MATH 720, Algebra I
Exercises 5
Due Fri 28 Sep

Exercise 1. Let $\varphi\colon G \to H$ be a group homomorphism.

(a) Show $\varphi([G,G]) \subseteq [H,H]$.
(b) Show $\varphi(G^{(n)}) \subseteq H^{(n)}$ for each $n \geqslant 0$.
(c) Show $\varphi(G_{(n)}) \subseteq H_{(n)}$ for each $n \geqslant 0$.
(d) If φ is an epimorphism, then $\varphi(G^{(n)}) = H^{(n)}$ and $\varphi(G_{(n)}) = H_{(n)}$ for each $n \geqslant 0$.
(e) Find an example where $\varphi([G,G]) \neq [H,H]$.

Exercise 2. Let G be a group and $x \in G$. The function $\varphi_x\colon G \to G$ given by $\varphi_x(y) = xyx^{-1}$ is an *inner automorphism of* G. Let $\mathrm{Inn}(G)$ denote the set of all inner automorphisms of G. Show $\mathrm{Inn}(G) \trianglelefteqslant \mathrm{Aut}(G)$.

Exercise 3. Assume that G acts on sets S and S'.

(a) For all $s \in S$ and all $g \in G$, show that $gG_sg^{-1} = G_{gs}$.
(b) If $\varphi\colon S \to S'$ is an isomorphism of G-sets, then $\varphi^{-1}S' \to S$ is an isomorphism of G-sets.

Exercise 4. Let G be a group.

(a) Show $G^{(n)} \subseteq G_{(n)}$ for each $n \geqslant 0$.
(b) If G is nilpotent, then G is solvable.