MATH 720, Algebra I
Exercises 10
Due Fri 02 Nov
Instructions: Complete four of the following problems. You may submit the fifth problem for extra credit.

Exercise 1. Let $\mathcal{C}$ be a category and fix an object A in $\mathcal{C}$. Show that the definition of $\operatorname{Hom}_{\mathcal{C}}(-, A)$ from Example 1.2.5 of the notes yields a contravariant functor $\mathcal{C} \to \underline{\text{sets}}$.

Exercise 2. Fix an integer $n \geqslant 2$. Show that the following conditions are equivalent.
(i) $\mathbb{Z}/n\mathbb{Z}$ is a field;
(ii) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain;
(iii) n is prime.

Exercise 3. Let G be a group with operation written multiplicatively, and let R be a ring with identity. Let $R[G]$ be the group ring from Example 3.1.5.
(a) Show that $R[G]$ is a ring with identity $1_{R[G]} = 1_R e_G$.
(b) Show that $R[G]$ is commutative if and only if R is commutative and G is abelian.

Exercise 4. Let R be a ring with identity. An element $r \in R$ is a *unit* if it has a two-sided inverse in R. Let $I \subseteq R$ be a left ideal (or a right ideal). Show that the following conditions are equivalent.
(i) $I = R$;
(ii) $1_R \in I$;
(iii) I contains a unit.

Exercise 5. Let R be a commutative ring with identity. Show that the following conditions are equivalent.
(i) R is a field;
(ii) R has exactly two ideals;
(iii) $\{0_R\}$ and R are the only ideals of R.