism.

Exercise 4. Let H and K be groups. An *automorphism* of H is a group isomorphism $f\colon H \to H$. Let $\mathrm{Aut}(H)$ denote the set of all automorphisms of H.
(a) Show that $\mathrm{Aut}(H)$ is a group under composition.
(b) Let $\phi\colon K \to \mathrm{Aut}(H)$ be a group homomorphism. For each $k \in K$ set $\phi_k = \phi(k)$. Define $H \times_\phi K$ to be the cartesian product of H and K with multiplication $(h,k)(h',k') := (h\phi_k(h'), kk')$. Show that $H \times_\phi K$ is a group, that $H \cong H \times \{e_K\} \trianglelefteq H \times_\phi K$, that $K \cong \{e_H\} \times K \leqslant H \times_\phi K$, and that under these identifications $\phi_k(h) = khk^{-1}$. (The group $H \times_\phi K$ is the *semi-direct product* of H and K with respect to ϕ.)
(c) Let $f\colon H \times K \to H \times_\phi K$ be given by $f(h,k) = (h,k)$ Show that $\phi = \mathrm{id}_H$ if and only if f is a group homomorphism. (Note that f is bijective, so it is a homomorphism if and only if it is an isomorphism.)
(d) Let G be a group and assume $H \trianglelefteq G$ and $K \leqslant G$. Assume that $HK = G$ and $H \cap K = \{e\}$. For $h \in H$ and $k \in K$, define $\phi_k(h) = khk^{-1}$. Show that $\phi_k \in \mathrm{Aut}(H)$, that the function $\phi\colon K \to \mathrm{Aut}(H)$ given by $\phi(k) = \phi_k$ is a group homomorphism, and that $G \cong H \times_\phi K$.